554

"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit.* lib. i. cap. 1. Not.

80269

"Meditationis est perscrutari occulta; contemplationis est admirari perspicua .... Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

———————

———"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."
                              J. B. Pinelli ad Mazonium.

ALERE   FLAMMAM.
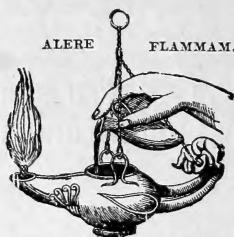
CONTENTS OF VOL. VI.

(SIXTH SERIES).

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PLATES.

ERRATA.

For γ in pages 20, 23, 24, 25, 26, read $\wp$.

Page 24, bottom line, for $l = \dfrac{1 - \sqrt{k'}}{1 - \sqrt{k}}$ read $l = \dfrac{1 - \sqrt{k'}}{1 + \sqrt{k'}}$.

INDEXED

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J U L Y 1903.

I. *On Electrostriction. By* LOUIS T. MORE, *Ph.D., Professor of Physics in the University of Cincinnati* [*].

[Plate I.]

IN a previous paper[†], I published an account of some ex-
periments made on the supposed elongation of a dielectric
placed in an electrostatic field. The conclusion reached was
that glass and hard rubber do not elongate perpendicularly
to the field, and that the effects observed by others were due
to different causes. This conclusion was something of a
disappointment, as I had undertaken the work for the purpose
of solving a problem based on this effect of electricity.

The laws of electrostriction had been so elaborately formu-
lated, and had been substantiated by so many experimenters,
that their results were accepted and an apparatus constructed
which would record changes of the given magnitude, allowing
for a margin of safety. In the early stages of the work,
when the tubes of glass operated on were charged, elongations
were observed and recorded in my notes which agreed fairly
with those of my predecessors. But combined with them
were, at times, irregularities and discrepancies. The motion
of the recording image was jerky, and the return of the tube
after discharge sometimes great and at other times small;
occasionally a tube was used which showed practically no
deflexion when charged. After considerable difficulty these

disturbances were located, and proved to be due principally to heating of the glass and to a slight bending of the tube, produced by a field of force not absolutely uniform in all directions.

As these discrepancies threw doubt on my results, the work of others was compared, and so many inconsistencies became evident that I abandoned my original plan and turned my attention to the subject of electrostriction itself.

From experiments made by Quincke* it could be inferred that rigidity and accuracy of alignment were very essential, and my apparatus had, as I thought, been constructed to obviate errors due to the neglect of these conditions. But repeated trials convinced me that the utmost care was necessary to eliminate their influence. I finally did get such uniform tubes in so accurate and rigid adjustment as to reduce them to a minimum. But with the elimination of these extraneous phenomena the entire elongation disappeared also. So I concluded that either the effect did not exist or was much smaller than was supposed.

Some time afterwards Dr. Sacerdote called my attention to his own memoir on the theory and to the experimental work of Cantone. He argued that the absence of the effect in my case resulted from employing an apparatus not sufficiently sensitive. His criticisms were afterwards published†. In a reply ‡ I attempted to show that Dr. Sacerdote had misunderstood certain important statements of mine, and that my apparatus did have a sensibility great enough to have indicated the effect, assuming the values given by Cantone. The results of others, it was stated, were not quantitatively reliable.

After some correspondence, Dr. Sacerdote has asked me to repeat my work, using a more sensitive apparatus. The delay in doing so has been somewhat great; but it has the advantage of allowing time for the discussion of the subject which has occurred.

It is hardly necessary to give an account of the experimental results on this subject, as they have already been described and discussed by myself and others, excepting the work of the more recent investigators.

Prof. Cantone §, alone and in collaboration with Sozzani ∥,

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* Quincke, Wied. *Ann.* x. pp. 161, 374, 513; xix. pp. 545, 705; xxviii. p. 529; xxxii. p. 530.
† Sacerdote, Phil. Mag. vol. i. p. 357 (1901).
‡ Phil. Mag. vol. ii. p. 527 (1901).
§ Cantone, *Rend. della R. Acc. dei Lincei*, t. iv. pp. 344, 471 (1888).
∥ Cantone and Sozzani, *Rend. della R. Ist. Lombardo*, t. xxxiii. (1900).

has published many experiments on the behaviour of glass condensers when charged. For the purpose he used cylindrical tubes about 70 centimetres long and 7 millimetres in diameter, ranging from 0·045 to 0·2 centimetre in thickness. The potentials to charge them occasionally were great enough to give a 10 millimetre spark. The armatures of the condensers were adherent to the glass, and the elongations were measured by the displacement of interference-bands. The greatest displacement noted is 2·6 bands, but the great majority were less than one band, and they were measured to one-hundredth. The elongations, reduced to unit dimensions, average

$$\frac{\delta l}{l} \times \frac{d^2}{V^2} = 6 \times 10^{-13}.$$

This result is calculated from the deflexions observed when the tubes were charged; the return, when the tubes were discharged, sometimes equalled the above, and was sometimes less than half. The time of charging also affected the amount of the elongation; but no law could be formulated, and no reasons are given for the time of charging determined on to give the correct value.

In spite of the carefulness which Cantone's experiments show, there are evidences that the values given are, at least, partly due to causes other than the one he ascribes them to.

In the first place, the amount of the elongation increases with the time of charging and to an unknown extent.

Secondly, they are partly due to heating of the tube. Cantone himself attributes the erratic return of the tubes to the heat evolved; and this varies between a complete return and a value of only one-half. If so, surely part of the elongation in charging must be due to the same cause.

Thirdly, the armatures touch the glass, and their attraction will elongate the tube, using Poisson's ratio, approximately one-third the observed amount.

Fourthly, tubes of the dimensions given are flexible, rarely straight, and do not have a uniform wall-thickness. All of these defects tend to strain the tube when it is charged, as the field is not uniform. The amount of this strain is large, and even with unusually accurate tubes it is difficult to avoid an apparent elongation greater than the supposedly true effect.

Unless these causes of possible elongation are eliminated, the elongation observed is subject to criticism.

In answer to a statement made in my paper, that the true effect could be obtained only when free charges were not placed on the dielectric; or, in other words, when the armatures of the condenser were separated from the dielectric and

the intervening spaces filled with a non-conducting fluid having a dielectric constant equal to that of the solid dielectric, Prof. Ercolini [*] made a series of experiments with such armatures.

The arrangement of his apparatus was similar to mine, except that the changes of length were measured by the displacement of interference-bands.

A comparative table of the dimensions of the apparatus will show their relative advantages.

TABLE I.

	Ercolini.	More.	
		Former.	Present.
Outer diameter of glass tube (A) ...	16·33 mm.	57·00 mm.	44·00 mm.
Inner ,, ,, ,, ,, ...	15·30 ,,	52·00 ,,	40·00 ,,
Thickness ,, ,, ,, ...	0·52 ,,	2·50 ,,	2·00 ,,
Distance between A and armatures .	2·00 ,,		
,, ,, armatures	4·52 ,,	7·50 ,,	4·00 ,,
Length of armatures	550·00 ,,	440·00 ,,	550·00 ,,
Value of least reading	0·01 fringe	$1·5 \times 10^{-5}$,,	3×10^{-6} ,,

Ercolini states that each of the tubes operated on experienced at the moment of charge an elongation, which disappeared with the discharge. But that these results were so small and disturbed by oscillations occurring when the spark passed that he could not obtain numerical results. Together with this elongation of the tube were *lateral displacements* shown by irregular movements of the bands, which could never be avoided. These were probably due to irregularities in the thickness of the glass and to lack of coincidence in the axes of the three tubes.

Just how the writer could distinguish between the regular and irregular displacement of the bands, it is difficult to see, since they must have occurred simultaneously ; nor does he take any notice of effect of the heating of the tubes.

Dr. Sacerdote [†] considers this case to require the special equation

$$\frac{\delta l}{l} = k_1 \frac{\overline{K} H^2}{8\pi}.$$

That is, the elongation is dependent only on the difference of potential, the dielectric constant of the glass, the variation of this constant with pressure k_1, and the distance between

* Ercolini, *Nuovo Cimento*, ser. 5, t. ii. p. 5 (1901).
† Ercolini, *Journ. de Phys.* sér. 4, t. i. p. 40.

the armatures ; but that it is independent of the nature of the non-conducting fluid.

These experiments are adduced by Ercolini and Sacerdote as proving that the coefficient k_1 is positive. That is, the dielectric constant increases with a tractile force acting perpendicularly to the field of force. In a note it is stated that the elongation was too small to be even noted by my apparatus. Possibly that is dismissing the matter too curtly unless the estimate of the sensibility of my apparatus made by Dr. Sacerdote is the correct one. In my own opinion, formed after long experience with it, this sensibility is at least $3\frac{1}{2}$ times as great as he assumes it to be ; and I thought such was evident from my explanation published after his first criticism. Even if this criticism were valid in respect to my former apparatus, it certainly is not to my present one. The only apparent advantage of Ercolini's arrangement is that he used thinner tubes ; but by doing so he increased the effect of the irregular movements he describes. Also it must be remembered that I used potentials great enough to rupture the tube, and a more intense field could hardly have been obtained. Yet, as will be shown later, no elongation was observed with non-adherent armatures when lateral effects were eliminated. The conclusions of Ercolini's work have been questioned by Corbino, and, I think, rightly.

The most recent work in this field is by Dr. Shearer[*]. The writer experimented on condensers of glass and hard rubber with adherent armatures. The method of measuring the effect was by the displacement of interference-bands. His results on rubber and glass may be summarized as follows,— the supposed elongation being for these substances of the order 10^{-8} and 10^{-13} per unit dimensions respectively.

Rubber plate condenser 40 cms. long and 3 mm. thick. Difference of potential sufficient to puncture glass 3 mm. thick. A displacement of approximately one-fourth of a band was observed, increasing gradually, and returning to the original position after each discharge. This effect disappeared after the first few charges when there was no spark-gap in the circuit. A very slight twist of the hard rubber, by reason of non-uniformity of the field, might easily cause this effect. When a spark-gap was inserted a considerable displacement of the bands, increasing greatly with the time, occurred. On looking between the condenser-plates the whole region appeared filled with a glow-discharge. It may be inferred that this motion was due to a gradual heating of the hard rubber.

[*] Shearer, Physical Review, vol. xiv. p. 89 (1902).

Glass rod with capillary bore, length 45 cms., diameter 5 mm. Potential from 14 to 80 kilovolts. No movement whatever was observed except when a spark passed.

Glass tubes 75 cms. long, wall-thickness from 0·08 to 0·12 cm. Potential in each case great enough to rupture the tube. No movement so large as one-half a band-width was observed, when the conditions were such as to avoid a direct heating effect. (With tubes of the same length and 0·13 cm. thick, Cantone records a displacement of 1·4 band for a spark-length of 1 centimetre.)

Glass tubes, silvered part of the length 73 cms., wall-thickness 0·24 cm. Potential produced a spark 5 cms. long, at which distance the tubes broke. The writer shows that, compared with Cantone's results, the expansion should have been far within the limits of possible readings with the instrument. No effects were observed other than those plainly due to temperature changes. Not the least motion of the bands could be detected when the potential was instantaneously raised by connexion being made with a set of highly charged jars.

Hard rubber tubes, length 100 cms., wall-thickness 2·5 mm. Potentials increased to 4 cms. spark, when tubes broke. According to Korteweg and Julius, the expansion of rubber is of the order 10^{-8}, instead of 10^{-13}, the coefficient of glass. When charged for one minute, there was a slow motion of 18 bands across the field. About ten minutes were required for the complete return. This motion was shown to be due to heating. A long piece of fine copper wire was wound spirally about a similar tube. The wire was inserted in a Wheatstone-bridge and the resistances balanced. The tube was then charged with a potential to give a 2-cm. spark, the displacement of the fringes noted, and the tube discharged. On again measuring the resistance of the wire it was found to be greater, and the increase was measured. Calculation of the rise in temperature of the wire proved that the heating of the tube was of the same order as the elongation of the tube, if it were due to heating alone.

Dr. Shearer concludes from his experiments that the observed changes in length may be readily accounted for without reference to other causes than heat and slight distortions due to an unsymmetrical field of force ; and that the production of changes of form and dimensions in matter by purely electrostatic charge is extremely improbable.

For the theoretical development of this subject the reader may refer to the memoirs of Sacerdote and Shearer, who have given complete bibliographies.

The greatest confusion in the correct interpretation of the formulæ arises, I think, from what may be called a too material view of the æther. It has proved so convenient a medium on which to hang our equations that we are apt to forget that it has, and must necessarily have, so far as we are concerned, a hypothetical existence; and a mechanical interpretation of one of its functions invariably breaks down either *per se*, or when it is compared with other functions. Witness Maxwell's attempt to explain mechanically displacement-currents of electricity, or the abandonment of the elaborately developed elastic-solid theory of light. But if the functions ascribed to the æther itself are contradictory, they become hopelessly involved when the æther is associated with matter. Even if Faraday's hypothesis, that an electric charge produces a strain in the æther, be granted, it does not at all follow that it will be accompanied by unbalanced mechanical forces in matter immersed in the æther. In fact, every experiment has failed which attempted to show a connexion between the two when the effects were static or kinetic, if the velocity was comparatively small. When the velocity of the kinetic energy involved is as great as or greater than that of light, the presence of matter in the æther does seem to affect the action. Certainly Kerr's effect, and others of the same kind, can be as readily explained as purely mechanical or thermal actions.

The most recent and complete of these theories of electrostriction is by Sacerdote, who gives for the relation between the electrical field and the elongation of a cylinder having adherent armatures

$$\frac{\delta l}{l} = (a + k_1) \frac{\overline{\text{K}} \text{V}^2}{8\pi d^2}.$$

In this equation a is the inverse of Young's modulus and k_1 the coefficient of variation of the dielectric constant $\overline{\text{K}}$, produced by a tractile force acting perpendicularly to the lines of force. The remaining letters have their usual significance. The value of k_1 has never been determined with any accuracy. Indeed, even its sign is a matter of dispute, some obtaining a positive value, and others a negative one. It is introduced into the equation on the ground that the relation is a reciprocal one. If the dielectric constant of a substance increases with a tractile force applied perpendicularly to the lines of force, it is assumed, when a dielectric is charged, that the field of force should produce a tractile force, accompanied by a corresponding elongation. But it should be remembered that this coefficient is not yet determined, and even if it were, the reciprocal relation may not exist. In a general way, the

dielectric constant is greater, the denser the substance ; it
may well be that if a gas, or a solid to a less degree, were
compressed, and the molecules thus become closer together,
the dielectric constant might increase.   It does not at all
follow that the increase is accompanied by a repulsion of
the molecules at right angles to the field, unless we make the
assumption that an electrical charge produces in some manner
a mechanical strain.   If the dielectric constant is shown by
experiment to increase with a tractile force, not because the
molecules of matter are incidentally brought closer together, but
because of the stress itself, then the reciprocal relation that
a charge causes a strain perpendicularly to the field will hold,
or *vice versâ.*   But until one or the other is indisputably
proven, to introduce either one into the equation to prove the
other is tacitly assuming the whole question of the mechanical
relations between æther and matter.

In a recent paper Dr. Sacerdote * discusses theoretically
the special case when the armatures are separated from the
solid dielectric by spaces filled with a non-conducting fluid.
He arrives at the conclusion that the modulus of elasticity
does not enter, the equation being

$$\frac{\delta l}{-} = k_1 \frac{\mathrm{\overline{K}V^2}}{8\pi d^2}.$$

The letters have here the same significance as before, and the
dielectric constant of the fluid is immaterial.

This is, however, only apparently so, for this constant would
be involved in the calculation of the field-strength in the solid
dielectric.   For if the dielectric constants are different, the
fall of potential between the armatures is no longer uniform.
The resultant field in the solid would be equivalent to a field
produced by the potentials of the armatures acting through a
homogeneous dielectric *plus* a field produced by free charges
assumed to be *on* the faces of the solid dielectric.   These latter
evidently require the treatment for a condenser with *adherent*
armatures, and the introduction of the coefficient of elasticity.

In this paper he again asserts that my experiments gave
nugatory results because of lack of sensibility, and Ercolini's †
show k_1 to be positive.   Granting this criticism to be true,
it should not be overlooked that I also experimented with
hard-rubber tubes and obtained no elongation, although the
effect is supposedly much greater than for glass. The results,
to be given later, using thinner glass, a less distance between

* Sacerdote, *Journal de Phys.* (3) t. x. p. 196
† Ercolini, *loc. cit.*

the armatures, and a more sensitive apparatus than Ercolini's, confirm my former ones.

As my previous experiments did not convince my critics, there was no alternative but to repeat them, and as conclusively as possible. In doing so the following points were kept in view : the apparatus should be more sensitive and more carefully adjusted ; extraneous effects guarded against or eliminated ; effect of heating measured ; and the mechanical attraction of the armatures, when adherent, calculated. The task has been an extremely difficult one ; but perhaps in all four particulars something has been done.

In a general way, the apparatus is similar to the one previously employed ; but it was certainly more efficient because of better construction. Whether the interferometer method is better or not, I do not know ; but the errors do not result from any lack of sensitiveness or accuracy of the magnifying machine used.

A brass tube A (Pl. I. fig. 1) was screwed in a heavy iron flanged collar, and this to a wooden base. The experimental glass tubes were placed coaxially over this brass tube and cemented to a brass collar C at the base. The brass tube thus formed one armature and the glass tube the dielectric of a condenser. For the other armature several devices served, either a sheet of tinfoil was pasted on the outer surface of the experimental tube ; or a glass tube, silvered on its inner surface, cemented in a position coaxially to the other two ; or a brass tube F capped with short glass tubes at the top and bottom, as shown in the figure. The brass tubes were seamless, quite straight, and of uniform thickness. The glass tubes were obtained from Messrs. Eimer and Amend, of New York, who kindly examined their entire stock in order that they might supply me with perfect pieces.

The dimensions of the tubes are given in the table.

TABLE II.—Experimental Glass Tubes.

No.	Length.	Outer Diam.	Inner Diam.	Thickness.
1	65 cms.	4·40 cms.	4·05 cms.	1·8 mm.
2	65 ,,	4·48 ,,	4·05 ,,	2·2 ,,
3	65 ,,	4·43 ,,	4·03 ,,	2·0 ,,
4	65 ,,	4·55 ,,	4·13 ,,	2·1 ,,
5	65 ,.	4·53 ,,	4·15 ,,	1·9 ,,
6	65 ,,	4·67 ,,	4·23 ,,	2·2 ,,

Inner brass tube A, Length 64 cms., outer diam. 3·9 cms.
Outer glass ,, F, ,, 62 ,, inner ,, 4·9 ,, base to silver 6 cms.
Outer brass ,, F, ,, 55 ,, ,, ,, 4·7 ,, ,, tube 5 cms.

The system of optical levers used for magnifying the expansion of the tubes is shown in Pl. I. fig. 2.   An incandescent electric lamp with a ground-glass globe was clamped behind a metal screen pierced with a round hole 5 millims. in diameter. Across this opening fine glass fibres were fastened horizontally, to serve for the movable image.   The light from the lamp, after reflexion at right angles by a totally-reflecting prism, passed through an achromatic lens to the mirror *m* mounted vertically on a little tripod-table.   Upon reflexion from this mirror it again passed through the lens and on, above the prism, to a micrometer-microscope.

The mirror (fig. 3), 1·5 cm. by 2 cms., furnished by Brashear, was silvered on the front face and plane to $\frac{1}{4}\lambda$.   The tilting-table of brass, 1·6 cm. by 1·6 cm. had three legs made of the finest needle-points, one of which rested on the experimental tube and the other two on the brass cap D (fig. 1).   By raising or lowering the cap, the beam of light could be reflected at any desired angle.   Small weights, which hung from an arm below the level of the table, increased the stability to such a degree that the image showed no oscillation from outside disturbances.   The prism and lens were of the best construction, and the micrometer-microscope was a new instrument obtained especially for this experiment.   The fibres and the microscope were each placed in the principal focus of the lens ; the image of these fibres in the microscope was remarkably distinct and sharp.

From the arrangement of the parts, an elongation of the experimental tube caused the image of the fibres to fall in the microscope.   The least change in length which could be observed was calculated from the following dimensions :—

TABLE III.—Dimensions of Magnifying Power.

Focal length of lens .... 100 cms.
Distance between feet of tripod I. .... 6 mm.
      ,,   ,,   ,,   ,,   II. .... 3 mm.
One division of micrometer on microscope ... 0·002 mm.

A deflexion of one division of the micrometer, using tripod I., is equal to a change of length in the tube of

$$\frac{6}{2 \times 1000} \times \cdot002 = 6 \times 10^{-6} \text{ mm.},$$

and with tripod II. one-half of the above, or 3×10^{-6} mm. My former apparatus had a sensibility of $1\cdot5 \times 10^{-5}$ mm. ;

that is, one-fifth that of the present form. And this minimum deflexion could be observed readily and measured accurately.

When electrically charged the inner armature and the tripod table were always grounded, and the outer one charged by a powerful static generator, capable of giving a spark eight inches long and of great volume. From one to four large leyden-jars were placed in the circuit to prevent irregularities in the charging. The potential-differences were measured by a spark-micrometer with balls two centimetres in diameter. With all the apparatus in the circuit the highest potentials were reached in from ten to fifteen seconds. To prevent any charge from reaching the tripod table, it was inclosed in a cage of fine copper gauze wire, which fitted over the top of the experimental tube and was grounded. When the tube was disconnected from the circuit, no disturbance reached the mirror, even though long sparks passed across the terminals of the spark micrometer.

The entire apparatus was fastened to heavy beams, bolted to masonry piers in a basement room. No experiments were made when the image showed any motion due to temperature changes in the room during a period of time five or six times that of charging.

A convenient method of testing the apparatus is to make a determination of the modulus of elasticity. The weights for compressing the tubes were placed on a table with three legs which rested on the glass tube only. Deflexions are expressed in divisions of the micrometer.

TABLE IV.—Modulus of Elasticity.

	Tube 3.	Tube 4.
Weight, 1000 gr.	60·5 div. 61·0 58·0 56·0 60·0 —— 59·0	45·0 div. 55·0 55·0 —— 51·7
Weight, 500 gr.	31·0 30·5 30·5 —— 30·7	25·0 25·0 —— 25·0

Employing the ordinary formula for Young's modulus, we have from the data,

	3.	4.
Length of tube	65 cms.	65 cms.
Area of cross-section	2·6 cms.	2·9 cms.
Change of length per kg.	60·0 div.	50·0 div.
Value of a division	6×10^{-6} mm.	6×10^{-6} mm.
Young's modulus (μ)	$6 \cdot 8 \times 10^{11}$	$7 \cdot 3 \times 10^{11}$

These results agree so closely with each other and with the accepted value for this modulus for glass that there should be no hesitation in accepting the readings of the apparatus.

Passing now to the experiments made when the condenser was electrically charged, it is more convenient to give the results in a series of tables without comment, and to discuss them later. They may be divided into two classes, those with adherent armatures and those when the armatures are separated from the dielectric by a non-conducting fluid.

Adherent Armatures.

Tube 5 was cemented in place and covered on the outer surface with tinfoil for a distance of 50 centimetres. The space between this tube and the grounded brass tube was filled with a concentrated solution of sodium carbonate in water, which served to ground the inner surface of the glass tube. A second glass tube, larger in diameter and without a metal coating, was cemented coaxially with the others, and the space between the glass tubes, about a millimetre, was filled with pure lard oil. The oil and outer tube merely prevented the charge from leaking off the tinfoil. A fine wire, connected with the generator and spark-micrometer, was attached to the foil ; first passing through a hole bored in the outer tube, half-way up. The hole was then closed with a rubber stopper.

Observations are recorded only for days when the room temperature was so constant that the image remained steady. The drum of the micrometer contained 250 divisions, and a change from a greater to a less number denotes an increase in the length of the tube. Both tilting-mirrors were used, but as the readings agreed they are all reduced to the dimensions of the one with feet 6 millimetres apart. Observations are recorded in divisions of the micrometer, one division being equivalent to 6×10^{-6} millimetres.

From the final reading 243 with a 15 mm. spark, the readings were made as rapidly as possible, shown by the final reading of one being the same as the initial of the next.

When the spark-length was altered, a considerable time necessarily elapsed between readings, as the difference between consecutive values indicate. Before this series of experiments, the image was perfectly stationary, and it was not until the tube had been charged that a continuous increase in length occurred.

The up and down deflexions were always accompanied by a slight horizontal motion. The deflexion begins slowly, and increases rapidly during the latter part of the charging, being especially rapid just before and at the time the spark passes. After the spark, the mirror oscillates rapidly and then comes to rest, nearly always at the final reading ; the return is slow and never to the full amount of the original deflexion.

TABLE V.—Tube 5.   Length of Armature, 50 cms.

Spark-length.	Initial Reading.	Final Reading.	Deflexion.	Remarks.
12 mm. ...	190 div.	175 div.	15 div.	1 jar in circuit.
	145	125	20	Time to charge $\frac{1}{4}$ minute.
	135	120	15 (?)	
Discharged and left standing 7 minutes.				
12 mm. ...	62	25	37	Time to charge 1 minute.
	18	233	35	
Discharged and left standing 1 minute.				
12 mm. ...	213	193	20	Time to charge 1 minute.
	173	163	10 (?)	
	163	130	33	
Discharged but time not measured.				
12 mm. ...	43	38	5	3 jars in circuit.
	25	18	7	Time to charge less than 15 seconds.

TABLE VI.—Tube 5.   Length of Armature, 50 cms.

Spark-length.	Initial Reading.	Final Reading.	Deflexion.	Remarks.
16 mm.	50	35	15	Discharged 5 min.
	193	167	26	Discharged 1 min.
	123	82	41	Discharged 10 min.
	75	57	18	

The tube was ruptured when the spark was increased to 2 cms.

TABLE VII.—Tube 2.   Length of Armature, 52 cms.

Spark-length.	Initial Reading.	Final Reading.	Deflexion.	Remarks.
10 mm. ...	88	75	13	1 jar in circuit.
	63	38	25 (?)	
	18	3	15	
15 mm. ...	218	160	58	
	30	243	37	For the remainder of experiment no time elapsed between successive charges.
	243	214	29	
	214	185	29	
20 mm. ...	148	100	48	Zero changed.
	10	230	30	
	230	199	31	
15 mm. ...	185	168	17	
	168	158	10	
	143	133	10	
10 mm. ...	102	97	5	
	90	84	6	

The first series of observations, Table V., shows the influence of the duration of the charge, of discharge, and of the repetition of the charge when the potential-difference is maintained the same.   When a quick charge is first given to the tube, it apparently increases in length about 15 divisions for a potential-difference of 12 mm. spark, or 110 c.g.s.   If the tube is charged more slowly, about a minute, the increase in length is more than double, and, on the other hand, a less duration of charge diminishes the effect to a very small displacement.   Table VI. shows the first effect of a charge at a greater potential to be the same as for a less.

In Table VII. the effect of varying the potential is given. With an increase of potential the elongation increases, but not uniformly, and a return to the lower potentials shows a much diminished value.

Before a true interpretation of this elongation normal to the field can be made, at least two corrections must be allowed for.   In the first place, since the armatures are in contact with the glass, their mechanical attraction will compress the glass in the direction of the field, and this compression will be attended by an expansion in the direction of the axis of the cylinder, according to Poisson's theory.   Finally, charging the armatures will certainly heat the tube to some extent.

The formula for the first correction in my former paper

is incorrect, as it should be

$$\delta l = \frac{1}{\mu}\frac{F}{S}\sigma l = \frac{1}{\mu}\frac{KV^2}{8\pi d^2}\sigma l,$$

where μ = modulus of elasticity = 7×10^{11}
    K = dielectric constant   = 7
    V = potential-difference = 110 c.g.s.
    σ = Poisson's ratio   = 0·25
    l = length   = 50 cms.
    d = thickness   = 0·2 cm.

or $\delta l = 1\cdot5 \times 10^{-6}$ cm. = 3 divisions of micrometer.

This effect is then, at least, one-fifth the elongation when a tube is charged for the first time, and about one-half the elongation if it has been previously charged to a higher potential.

There can be no doubt that the charges on the armatures are sufficient to heat the glass an appreciable amount, for as soon as the tube has been discharged, the image begins to move continuously in the direction to show an elongation. The values in Table VI. give the magnitude of this effect. With a spark-gap of 16 millimetres, an initial deflexion of 15 divisions was obtained, the same as for a 12 millimetre spark. The tube was then discharged and allowed to stand five minutes, during which time the image moved 92 divisions (from 35 to 193). A second charge gave a deflexion of 26 divisions and a discharge for one minute 44 divisions. The change of seven divisions during the ten minutes following has no significance, as it was necessary to readjust the apparatus. But the effect of heat in expanding the tube is even more apparent from the values obtained when the tube was charged. The first reading, with an unheated tube, is 15 divisions. If it is then allowed to cool for ten minutes before charging again, the elongation is 18, and for five minutes 26; while if the previous cooling is only for one minute the expansion is 41. It seems evident that the increase of length, due to heat evolved during the charging, is of an order equal to the observed effect.

It is easy to calculate the rise in temperature which would expand the tube the entire amount, assuming 15 divisions or 9×10^{-6} centimetre for a 12 millimetre spark. The formula is

$$\delta t = \frac{\delta L}{\lambda L},$$

where δL = observed expansion = 9×10^{-6} cms.
    λ = coeff. of expansion = 9×10^{-6}
    L = length of tube   = 60 cms.,
then δt = 0°·017 C.

The rise in temperature of the glass is from two **causes** ; the ohmic current through the glass from armature to armature, and the discharge through the oil and along the surface of the glass to the top and bottom of the apparatus.   Of the two the latter is undoubtedly the greater, as the oil is a better conductor.   The liquid is always put in violent commotion by the charge and, at times, the discharge vertically is so great as to cause a glow-discharge along the glass from the surface of the oil to the wire-gauze cage on the top of the tube.   The deflexions also indicate the same thing as they lag behind the potential, showing that the passage of heat from the oil to the glass requires a small interval of time.

Although the rise in temperature of the glass itself cannot be measured, it can for the oil.   For this purpose one junction of a platinum-iron thermoelectric element was inserted in the oil and the other in ice.   The galvanometer was first calibrated, and gave a permanent deflexion of 2 centimetres for an increase of 6°.   The tube was then charged ten times to a potential giving a 12 millimetre spark, when the deflexion was 0·3 centimetre, an increase of 0°·9.   On another day, the same number of similar charges indicated a rise of 0°·4. This is much greater than is required to expand the glass by the observed amount, and will account for the continued increase in the length of the tube after it has been charged and discharged a number of times.

Non-adherent Armatures.

To experiment with non-adherent armatures, a new glass tube, number 3, was cemented in place and coaxially with it a larger glass tube, 4·9 centimetres inner diameter, which had been silvered on its inner surface a length of 50 centimetres. The sodium carbonate solution was not used, but, instead, both spaces between the three tubes were filled with lard oil. Electrical connexion was made with the silvered surface by a strip of tinfoil, carried through a hole in the outer tube, which was then closed by a rubber stopper.   The other arrangements were as before.   Both mirrors were used, but deflexions have been reduced so that one division equals 6×10^{-6} mm. as before.

The tube was now charged to a potential producing a 12 millimetre spark, with one jar in the circuit.   The image moved to the left, entirely out of the field of the microscope, and this motion was accompanied by a large apparent elongation of the tube, too irregular and jerky to measure.

No care that I could employ to make the three tubes rigid and parallel would overcome this difficulty until the

tops of the tubes were kept in their proper positions by wedges. This was accomplished by inserting three little wedges of wood, or, preferably, of glass, between the experimental tube and each of the armatures at the top. Lest the friction might lessen the full effect of the expansion, new determinations for the modulus of elasticity were made and gave the same values recorded in the previous table for that experiment.

With the wedges in place, a potential of 12 millimetre spark produced some lateral bending, but not nearly so much and the motion was uniform. It was accompanied by an apparent elongation of 60 divisions. That this elongation was due to the side-wise motion was proved by having an assistant press gently against the top of the tube in a horizontal direction until the lateral motion was the same as before— the mirror again indicated an elongation of 60 divisions.

It was noted, during the experiments with adherent armatures, that after a tube had been charged a few times the image moved steadily to indicate a gradual elongation of the tube. This motion was ascribed to the effect of heat. Experiment shows this effect to be more pronounced with the present arrangement.

The room temperature being constant, a series of charges were given the condenser by exciting the generator continuously until a number of sparks passed, after which it was short-circuited and the image observed ; then the deflexion was recorded at successive intervals of time.

Spark-length, 12 *mm.—Number of Sparks,* 10.

Deflexion, 1·5 minutes after tube was discharged ...　50　div.
,,　　4·0　　,,　　later　...........................　100　,,

,,　　5·5　　,,　　...........................　150　,,

Spark-length, 12 *mm.—Number of additional Sparks,* 10.

Deflexion,　1·5　minutes after discharge　......　130　div.
,,　　4·0　　,,　　later.....................　225　,,
,,　　5·0　　,,　　,,　.....................　245　,,
,,　　5·0　　,,　　,,　.....................　285　,,
,,　　1·0　　,,　　,,　.....................　80　,,

,,　　16·5　　,,　　.....................　965　,,

The outer tube broke with further charging. It was not altogether satisfactory, as it was not accurately cylindrical and straight and was unnecessarily large. It was replaced by a

brass tube 55 cms. long, 4·7 cms. inner diameter, and 2 mm. thick. To insulate it, two glass caps, slightly larger in diameter, were added to the ends, as shown in fig. 1. When the glass wedges were in place, I succeeded in overcoming the lateral motion entirely.

With a spark-gap of 12 mm., the image remained quite at rest until the spark passed, when there was a sudden motion of 25 divisions. Keeping the potential constant, a number of charges were given, the jerk as the spark passed grew successively smaller, and finally ceased altogether. The spark-gap was increased to 2 cms., but produced no deflexion. Higher potentials could not be used as the upper cap broke.

When the silvered glass armature was used, the electricity leaked rapidly from the sharp edges of the silvering, causing a violent motion of the oil and a glow-discharge between the surface of the fluid and the wire cage covering the mirror. The edges of the brass-tube armature were well-rounded and the discharge materially lessened ; so much so that the generator reached its highest potential in far less time, and the tube retained a high charge for many minutes. The heating was correspondingly less and slower.

Heating Due to 10 *Sparks of* 12 *Millimetres.*

Deflexion	2 minutes after discharge	......	22 div.
,,	2 ,,	later....................	20 ,,
,,	6 ,,	,, ...	34 ,,
,,	10 ,,	....	76 ,,

The following experiments were made when the room temperature was constant, and sufficient time was allowed between two chargings for the tube to cool off and the image to come to rest again.

With the spark-gap set for 12 mm. a spark passed through the oil from the upper edge of the armature to the wire cage. Just before and at the time of the discharge, the tube expanded 27 divisions. The generator was short-circuited, and the heat effect for one minute was 11 divisions.

The discharge was a second time through the oil, the accompanying expansion, 13 divisions, and the motion during one minute after discharge, 11 divisions.

Spark-gap 10 mm., the discharge was across the spark-micrometer. There was no motion whatever of the image. Heating effect for one minute after discharge, 9 divisions.

Spark-gap 11 mm. No effect except a slight and rapid

oscillation as the spark passed. Heating for one minute, 15 divisions.

Spark-gap 12 mm. Discharge was not through the oil, and there was no deflexion except the oscillation described before.

Spark-gap 15 mm. Discharge was through the oil, but there was no deflexion except a permanent displacement when the spark passed, caused by the jar of the explosion.

The experiments described confirm the results of my former paper, that the elongation which may occur when a dielectric is charged can be fully accounted for by causes other than electrical stresses. Of these, heating is undoubtedly the most important, although the mechanical attraction of adherent armatures and the slightest non-uniformity in the field also produce appreciable errors.

The heat developed is probably, at least in my experiments, due more to currents along the surface of the dielectric than those through it, if the substance is a good insulator. As shown by my observations, the elongations, if they occur, lag behind the potential, and this effect is much diminished when a brass tube with rounded edges is used for the charged armature instead of tinfoil or silvered glass. When such an armature, separated from the dielectric, is charged, the heating effect is apparently so much retarded as not to interfere with rapid readings, and as a consequence no elongation occurs.

Until these extraneous causes can be eliminated or balanced, and an elongation observed which is indisputably due to the electrical field, I am compelled to believe either that the effect of electrostriction on matter is zero or that it is very much smaller than has been supposed.

University of Cincinnati,
    February 1903.

II. *On the Potential and Lines of Force of a Circular Current.* By H. Nagaoka, *Professor of Physics, Imperial University, Tokyo*[*].

1. THE potential of a circular electric current or of a vortex ring of infinitely small section was investigated by Lord Kelvin[†], Maxwell[‡], Hicks[§], and Minchin[||]. They express the solid angle subtended by the circle by means

[*] Communicated by the Author.
[†] Lord Kelvin, Trans. R. S. E. (1869).
[‡] Maxwell, 'Treatise on Electricity and Magnetism,' vol. ii. chap. 14.
[§] Hicks, Phil. Trans. 1881, p. 628.
[||] Minchin, Phil. Mag. xxxv. (1893).

of elliptic integrals or in terms of zonal harmonics. The lines of force of a circular current (or stream-lines about a vortex ring) are usually obtained from the expression for the mutual potential energy (denoted by M) of two coaxial circular currents. By using F. Neumann's formula, M may be expressed by elliptic integrals, or developed in terms of zonal harmonics, which is sometimes advantageous in calculating the action between thick coils. Maxwell has also given a table of the coefficients of mutual induction when the coils are near each other. In these calculations we are always in need of Legendre's tables. It is very curious that so little use has been made of Jacobi's q-series. Mathy* uses Weierstrass' γ-function in evaluating M, but he seems to incline to the use of a hypergeometric series rather than to the reduction of these integrals to a rapidly converging q-series, to which the expression can be easily transformed.

The problem can, however, be attacked from another point of view. In the following I proceed by finding the Newtonian potential of a uniform circular disk, and derive the expression for the potential and the lines of force by simple differentiation. Finally, M is expressed by means of a simple q-series, of which a single term will generally suffice to secure a practically accurate value; the force between two coaxial coils can also be expressed in a similar manner.

2. The potential U of a homogeneous body of rotation (about z-axis) satisfies Laplace's equation outside the body, which in this case is given by

$$\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} + \frac{1}{x}\frac{\partial U}{\partial x} = 0,$$

x being radial coordinate. Thus

$$\frac{\partial^2 U}{\partial z^2} = -\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial U}{\partial x}\right),$$

and

$$\frac{\partial^2 U}{\partial x \partial z} = \frac{1}{x}\frac{\partial}{\partial z}\left(x\frac{\partial U}{\partial x}\right).$$

If the potential ϕ of a certain distribution symmetrical about the z-axis be derivable from U by differentiation with respect to z, so that

$$\phi = -\frac{\partial U}{\partial z}, \quad \cdots \quad \cdots \quad \text{(I.)}$$

and if

$$\psi = -x\frac{\partial U}{\partial x}, \quad \cdots \quad \cdots \quad \text{(II.)}$$

* Mathy, *Journal de Physique*, x. p. 33 (1901).

then
$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial z} = 0,$$

and
$$\psi = \text{const.}$$

must represent lines of force.

The potential of a circular current is equivalent to that of a magnetic shell, which is derivable from the Newtonian potential of a uniform disk, by differentiation with respect to a normal.

3. *Potential of a uniform circular disk.*—Let the surface-density be unity ; taking x, y axes in the plane of the disk, and z axis perpendicular to it through the centre, the potential at point x, 0, z is evidently given by

$$U = \int_0^a \int_0^{2\pi} \frac{\rho \, d\rho \, d\theta}{\sqrt{x^2 - 2x\rho \cos \theta + \rho^2 + z^2}}, \quad \cdot \quad \cdot \quad (1)$$

ρ and θ being polar coordinates, and a the radius of the disk.

Writing $R^2 = x^2 - 2x\rho \cos \theta + \rho^2$, and making use of Lipschitz's integral concerning the Bessel's function, we obtain

$$\frac{1}{\sqrt{x^2 - 2x\rho \cos \theta + \rho^2 + z^2}} = \frac{1}{\sqrt{R^2 + z^2}} = \int_0^\infty e^{-\lambda z} J_0(\lambda R) \, d\lambda. \quad (2)$$

The addition theorem of Bessel's function gives

$$J_0(\lambda R) = J_0(\lambda x) J_0(\lambda \rho) + 2 \sum_1^\infty J_n(\lambda x) J_n(\lambda \rho) \cos n\theta. \quad \cdot \quad (3)$$

Substituting the two expressions (2) and (3) in (1), and integrating between the limits 0 and 2π,

$$U = 2\pi \int_0^\infty \int_0^a e^{-\lambda z} \rho J_0(\lambda x) J_0(\lambda \rho) \, d\lambda \, d\rho.$$

But since
$$\rho J_0(\lambda \rho) = \frac{d(\rho J_1(\lambda \rho))}{\lambda \, d\rho},$$

$$U = 2\pi a \int_0^a \frac{e^{-\lambda z}}{\lambda} J_0(\lambda x) J_1(\lambda a) \, d\lambda. \quad \cdot \quad \cdot \quad (4)$$

This expression for the Newtonial potential of a uniform circular disk seems to have been first obtained by H. Weber[*].

4. *Potential and lines of force of a circular magnetic shell.*—The potential of a circular magnetic shell of unit strength is evidently given by differentiating (4) with respect to z.

[*] H. Weber, Crelle's Journal lxxv. p. 88.

Thus

$$\phi = -\frac{\partial U}{\partial z} = 2\pi a \int_0^\infty e^{-\lambda z} J_0(\lambda x) J_1(\lambda a) d\lambda, \quad . \quad . \quad (A)$$

and by (II.) the lines of force are given by the function

$$\psi = -x\frac{\partial U}{\partial x} = 2\pi a x \int_0^\infty e^{-\lambda z} J_1(\lambda x) J_1(\lambda a) d\lambda. \quad . \quad (B)$$

The two expressions (A) and (B) can be greatly simplified by using the addition theorem for $J_0(\lambda R)$ and $J_1(\lambda R)$. From (3) we easily find

$$J_0(\lambda x) J_1(\lambda a) = \frac{1}{\pi} \int_0^\pi \frac{J_1(\lambda R)}{R}(a - x\cos\theta)d\theta,$$

$$J_1(\lambda x) J_1(\lambda a) = \frac{2}{\pi} \int_0^\pi J_0(\lambda R) \cos\theta d\theta.$$

Remembering that

$$\int_0^\infty e^{-\lambda z} J_1(\lambda R)d\lambda = \frac{1}{R} - \frac{z}{R\sqrt{R^2 + z^2}},$$

$$\int_0^\infty e^{-\lambda z} J_0(\lambda R)d\lambda = \frac{1}{\sqrt{R^2 + z^2}},$$

we easily find by simple substitution

$$\phi = 2a \int_0^\pi \frac{a - x\cos\theta}{R^2} d\theta - 2az \int_0^\pi \frac{(a - x\cos\theta)}{R^2\sqrt{R^2 + z^2}} d\theta, \quad (5)$$

$$\psi = 4ax \int_0^\pi \frac{\cos\theta d\theta}{\sqrt{R^2 + z^2}}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Since

$$a \int_0^\pi \frac{a - x\cos\theta}{R^2} d\theta = \pi$$

(5) becomes

$$\phi = 2\pi - 2az \int_0^\pi \frac{(a - x\cos\theta)d\theta}{(a^2 + x^2 - 2ax\cos\theta)\sqrt{a^2 + x^2 + z^2 - 2ax\cos\theta}}. \quad (A')$$

The above expression (A') represents the solid angle subtended by the disk at point x, 0, z.

Evidently the coefficient of mutual induction M of two parallel coaxial coils is connected with ψ by the relation

$$\pi\Psi = M. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Consequently (6) gives

$$M = 4\pi a x \int_0^\pi \frac{\cos\theta d\theta}{\sqrt{a^2 + x^2 + z^2 - 2ax\cos\theta}} \quad . \quad . \quad (B')$$

This expression coincides with that obtained from F. Neumann's formula.

5. *Solid angle subtended by a circle.*—Denoting the integral entering in (A') by Ω, so that

$$\Omega = 2az \int_0^\pi \frac{(a - x\cos\theta)\,d\theta}{(a^2 + x^2 - 2ax\cos\theta)\ \sqrt{a^2 + x^2 + z^2 - 2ax\cos\theta}}, \quad (8)$$

and putting $\cos\theta = \text{A}s + \text{B}$, where

$$\text{A} = \left(\frac{2}{ax}\right)^{\frac{1}{3}}, \quad \text{B} = \frac{a^2 + x^2 + z^2}{6ax}, \quad . \quad . \quad . \quad (9)$$

we easily find that

$$\Omega = \text{A}z \int_{\omega_2}^{\omega_3} du + \dot{z}(a^2 - x^2) \int_{\omega_2}^{\omega_3} \frac{du}{a^2 + x^2 - 2ax\,(\text{A}s + \text{B})}, \quad (10)$$

where, following Weierstrass' notation,

$$u = \int_s^\infty \frac{ds}{\sqrt{\text{S}}}$$

$$\sqrt{\text{S}} = \sqrt{4s^3 - g_2 s - g_3} = \sqrt{4(s - e_1)(s - e_2)(s - e_3)}$$
$$= \sin\theta\ \sqrt{a^2 + x^2 + z^2 - 2ax\cos\theta}$$

$$e_1 = \frac{2\text{B}}{\text{A}}, \ e_2 = \frac{1 - \text{B}}{\text{A}}, \ e_3 = -\frac{1 + \text{B}}{\text{A}} \quad . \quad . \quad (11)$$

$$s = \gamma(u)$$

$$\left.\begin{array}{l} k^2 = \dfrac{4ax}{(a + x)^2 + z^2} \\[2mm] k'^2 = \dfrac{(a - x)^2 + z^2}{(a + x)^2 + z^2} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Let
$$\gamma(\alpha) = \frac{2(a^2 + x^2) - z^2}{6ax\text{A}},$$

then
$$\gamma'(a) = -\ \sqrt{4\gamma^3(\alpha) - g_2\gamma(\alpha) - g_3} = -i\frac{z(a^2 - x^2)}{2ax},$$

and
$$\Omega = \omega_1 \text{A}z + i \int_{\omega_2}^{\omega_3} \frac{\gamma'(\alpha)}{\gamma(\alpha) - \gamma(u)}\,du, \quad . \quad . \quad . \quad . \quad (13)$$

$$= \frac{2\omega_1 z}{e_2 - e_3} + 2i\left\{\eta_1\alpha - \omega_1\frac{\sigma'}{\sigma}(\alpha)\right\}. \quad . \quad . \quad . \quad (14)$$

Thus the potential of a circular current is given by

$$\phi = 2\pi - 2\left[\frac{\omega_1 z}{e_2 - e_3} + i\left\{\eta_1 \alpha - \omega_1 \frac{\sigma'}{\sigma}(\alpha)\right\}\right], \quad . \quad (A'')$$

or

$$= 2\pi - \frac{\pi \vartheta_2^2(0)z}{\sqrt{e_2 - e_3}} + i\frac{\vartheta_1'(v_\alpha)}{\vartheta_1(v_\alpha)},$$

where $v_\alpha = \dfrac{\alpha}{2\omega_1}.$

The form of integral (A′) is somewhat different from that given by Hicks and Minchin, but it leads to the same result (A″). The process of reduction from the expressions given by the above-mentioned authors is more laborious.

For the convenience of calculation the following values of $\gamma(\alpha) - e_\lambda$ are tabulated for $\lambda = 1, 2, 3$:—

$$\left.\begin{array}{l}\gamma(\alpha) - e_1 = -\dfrac{1}{A}\dfrac{z^2}{2ax} = -\dfrac{z^2}{4ax}(e_2 - e_3) \\[2mm] \gamma(\alpha) - e_2 = \dfrac{1}{A}\dfrac{(a-x)^2}{2ax} = \dfrac{(a-x)^2}{4ax}(e_2 - e_3) \\[2mm] \gamma(\alpha) - e_3 = \dfrac{1}{A}\dfrac{(a+x)^2}{2ax} = \dfrac{(a+x)^2}{4ax}(e_2 - e_3)\end{array}\right\} \quad . \quad . \quad (15)$$

Thus $e_1 > \gamma(\alpha) > e_2$, showing that α is a purely imaginary quantity.   In addition to this

$$e_2 - e_3 = \frac{2}{A} = 2\left(\frac{ax}{2}\right)^{\frac{1}{3}}$$

$$e_1 - e_3 = \frac{1}{A}\frac{(a+x)^2 + z^2}{2ax} = \frac{(a+x)^2 + z^2}{4ax}(e_2 - e_3)$$

$$e_1 - e_2 = \frac{1}{A}\frac{(a-x)^2 + z^2}{2ax} = \frac{(a+x)^2 + z^2}{4ax}(e_2 - e_3).$$

In practical calculation the most painstaking part is the evaluation of α, for which we put, following Schwarz[*]

$$\frac{\sqrt[4]{e_1 - e_3}\sqrt{\gamma(\alpha) - e_2} - \sqrt[4]{e_1 - e_2}\sqrt{\gamma(\alpha) - e_3}}{\sqrt[4]{e_1 - e_3}\sqrt{\gamma(\alpha) - e_2} + \sqrt[4]{e_1 - e_2}\sqrt{\gamma(\alpha) - e_3}} = lt,$$

where $l = \dfrac{1 - \sqrt{k'}}{1 - \sqrt{k}}$

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[*] Schwarz, *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen*, p. 71, Berlin, 1893.

and
$$L_0 = 1 + \left(\frac{1}{2}\right)^2 l^4 + \left(\frac{1\cdot3}{2\cdot4}\right)^2 l^8 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 l'^2 + \dots$$

$$L_{0,1} = \left(\frac{1}{2}\right)^2 l^4 + \left(\frac{1\cdot3}{2\cdot4}\right)^2 l^8 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 l'^2 + \dots$$

$$L_{0,2} = \left(\frac{1\cdot3}{2\cdot4}\right)^2 l^8 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 l'^2 + \dots$$

. . . .

A simple calculation will show that

$$\frac{\alpha}{2\omega_1}\pi i = \tilde{v}_a \pi i$$
$$= -\tfrac{1}{2}\log(t + \sqrt{t^2-1}) - \tfrac{1}{2}\sqrt{t^2-1}\left(\frac{L_{0,1}}{L_0}t + \frac{2}{3}\frac{L_{0,2}}{L_0}t^3 + \dots\right).$$

Since $L_{0,1}, L_{0,2}, \dots$ are small quantities, the second terms will be generally very small compared to the first. Further, we notice that

$$i\left\{\eta\alpha - \omega_1\frac{\sigma'}{\sigma}(\alpha)\right\}$$
$$= \frac{\pi}{2}\left\{\frac{e^{v_\alpha\pi i} + e^{v_\alpha\pi i}}{e^{v_\alpha\pi i} - e^{v_\alpha\pi i}} + \sum'\frac{2q^{2n}e^{-2v_\alpha\pi i}}{1-q^{2n}e^{-2v_\alpha\pi i}} - \sum'\frac{2q^{2n}e^{2v_\alpha\pi i}}{1-q^{2n}e^{2v_\alpha\pi}}\right\}.$$

Thus the calculation of ϕ can be easily undertaken without the use of special tables.

6. *Evaluation of* M *or the coefficient of mutual induction of two coaxial parallel coils.*—As was already noticed, $M = $ const. gives magnetic lines of force about a circular current, or stream-lines about a circular vortex ring of infinitely small section. Reverting to (B'), (9), (11), and (12), we find that

$$M = 4\pi ax\int_0^\pi \frac{\cos\theta\,d\theta}{\sqrt{a^2 + x^2 + z^2 - 2ax\cos\theta}}$$
$$= 4\pi ax A\int_{\omega_2}^{\omega_3}(A\gamma(u) + B)du. \quad . \quad . \quad . \quad (16)$$

Thus the expression for M can be written in either of the following forms:—

$$M = 4\pi ax\frac{A^2}{\omega_1}\left(\frac{e_1\omega_1^2}{2} - \eta_1\omega_1\right), \quad . \quad . \quad . \quad (B_1'')$$

or

$$M = 4\pi ax\frac{A^2}{\omega_3}\left(\frac{e_1\omega_1\omega_3}{2} - \eta_1\omega_3\right). \quad . \quad . \quad (B_2'')$$

Expressing $e_1\omega_1{}^2$ and $\eta_1\omega_1$ by means of ϑ-functions, we obtain

$$M = \frac{2\sqrt{ax}}{\vartheta_2{}^2(0)}\left\{\frac{\vartheta_1{}'''(0)}{\vartheta_1{}'(0)} - \frac{\vartheta_2{}''(0)}{\vartheta_2(0)}\right\}, \quad . \quad . \quad (17)$$

or

$$M = -\frac{2\sqrt{ax}}{\vartheta_2{}^2(0)}\left\{\frac{\vartheta_0{}''(0)}{\vartheta_0(0)} + \frac{\vartheta_3{}''(0)}{\vartheta_3(0)}\right\} . \quad . \quad . \quad (18)$$

Similarly (B_2'') can be written

$$M = \frac{8\sqrt{ax}}{\pi\vartheta_0{}^2(0/\tau_1)}\left\{\pi^2 - \frac{\log\left(\frac{1}{q_1}\right)}{4}\left(\frac{\vartheta_2{}''(0/\tau_1)}{\vartheta_2(0/\tau_1)} + \frac{\vartheta_3{}''(0/\tau_1)}{\vartheta_3(0/\tau_1)}\right)\right\}, \quad (19)$$

where $\tau_1 = -\dfrac{\omega_1}{\omega_3}$.

The expressions (17), (18), and (19) are of great practical importance, as will be shown in another section.

7. *Expression for* $\dfrac{\partial M}{\partial z}$.—In addition to M, we shall have to find $\dfrac{\partial M}{\partial z}$, which represents the force acting between two coaxial coils.

Since

$$\frac{\partial M}{\partial z} = 4\pi axz\int_0^\pi \frac{\cos\theta\, d\theta}{(a^2 + x^2 + z^2 - 2ax\cos\theta)^{\frac{3}{2}}}, \quad . \quad (20)$$

we easily find that

$$\frac{1}{4\pi ax}\frac{\partial M}{\partial z} = \frac{A^4 z}{4}\int_{\omega_2}^{\omega_3}\frac{A\gamma(u) + B}{A\gamma(u) - B}du$$

$$= \frac{A^4 z}{4}\left\{\frac{3e_1}{2(e_1 - e_2)(e_1 - e_3)}(\eta_1 + e_1\omega_1) - \omega_1\right\}, \quad (21)$$

which, expressed in ϑ-functions, gives

$$\frac{\partial M}{\partial z} = -\frac{\pi z}{\sqrt{ax}}\left\{\vartheta_2{}^2(0) + \frac{1}{2\pi^2}\left(\frac{1}{\vartheta_0{}^4(0)} + \frac{1}{\vartheta_3{}^4(0)}\right)\vartheta_2{}''(0)\vartheta_2(0)\right\}. \quad (22)$$

8. M *expressed in* q-*series*.—For reducing the ϑ-functions in (17) and (18), we can conveniently make use of the expressions given by Jacobi[*]. As the result of expansion we find

$$\frac{M}{4\pi\sqrt{ax}} = 4\pi q^{\frac{3}{2}}(1 + 3q^4 - 4q^6 + 9q^8 - 12q^{10} + \dots). \quad (23)$$

[*] Jacobi, *Fundamenta Nova*, pp. 104, 105; *Gesammelte Werke*, i. p. 161.

Putting $\qquad 3q^4 - 4q^6 + 9q^8 - 12q^{10} + \ldots = \epsilon,$

we can briefly write

$$\frac{M}{4\pi \sqrt{a.x}} = 4\pi q^{\frac{3}{2}}(1+\epsilon). \quad . \quad . \quad . \quad (24)$$

Since ϵ is a very small quantity we can, with tolerable accuracy, put

$$M \doteq 16\pi^2 \sqrt{a.x} q^{\frac{3}{2}}. \quad . \quad . \quad . \quad . \quad (25)$$

Expressing (19) in terms of q_1

$$\frac{M}{4\pi \sqrt{ax}} = \frac{1}{2(1-2q_1+2q_1{}^4-2q_1{}^9)^2}\Big[\log \text{nat.}\Big(\frac{1}{q_1}\Big)\Big\{1+8q_1$$
$$(1-q_1+4q_1{}^2-5q_1{}^3+6q_1{}^4-4q_1{}^5+8q_1{}^6-13q_1{}^7+..)\Big\} - 4\Big]. \ (26)$$

The above expression is useful when the coils are very near each other. In such cases q_1 is a very small quantity, so that by putting

$$32q_1{}^3 - 40q_1{}^4 + 48q_1{}^5 - 32q_1{}^6 + 64q_1{}^7 - 104q_1{}^9 + \ldots\ldots = \epsilon_1$$

$$\frac{M}{4\pi \sqrt{ax}} = \frac{1}{2(1-2q_1+2q_1{}^4+\ldots)^2}\Big[\log \text{nat.}\Big(\frac{1}{q_1}\Big)\Big\{1+8q_1$$
$$(1-q_1)+\epsilon_1\Big\} - 4\Big]. \quad . \quad (27)$$

For $k > \sin 70°$, $q_1{}^4$ is negligibly small, and

$$\frac{M}{4\pi \sqrt{ax}} = \frac{1}{2(1-2q_1)^2}\Big[\log \text{nat.}\Big(\frac{1}{q_1}\Big)\Big\{1+8q_1$$
$$(1-q_1+4q_1{}^2)\Big\} - 4\Big]. \quad . \quad (28)$$

Finally, (22) gives

$$\frac{\partial M}{\partial z} = \frac{192\pi^2 z}{\sqrt{ax}} q^{\frac{5}{2}}(1+20q^2+225q^4+1840q^6+12120q^8+..)(29)$$

It will not be altogether out of place to digress on the practical utility of the several formulæ above deduced.

In the first place, there is no need of finding γ connected by the relation $k = \sin \gamma$, $k' = \cos \gamma$. Instead of finding γ we shall have to calculate

$$q = \frac{l}{2} + 2\Big(\frac{l}{2}\Big)^5 + 15\Big(\frac{l}{2}\Big)^9 + \ldots\ldots$$

where $\qquad l = \dfrac{1 - \sqrt{k'}}{1 + \sqrt{k'}},$

or
$$q_1 = \frac{l_1}{2} + 2\left(\frac{l_1}{2}\right)^5 + 15\left(\frac{l_1}{2}\right)^9 + \ldots\ldots$$

where
$$l_1 = \frac{1 - \sqrt{k}}{1 + \sqrt{k}}.$$

In these expansions it is generally sufficient to retain only the first term [*], as the table for $q - \frac{l}{2}$ given below will show. For practical purposes a single calculation according to (25) and (27) gives at once the value of **M**. In order to show the rapid convergence of these q-series it will be sufficient to indicate the smallness of the corrections ϵ, ϵ_1, and $\epsilon_1' = -8q_1^2 + \epsilon_1$, entering in the above two formulæ. For this purpose the following short tables of corrections were calculated :—

Table of $\epsilon = 3q^4 - 4q^6 + 9q^8 - 12q^{10}$ and $q - \frac{l}{2}$.

q.	Approximate value of $\gamma = \sin^{-1}k$.	ϵ.	$q - \frac{l}{2}$.
0·01	22·6°	0·0000000	0·00000000
0·02	31·6	0·0000005	0·00000001
0·03	38·1	0·0000024	0·00000005
0·04	43·5	0·0000077	0·00000020
0·05	48·0	0·0000187	0·00000063
0·06	51·9	0·0000387	0·00000156
0·07	55·3	0·0000726	0·00000336
0·08	58·4	0·0001218	0·00000655
0·09	61·1	0·0001947	0·00001181
0·10	63·6	0·0002961	0·00002000
0·11	65·9	0·0004323	0·00003220
0·12	67·9	0·0006105	0·00004974
0·13	69·8	0·0008382	0·00007421
0·14	71·5	0·0011234	0·00010746

Table of $\epsilon_1 = 32q_1^3 - 40q_1^4 + 48q_1^5 - 32q_1^6 + \ldots$ and $\epsilon_1' = -8q_1^2 + \epsilon_1$.

q_1.	Approximate value of $\gamma = \sin^{-1}k$.	ϵ_1.	ϵ_1'.
0·010	67·4°	0·0000316	−0·0007684
0·009	68·5	0·0000231	−0·0006249
0·008	69·7	0·0000162	−0·0004958
0·007	71·0	0·0000109	−0·0003811
0·006	72·4	0·0000069	−0·0002811
0·005	73·9	0·0000040	−0·0001960
0·004	75·6	0·0000020	−0·0001260
0·003	77·5	0·0000010	−0·0000710
0·002	79·8	0·0000003	−0·0000317
0·001	82·8	0·0000000	−0·0000080

[*] For the table of q see Jacobi, Crelle's *Journal*, xxvi. p. 93 *Gesammelte Werke*, i. p. 363.

The table shows that the error in M calculated by the approximate formula (25) is only 0·001 per cent. for $\gamma = 45°$, and 0·09 per cent. for $\gamma = 70°$. When the coils are near each other the approximation can be carried still further by using (27) ; thus dispensing with Legendre's tables of elliptic integrals. It will not be out of place to give a numerical value for a single instance in order to show the rapid convergence of (23). For $k = \sin 70°$

$$q = 0·1309845\left(= \frac{l}{2}\right) + 0·0000771\left(= 2\left(\frac{l}{2}\right)^5\right)$$
$$+ 0·0000002\left(= 15\left(\frac{l}{2}\right)^9\right)$$

$$= 0·1310618,$$

$$\log \frac{M}{4\pi\sqrt{ax}} = \bar{1}·7754242 (= \log 4\pi q^{\frac{3}{2}}) + 0·0003758 (= \log(1+\epsilon))$$

$$= \bar{1}·7758000,$$

which coincides with the value given by Maxwell. It is to be noticed that the above is the most unfavourable case where (24) may be applied.

9. It will be worth while to mention that the expression for the solid angle subtended by a circle, in terms of zonal harmonics, can be deduced from the formula already obtained. The potential of a magnetic shell ϕ is given by (A)

$$\phi = 2\pi a \int_0^\infty e^{-\lambda z} J_0(\lambda x) J_1(\lambda a)\, d\lambda.$$

By expanding $J_1(\lambda a)$ according to ascending powers of λa, and remembering that

$$\int_0^\infty e^{-\lambda z} J_0(\lambda x) \lambda^{2m+1} d\lambda$$
$$= 1 . 2 . 3 .. (2m+1)(x^2+z^2)^{-m+1} P_{2m+1}\left(\frac{z}{\sqrt{x^2+z^2}}\right),$$

where P_{2m+1} denotes zonal harmonics of $2m+1$-th order, we arrive at the ordinary expression for the solid angle in terms of spherical harmonics. M can be similarly expressed by using (B').

It is needless to remark that such expressions converge very slowly. What I want to show in the present paper is that we may sometimes arrive at a convenient and practical result by using the q-series, instead of falling into the grooves of spherical harmonics.

III. *Note on the Application of* Cornu's *Spiral to the Diffraction-Grating.—A Geometrical Method of obtaining the Intensity Formula for a Flat Diffraction-Grating. By* ARTHUR L. KIMBALL, *Ph.D., Professor of Physics in Amherst College*.*

IT is well known that the graphic method of Cornu furnishes a useful method of discussing many diffraction phenomena, notably the case of a single narrow slit; but, so far as I am aware, the formula for the intensity of the light diffracted by a grating of n lines has not been obtained without involving the summation by analytic process of the trigonometric series

$$\sin x + \sin (x+y) + \sin (x+2y) + \&c. \ldots \sin (x+(n-1)y).$$

In the following note it is shown how, by a simple geometrical device, the result may be obtained at a stroke; and the method besides giving immediately the desired summation makes the discussion of the result particularly direct and simple. When a flat wave falls on a flat diffraction-grating made up of alternate bars and spaces, the graphic expression of the amplitudes and phases of vibration due to the waves through the several slits, is a series of chords of Cornu's spiral.

In case the resultant amplitude is sought at a point at an infinite distance from the grating, or for a point in the focal plane of the observing telescope, Cornu's spiral becomes a circle.

Thus the chord cd will represent the amplitude and phase of motion due to the first slit, ef that due to the second, gh
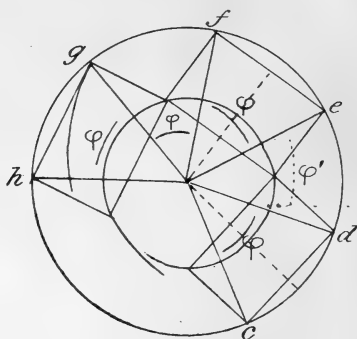
that of the third, &c., and if the slits are all equal and equally far apart, the amplitudes will be a series of equal chords at equal distances apart around the circle, and the resultant amplitude is the geometrical sum of these chords.

* Communicated by the Author.

Suppose the incident wave-front is parallel with the grating, and that the direction of the point P, where the resultant amplitude is sought, makes an angle θ with the normal to the grating, in a plane at right angles to the slits. Then if a is the width of the slits, and b is the width of the opaque bars, the angle ϕ which expresses the difference in phase at P of light from the two edges of a slit is

$$\phi = \frac{2\pi a \sin \theta}{\lambda}$$

and the difference of phase of light from the two edges of a bar is

$$\phi' = \frac{2\pi b \sin \theta}{\lambda}.$$

The arc cd is proportional to a, hence $\phi = k \dfrac{a}{R}$, where R is the radius of the circle $cdef$, and k is a constant.

$$\frac{2\pi a \sin \theta}{\lambda} = k \frac{a}{R} \therefore R = \frac{k\lambda}{2\pi \sin \theta}.$$

The length of the chord cd expressing the amplitude due to a single slit may be represented by l, and we have

$$l = 2R \sin \tfrac{1}{2}\phi = ka \cdot \left(\frac{\sin \dfrac{\pi a \sin \theta}{\lambda}}{\dfrac{\pi a \sin \theta}{\lambda}} \right)$$

Construct a circle, radius r (in the above diagram smaller than R), such that chords equal and parallel to chords cd, ef, &c. will meet on its circumference.

The angle between the normals to consecutive chords is $\phi + \phi'$, and since they meet on its circumference we have

$$\frac{l}{2r} = \sin \tfrac{1}{2}(\phi + \phi'), \text{ hence } r = \frac{l}{2\sin \tfrac{1}{2}(\phi + \phi')}.$$

But in the circle of radius r the n chords subtend an arc $n(\phi + \phi')$, and the resultant amplitude is therefore

$$A = 2r \sin \frac{n(\phi + \phi')}{2} = l \frac{\sin \dfrac{n(\phi + \phi')}{2}}{\sin \dfrac{\phi + \phi'}{2}}$$

and the intensity $I = A^2$.

Substituting values of ϕ and ϕ' we find

$$A = l \cdot \frac{\sin\left(\dfrac{n\pi(a+b)\sin\theta}{\lambda}\right)}{\sin\left(\dfrac{\pi(a+b)\sin\theta}{\lambda}\right)}$$

the square of which is the usual expression for intensity.

The discussion of the maxima and minima becomes exceedingly simple from the point of view of the above construction.

It will be observed that as $\phi + \phi'$ increases r varies back and forth between the limits *infinity* and $l/2$.

The *principal maxima* will be when the chords l are all in the same phase, and hence form a straight line of length nl; this can only occur when the radius r is infinitely great, that is when $\phi + \phi' = 2\pi p$ or $(a+b)\sin\theta = p\lambda$ where p is any whole number or zero. The corresponding intensities are n^2l^2.

After passing such a maximum, as $\phi + \phi'$ increases r becomes smaller until the curve subtended by the n chords reaches just completely around the circle. The intensity is then zero, and the arc $n(\phi + \phi')$ equals $2\pi np + 2\pi$, or differs by 2π from that corresponding to the maximum.

As $\phi + \phi'$ still farther increases r continues to decrease till it reaches a value where the curve of chords reaches one and a half times around the circle, or where $n(\phi + \phi') = 2\pi np + 3\pi$. This is *very nearly* a subordinate maximum, though not *exactly*, because, since r is decreasing, the maximum value will be the chord of an arc a little less than 3π for which the radius is greater. It is easily found by differentiation in the usual way.

Taking the approximate value $n(\phi + \phi') = 3\pi$, the amplitude being $2r$ becomes

$$\frac{l}{\sin\left(\dfrac{3\pi}{2n}\right)}$$

and as n is large we have

$$A = \frac{2nl}{3\pi} \text{ or } I = n^2l^2\left(\frac{4}{9\pi^2}\right),$$

that is, the first subordinate maximum has about $\frac{1}{22}$ the intensity of the principal maximum.

As r further decreases there will be a subordinate maximum every time the curve of the n chords reaches around the circle a whole number of times plus an additional half turn. Between two consecutive principal maxima there will thus be a series of $n-2$ subordinate maxima with absolute minima

between them, and the intensity of the smallest of these maxima will be l^2.

The dependence of the width of the lines in the diffraction-spectrum on the effective width of the grating is easily shown as follows :—Suppose it is assumed that the first maximum practically fades out of sight when $n(\phi + \phi')$ differs from $2\pi n p$, its value at the principal maxima, by $\pm \dfrac{3\pi}{2}$, in which case the intensity is less than one-tenth that of the maximum Then,

$$\frac{n\pi(a+b)\sin\theta}{\lambda} = 2\pi n p \pm \frac{3\pi}{2},$$

and

$$\sin\theta = \frac{2\lambda p}{a+b} \pm \frac{3\lambda}{2n(a+b)}$$

$$d\theta = \frac{d(\sin\theta)}{\cos\theta} = \frac{3\lambda}{2n(a+b)\cos\theta} = \frac{3\lambda}{2w\cos\theta}.$$

where w represents the width of the grating.

The angular width of a line in the spectrum is $2d\theta$ or

$$\alpha = \frac{3\lambda}{w\cos\theta}$$

assuming that the line fades out where the intensity is one-tenth of the maximum.

In ruled gratings, whether transmitting or reflecting, the value of l depends on the nature of the grooves cut by the ruling diamond as well as on the angle θ, and therefore the expression given above for l, which assumes a simple slit, does not apply. But the summation holds good in any case if the ruling is uniform.

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IV. *Measurements of Small Resistances.*
By ALBERT CAMPBELL, B.A.[*]

THE object of this paper is to give a brief account of a number of measurements of a set of low-resistance standards belonging to the National Physical Laboratory ; these tests were made partly with a view to comparing various methods of measurement, and it is from this point of view that they derive their main interest. The resistances were of the oil-cooled type and had separate potential terminals. They were made of manganin, and their nominal values were

approximately 0·1, 0·01, and 0·001 international ohm re-
respectively. For convenience of reference I shall allude to
them as X, Y, and Z. Three other standard resistances of
1, 1, and 10 ohms, whose values have been accurately deter-
mined, formed the starting-point in each of the methods.
Let them be named P, Q, and R respectively. The proce-
dure was to find first the ratio of X to P, then that of Y to
X, and, finally, that of Z to Y ; this gave X, Y, and Z in
terms of P. The following methods were used :—

    (1) Shunt Potentiometer.
    (2) Kelvin Bridge.
    (3) Two-step Bridge.
    (4) Differential Galvanometer.
    (5) Matthiessen and Hockin's Method.

As most of these methods are well known I shall discuss each
very briefly.

 (1) *Shunt Potentiometer.*—As shown in fig. 1 the potentio-
meter consisted of the known standards R and Q, one or

Fig. 1.
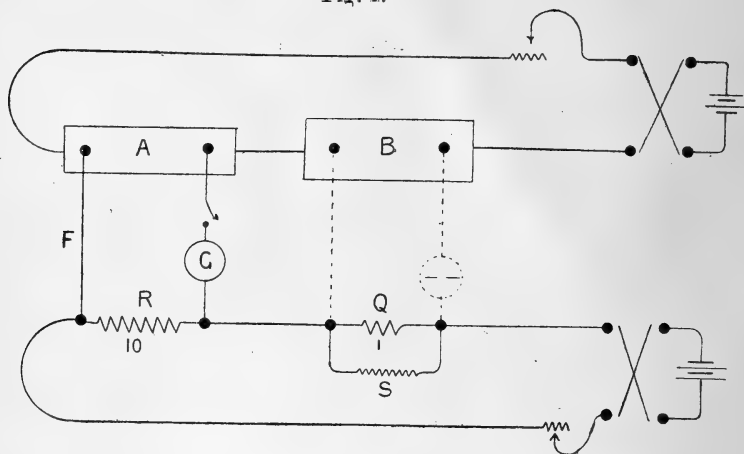
other of them (say Q) being shunted by a high resistance S,
while A and B were the resistances whose ratio was to be
found. A special double key was arranged so that the cross-
connexions (the wire F and the galvanometer G) could be
quickly switched over from A and R to B and Q. The main
current through A and B was kept constant, and the current
through R and Q was adjusted till a balance was obtained in
the first position (A against R). The switch was then turned
to the second position, and Q was shunted so as to balance
again, the process being repeated with successive adjustments

until a balance was obtained in each position. To eliminate thermoelectric errors the two batteries were simultaneously reversed and the same process repeated ; from the mean value of S, B was found in terms of A. The main advantage of the method lies in the simple and straightforward way of using the potential terminals, no current being taken from them when a balance is attained. The main defect is that large steady currents must be maintained, and this almost necessitates running the currents for a considerable time and altering the temperature of the coils.

In connexion with this I may mention the following simplification of the method, proposed by Mr. S. W. Melsom for the purpose of comparing two nearly equal resistances. In fig. 2 R and R′ are the resistances to be compared, while D

Fig. 2.
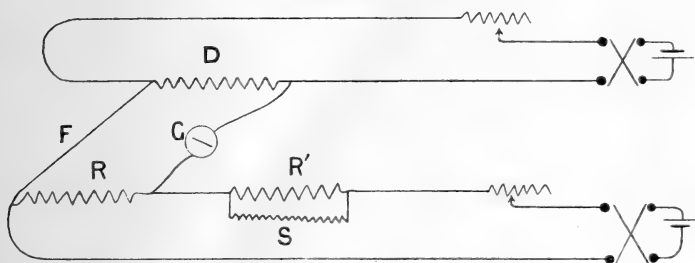
may be any unknown resistance of convenient magnitude. The double switch is arranged to switch the ends of F and G from R to R′. Thus a standard R can be copied at R′ without the use of any other known resistances.

Thomson Bridge.—The setting up of this was similar to that used at the German Reichsanstalt, the connexions being

Fig. 3.
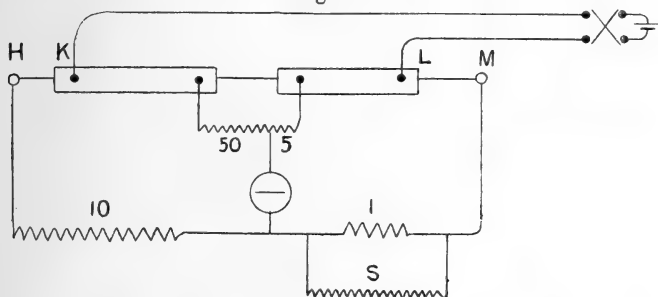
as in fig. 3. As the current entered and left by the potential leads at K and L, the leads HK and LM were included in the

D 2

higher arms, and hence had to be measured. This was done with sufficient accuracy by passing a current through them, and measuring the potential-drop on the galvanometer. To eliminate thermoelectric effects the galvanometer circuit was kept closed, and the balancing was done by reversing the battery.

" *Two-Step* " *Bridge.*—This is shown in fig. 4. A suitable small resistance, whose value need not be known exactly, is inserted at U, and is adjusted by shunting until the galvanometer balances in position (*a*). The galvanometer is then brought into position (*b*), and a balance obtained by another

Fig. 4.
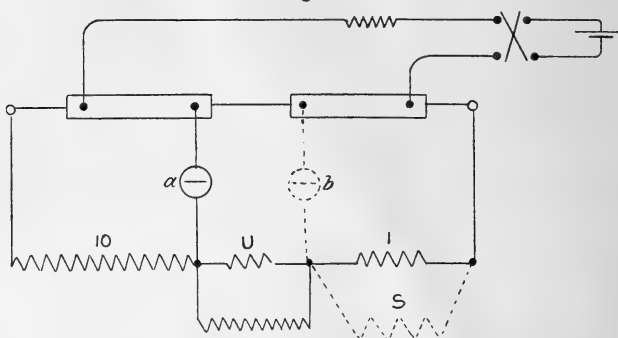
shunt at S. With a suitable galvanometer this method is considerably more sensitive than the Thomson Bridge, but it is less convenient since it necessitates balancing twice.

Differential Galvanometer.—The method of using this is so well known [*] that very little description is necessary. If the galvanometer circuits (100 ohms each) be called α and β, α had in series with it a small adjustable resistance z, while in series with β was a non-inductive copper coil of 900 ohms. This coil was placed near the galvanometer, the whole being in a double-walled box lagged with wadding. The α and β circuits were first connected in series and opposition, their magnetic effects on the needle being adjusted to balance in the usual manner by a small coil in series with one of them Then, by adjusting z, α and β were set to balance on the 10 and 1 ohm coils. Without altering this adjustment a balance was now obtained on the coils (A and B) to be compared, by shunting one of them ; and thus the ratio of A to B was determined. This method was only used for the 0·1 ohm

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[*] See Heaviside's Papers, vol. i. ; also C. W. S. Crawley, Journ. Instn. of Electrical Engineers, April 1901.

coil, but it would have been equally convenient for the lower ones also.

Matthiessen and Hockin's Method.—The connexions are shown in fig. 5. By altering the resistances R_1 and R_2 a

Fig. 5.
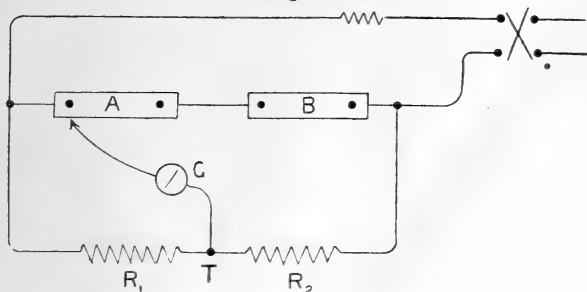
balance was obtained with the galvanometer circuit connecting T with each of the potential points of A and B in succession. Let a, b, c, d be the respective values of $\dfrac{R_1}{R_1 + R_2}$ for the four positions; then

$$\frac{A}{B} = \frac{b-a}{d-c}.$$

Although great care was taken to build up R_1 and R_2 from standard resistances, the method was found to be *much* less accurate than any of the four preceding ones. For example, the ratio of coil Y to coil Z was given as 10·0032 by one set of readings, and as 10·0057 by another. An examination of the formula shows that the degree of accuracy of the result may be much lower than that of the separate observations of a, b, c, and d.

In Table I. are given the collected results, the numbers being the observed resistances of the coils X, Y, and Z reduced to 17° C.

TABLE I.

Method.	X. Microhms.	Y. Microhms.	Z. Microhms.
(1) Potentiometer	100007	$10001·3_7$	$999·94_7$
(2) Thomson Bridge	$100005·_2$	$10001·2_2$	$999·97_2$
(3) Two-step ,, ...	$100005·_3$	$10001·2_3$	$999·93_4$
(4) Differential Galvanometer	$100007·_4$	—	—

In Table II. are given the corresponding sensitivities expressed in galvanometer scale-divisions (1 mm. at 1000 mm.) for 1 part in 100,000.

TABLE II.

Method.	X.	Y.	Z.
(1)	0·7	0·6	0·4
(2)	0·8	0·8	0·28
(3)	1·2	1·2	0·42
(4)	0·3	—	—

In conclusion I may remark that, within the limits of accuracy indicated by the numbers in Table II., the results given in Table I. show satisfactory agreement.

APPENDIX.

On the Sensitivity of Potentiometers for Measuring Resistance.

To obtain the best results with a potentiometer it is usually desirable that it should be of such resistance as to give, with the available galvanometer, the maximum sensitivity. If several galvanometers are available, it is important to have a means of predicting which will be the most suitable in any particular case. I have therefore investigated the general formula for the sensitivity of a potentiometer when used for measuring resistances with a galvanometer of given resistance, extending it to the more general case by the common assumption that the "current-sensitivity" of a galvanometer is proportional to the square root of its resistance.

In fig. 6 let the resistances have the values shown, viz. $\frac{1}{a}$, $\frac{1}{n}$, $\frac{1}{x}$, $\frac{1}{m}$, and $\frac{1}{g}$; $\frac{1}{m}$ and $\frac{1}{n}$ including the resistances of the adjacent batteries whose E.M.F.'s are E and e respectively. Let 0, V, and v be the potentials at the points so marked. Let c = current through galvanometer, so that $c = (v - V)g$.

$\frac{1}{a}$ is the part of the circuit under test, while $\frac{1}{x}$ is the part of the potentiometer in use.

Fig. 6.
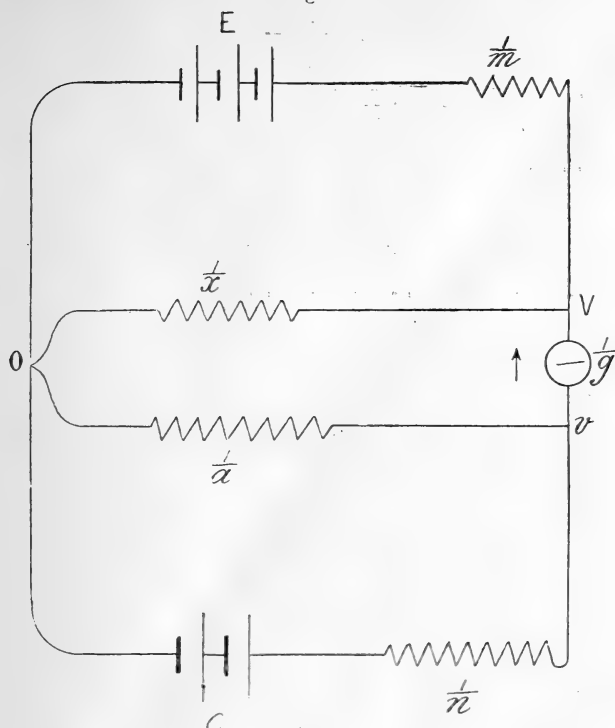
We have

$$(V - E)m + Vx = (v - V)g = -(v - e)n - va.$$

Hence

$$(m + x)V - c = mE,$$

and

$$g(a + n)V + (g + a + n)c = gne;$$

$$\therefore \quad c = \frac{gnex + gmn(e - E) - gamE}{(g + a + n)x + (m + g)(a + n) + mg}.$$

The sensitivity

$$= \left[\frac{\Delta c}{\dfrac{\Delta x}{x}} \right]_{c=0} = - \left[x \frac{\partial c}{\partial x} \right]_{c=0}$$

$$= - \frac{gnex}{(g + a + n)x + (m + g)(a + n) + mg} = -\sigma \ (\text{say}).$$

When $c = 0$,

$$nex + mn(e - E) - amE = 0;$$

$$\therefore \quad m = \frac{nex}{(n + a)E - ne};$$

$$\therefore \quad \sigma = \frac{gnex}{(g + a + n)\left[x + \dfrac{nex}{(n + a)E - ne}\right] + (a + n)g}.$$

Let resistances be written instead of conductances, *i. e.* let $\dfrac{1}{x} = X$, $\dfrac{1}{m} = M$, &c. Also let $k = A/N$.

Then

$$\sigma = \frac{ke}{1 + k} \cdot \frac{1}{[A + G(1 + k)]\dfrac{E}{(1 + k)E - ke} + X} \cdot \quad \cdot \quad \cdot \quad (1)$$

This is the complete formula.

If $e = E$, it takes the simple form

$$\sigma = \frac{kE}{1 + k} \cdot \frac{1}{A + X + G(1 + k)}. \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

It can be seen from (1) that since k is always positive, G has always more relative influence than X.

When the galvanometer resistance has not been fixed, if we make the usual assumption mentioned above, then the sensitivity is proportional to $\sigma \sqrt{G}$, and can be calculated for any case. The following table shows the result in two, possibly extreme, cases (when $E = e$).

TABLE.

k.	G.	A.	X.	σ varies as	$\sigma\sqrt{G}$ varies as
0·1	100	0·01	100	$\dfrac{1}{2100}$	$\dfrac{1}{210}$
0·1	1	0·01	0·1	$\dfrac{1}{22}$	$\dfrac{1}{12\cdot1}$

The values of $\sigma \sqrt{G}$ show that for measuring a potential-drop on a resistance of 0·01 ohm (with the battery voltage eleven times the voltage measured) a potentiometer of 0·1 ohm and galvanometer of 1 ohm give nearly twenty times greater sensitivity than a potentiometer and galvanometer of 100 ohms each.

This example is sufficient to indicate the importance of choosing the resistances suitably.

At the same time it should be remarked that in several ways the low-resistance potentiometer is less easy to construct and work; for instance, it may require much larger steady currents; and these practical considerations may sometimes outweigh the advantages of greater sensitivity.

V. *A Direct-reading Potentiometer for Thermoelectric Work.* *By* J. A. Harker, *D.Sc.*[*]

[Plate II.]

TO anyone who has used a potentiometer for thermoelectric measurement of temperature, it will be obvious that nearly all the ordinary types of instrument have material disadvantages when employed for this purpose.

Most of the higher class potentiometers are primarily designed for direct comparison of E.M.F.'s of the order of a volt and upwards, the readings being made in most cases to about ·0001 volt. With the thermojunction materials now employed for high-temperature measurement, this quantity 100 microvolts represents 6° to 10° C.; and in order to attain higher sensitivities at low ranges, special accessory apparatus is required to reduce the standard E.M.F. applied to the potentiometer to $\frac{1}{10}$ or $\frac{1}{100}$ of its normal value. This addition is generally accompanied by a considerable loss of proportionate sensitivity.

It is obvious that in thermoelectric work the circuit in which the E.M.F. to be measured is generated has generally a total resistance of quite a different order of magnitude to the cases most usually met with; and so far from a high resistance in the potentiometer being an advantage as in many cases, it is in this case detrimental to the attainment of a high sensitiveness.

The diameter and length of the wire used for thermojunctions of course varies greatly; but we shall be within the mark in saying that the total resistance of the thermojunction wire seldom exceeds 5 ohms, and that therefore a low resistance in the potentiometer is a great advantage. In the form to be described, intended for the direct measurement of E.M.F.'s up to 21,000 microvolts, the maximum resistance capable of being placed in series with the galvanometer and thermojunction is 2·1 ohms.

The instrument described represents a form which in subsequent work has proved very convenient, and which was designed and made in the National Physical Laboratory workshops, after consultation with the Director, Dr. Glazebrook, to whom I am indebted for several valuable suggestions.

Experience with several forms of Carey-Foster and Callendar-Griffiths resistance-bridges had convinced me that to calibrate to the necessary accuracy a slide-wire of such size as to make a resistance of 0·1 ohm possible with a moderate length was a difficult piece of work ; and therefore several arrangements were tried with a view of rendering possible the use of a short wire of large cross-section. It is, of course, a considerably easier matter to make a thick wire much more uniform from end to end than a thin one, and the calibration to the accuracy required, especially if the wire be very short, becomes comparatively simple.

The final arrangement decided upon, after some preliminary experiments on models, I believed at first to be novel ; but on searching the literature I find it has been previously used for potentiometers, and is nothing more than a low-resistance form of the Kelvin-Varley slide used in telegraphic work. A diagrammatic representation of the connexions is given in Pl. II. fig. 1, and a plan of the top of the instrument in fig. 2. The balancing-coils on which the fall of potential is adjusted to a definite value are in two rows—the centre row of the box being twenty coils of 0·1 ohm each, and in series with these there is a second row immediately behind the slide-wire consisting of eleven coils of ·01 ohm. By means of an arrangement of thick copper bus-bars connected with the ends of the slide-wire, which has a total resistance of ·02 ohm, any two adjacent coils of this latter series may be put in parallel with the slide-wire. The eleven coils of ·01, two of which are thus shunted, are equivalent to exactly ·1 ohm.

For all ordinary thermoelectric work the fall of potential along these two sets of coils is adjusted so that each of the back row represents 1000 microvolts, each ·01 being therefore 100 microvolts. The slide-wire is provided with a scale of 200 divisions figured as 100, which are practically millimetres ; and as the fall along the wire is 100 microvolts, 0·1 microvolt can easily be estimated. It will be seen that the slide-wire thus connected acts like a vernier to the small coils.

The adjustment of the E.M.F. is made by a standard Clark or Weston cell and the auxiliary set of coils in the back row, a feature of the instrument being that without any external alteration either form of standard may be used at

will. The five coils to the left are permanently connected in series, but are arranged so that any coil may be cut out of circuit when required. Their values are 100, 40, 1, ·5, and ·5 ohms respectively. Those to the right are a set of ten simple series coils of ·01 ohm each arranged so that a connexion can be taken from any one of them to the long bus-bar just in front. Suppose we wish to use as a standard a Clark cell whose E.M.F. at the prevailing temperature is 1·4333 volts. It is obvious that we may make the E.M.F. over the ·1 ohm balancing-coils have the desired value of 1000 microvolts by putting into circuit the coils 100, 40, 1, ·5, and 3 of the ·01 series, leading from the third hundredth to no. 17 of the balancing set, when altogether we shall have—

$$
\left.
\begin{array}{r}
100 \\
40 \\
1 \\
·5 \\
·03
\end{array}
\right\} \text{ in back row.}
$$

$$
\left.
\begin{array}{r}
1·7 \\
·1
\end{array}
\right\} \text{ in balancing-coils and bridge-wire.}
$$

With this connexion the maximum E.M.F. measurable is 18,000 microvolts.

Should a Weston cell of E.M.F. 1·0186 be substituted for the Clark, the only alteration necessary would be to short-circuit coils 40, 1, and ·5, and to move the connector from the third to the sixth of the set of hundredths.

The compensating current is furnished by a small 30-ampere-hour secondary cell in series with which is a dial resistance capable of variation from ·01 up to 200 ohms by steps of ·01.

The four thermojunction circuits provided are connected to a selector-switch, by means of which they or the difference of any two of them may be successively brought into circuit, and the change over from the standard cell-connexion required in the preliminary adjustment is made at the two-way switch at the front left-hand corner, which puts in the galvanometer to the circuit required.

Throughout the whole apparatus the only metals present carrying current are high-conductivity copper and manganin, and all connexions on the top of the box are made by means of mercury-cups. There are doubtless objections to the use of mercury in any apparatus intended to be permanent ; but as it was essential that in this instrument the contact-resistance of certain portions should be very low and constant, it was thought to be, on the whole, preferable to use well-fitting

amalgamated contacts throughout, rather than any form of rubbing contact or plug.

The interconnexions are made by means of a system of copper bus-bars and a form of copper mercury-cup, designed so as to be made without undue waste of material on an automatic lathe.

A pair of these cups with a connector for short-circuiting them are shown in full size in the diagram. It will be seen that the excess of mercury squeezed out of the lower hole may remain in the widened portion without overflowing.
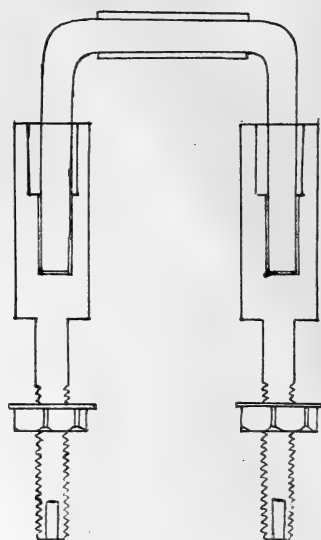
The slide-wire, which is of manganin, is protected from oxidation by gilding, as in some of the Crompton-Fisher potentiometers. Its ends are hard-soldered to massive copper blocks, which are screwed to the transverse ends of the two bus-bars above and below the row of ·01 ohm balancing-coils.

In order to make the bridge-wire exactly equivalent to the two coils, it was deemed better not to attempt to alter the cross-section by drawing or scraping, but to make the whole very slightly above ·02 ohm, and to reduce this to the desired exact value by inserting in the interior of the ebonite handle, holding together the two U-pieces forming the movable slide-wire connexion, a small shunt of about 200 times the resistance of the wire. In case of subsequent alteration of the relation of the slide-wire resistance to that of the coils, this shunt could

be altered as required, instead of a new slide-wire being necessary.

The blocks at the ends of the slide-wire, in addition to being firmly screwed, were connected to the bus-bars by mercury-cups and U-pieces, as it was not thought advisable to trust to the screws alone for a contact whose resistance must be negligible compared to ·02 ohm.

One great advantage of this form of instrument is that all the coils could be arranged so as to have their resistance adjusted, when in position in the box, by a method which could be made to include such of the contacts as were actually in circuit. The finest size of wire used will easily stand 1 ampere continuously through it; so adjustment even of the low coils to one part in a hundred thousand is not difficult.

The three sets of ·01 and ·1 ohm were made each of one continuous wire by selecting appropriate specimens of manganin, whose resistance from end to end was fairly uniform, and soldering to these by silver solder at equal distances short copper wires 1·5 mm. diam.

The whole was then bent to approximately the shape required, varnished with several coats of good shellac, and annealed by repeated heatings for some time to about 140° C.

The copper wires were then soft-soldered into place, the rough adjustment of the coils to about 1 part in 1000 being previously made by an appropriate method. After a week or two the fine adjustment was made by successive scraping, the contacts being employed as in actual use, and the whole being supported for this purpose at a convenient height by a suitable spider-frame.

When complete, two more coats of shellac were applied all over the interior of the box to protect against any possible inward leakage of mercury.

It is not of great importance that the coils of a potentiometer of this kind should be of negligible temperature-coefficient, so long as this is the same for all the different sizes of wire employed; but it is desirable that the series-resistance, used to reduce the E.M.F. of the accumulator to that of the standard cell, should be of the same material as the potentiometer, so that the effect of temperature changes will be compensated. Placing the standard cells and also the accumulator in double-walled boxes packed with cork-clippings materially adds to the convenience of working, obviating the necessity for so frequently checking the compensating current.

Mercury reversing-keys in the circuit both of the battery and thermojunctions are a convenience, and where many

junctions are in use together, it is better to have a larger selector-switch with multiple ways, instead of using the four-way one provided on the potentiometer proper.

All terminals of the potentiometer are arranged in one row at the back, and a cover over the box is provided, which, however, does not come over the terminals, so that the wires need not be detached when the instrument is not in use.

I am indebted for suggestions and help to Mr. Keeling, and the making and adjustment of the coils of the instrument has been mainly done by Mr. Melsom, while the rest of the constructional work has been carried out very efficiently by the mechanic to the physical department, Mr. Murfitt.

Teddington, Feb. 1903.

VI. *On Refraction at a Cylindrical Surface.*
By Arthur Whitwell, *M.A., A.R.C.Sc.I.*[*]

[Plate III.]

THE object of this paper is to describe and illustrate the position and form of the focal areas produced by the refraction at a cylindrical surface, bounding two media of different refractive indices, of light diverging from or converging to a point.

In general, when light diverging from or converging to a point falls on any surface bounding two media of different refractive indices, if a plane can be drawn through the point to cut an element of, or the whole surface, symmetrically, then all the light will really or virtually pass through a focal line or focal area in this plane. It is usual, when considering elements of the surface only, to use the term focal line, but it should always be remembered that these focal lines are in general elements of area. In the case we are about to consider there are two planes, about which the surface is symmetrical, which can be drawn through the radiant-point. One of these planes will contain the axis of the cylinder, and the other will be normal thereto.

We will consider first the plane containing the radiant-point and the axis of the cylinder.

Let fig. 1 (Pl. III.) be a plan and elevation of the cylinder of radius r, and let the radiant-point o be at a distance a from the axis. Draw the elevations and plans of two symmetrical incident rays and of the corresponding normals and refracted rays, join $o\,c$ and produce to meet the refracted ray in a point g. The two refracted rays will meet at the point g,

and the locus of the point g will be the focal line. A plane containing the points o, a, c will contain an incident ray and the normal, and will therefore also contain the corresponding refracted ray. This plane will cut the plane of the figure (1) in the line ocg, and the refracted ray will in general cut this line ocg in some point g. Again, a plane containing the points o, b, c will contain the corresponding symmetrical incident ray, normal, and refracted ray, and will cut the plane of the figure in the line ocg. From considerations of symmetry it follows that the two refracted rays will intersect in the same point g on the line $o\,c\,g$. If we consider only a thin slice of the cylinder parallel to the plane of the paper in fig. 1, the elevations of the lines oa, ac, and ag may be taken as their true lengths.

Let the vertical aperture $= h$,
The angle of incidence $= \theta$,
     ,,   of refraction $= \phi$,
     ,, agc $= \beta$,
     ,, com $= \psi$,
and the length cg $= d$.

Then we have

$$\beta = \psi - \phi$$
$$\sin \theta = \mu \sin \phi$$
$$d \sin \beta = r \sin \phi$$
$$d = \frac{r \sqrt{a^2 + h^2}}{h \cot \phi - a},$$

from which we get

$$d = \frac{r \sqrt{a^2 + h^2}}{\sqrt{\mu^2 (a - r)^2 + (\mu^2 - 1) h^2} - a}. \quad . \quad . \quad (1)$$

Taking the radiant-point as origin, the equation to the locus is

$$x = a + d \cos \psi,$$

or

$$x = a + \frac{ra}{\sqrt{\mu^2 (a - r)^2 + (\mu^2 - 1) h^2} - a},$$

and putting in the value of

$$h = \frac{ay}{x},$$

we get finally

$$y^2 = \frac{x^2}{a^2 (\mu^2 - 1)} \left\{ \frac{(ax + ra - a^2)^2 - (\mu (a - r)(x - a))^2}{(x - a)^2} \right\}. \quad (2)$$

It may be stated here that the equation represents not only the locus of the intersections of the real refracted rays, but also of the false refracted rays. These false rays are equally inclined to but on the opposite side of the normal to the real rays. This arises from the fact that we have to square to get rid of the radical sign in the denominator of the expression for d; for this reason it is immaterial, as far as the equation arrived at is concerned, whether we take the $+$ or $-$ sign before the radical quantity in the denominator of d.

The curves represented by this equation, taking $\mu = \dfrac{3}{2}$ and $r = 2$ and various values of a, are shown plotted in fig. 4, the light proceeding from left to right. When the radiant-point is at infinity, the focal line will be the straight line I. at a distance $\dfrac{\mu r}{\mu - 1}$ from the surface, or $\dfrac{r}{\mu - 1}$ from the axis of the cylinder. As the radiant-point moves up to the surface, the focal line gradually bulges out to the right at its centre, and its upper and lower ends bend towards and become asymptotic to the axis as shown by curve II., for which $a = 10$. When the distance of the radiant-point from the surface $= \dfrac{r}{\mu - 1}$, or when $a = \dfrac{\mu r}{\mu - 1}$, the centre of the focal line breaks, and its two ends become parallel to, but at an infinite distance from, the axis of x. The curve III. has parabolic asymptotes, the equation of which, referred to the radiant-point as origin, is

$$y^2 = 3 \cdot 2\, x + 22 \cdot 4.$$

This point, $a = \dfrac{\mu r}{\mu - 1}$, will be recognized as the principal focus for light proceeding from right to left. When the radiant-point is inside this focus, the curve, IV., has two branches and a pair of rectilineal asymptotes, the axis of the cylinder remaining an asymptote. The branch on the left is of course virtual.

As the radiant-point moves to the right, the angle which the rectilineal asymptote makes with the axis of x increases from zero to a maximum, and then diminishes to zero. When the radiant-point is on the surface there is no focal line; an incident cone of light produces a refracted cone, the ratio of the sines of the semi-angles of the cones being $= \mu$.

I have not plotted the false focal lines in curves I.–IV.; they all lie between the surface of the cylinder and the axis, to which they are asymptotic.

When a lies between r and $\dfrac{\mu r}{\mu + 1}$, the real focal line lies between the surface and axis of the cylinder, and the false focal line has two branches and a pair of rectilineal asymptotes. When $a = \dfrac{\mu r}{\mu + 1}$ the real line is shown by curve V., and the false line has parabolic asymptotes, the equation of which is

$$y^2 = 3 \cdot 2x + 7.$$

A graphical method of drawing the asymptotes may be obtained by considering equation (1),

$$d = \frac{r\sqrt{a^2 + h^2}}{\sqrt{\mu^2(a-r)^2 + (\mu^2 - 1)h^2} - a}.$$

If $\mu^2(a-r)^2 + (\mu^2 - 1)h^2 = a^2$, d is infinite, and this relation between a and h gives us the circle shown in fig. 4.

To obtain the asymptotes, draw an ordinate to the circle through the radiant-point, and project horizontally to a point on the axis of the cylinder. The asymptote will pass through this point on the axis, and through the radiant-point. When $a = 0$ the focal line will be the axis itself, and this is the only case in which the focal line will be a mathematical straight line whether the aperture be large or small.

When a becomes negative the curve is still asymptotic to the axis (see curve VI., for which $a = -10$), and as a increases the curve gradually moves to the right and approaches the line I., which it reaches when a is infinite.

The set of curves shown in fig. 5 are for the case in which light proceeds from a denser to a rarer medium, and are obtained by putting $\mu = \dfrac{2}{3}$ in equation (2).

When the radiant-point is at infinity the focal line is a straight line, VII., at a distance $\dfrac{r}{\mu - 1}$ from the surface, or $\dfrac{\mu r}{\mu - 1}$ from the axis: it is virtual. As the radiant-point moves to the right, the curve becomes of the form shown by VIII., for which $a = 10$. In this curve, when the vertical aperture $h = \pm \dfrac{\mu(a-r)}{\sqrt{1-\mu^2}}$, total reflexion occurs, and the focal line cuts the surface; the continuation of the curve inside the cylinder is the false focal line.

As a diminishes, the curve becomes smaller (curve IX. is for $a = 6$), and finally diminishes to a point when $a = r$; in

this case a cone of light having a semi-vertical angle equal to the critical angle is refracted, the remainder of the incident light being totally reflected, and the refracted cone having a semi-angle of $\frac{\pi}{2}$. When the radiant-point is inside the cylinder, or when the light is convergent, the curve becomes of the form X., for which $a=1$. In this case the part of the curve inside the cylinder is the real, and the part outside the false focal line.

When $a = \frac{\mu r}{\mu+1}$ the curve, XI., has parabolic asymptotes, the equation of which is

$$y^2 = -7{\cdot}2\,x - 13.$$

When a lies between $\frac{\mu r}{\mu+1}$ and $\frac{\mu r}{\mu-1}$ the curve has two branches and a pair of rectilineal asymptotes. Curve XII. is for $a = {\cdot}5$, and the branch on the right and that part of the branch on the left which is outside the cylinder are the false focal line, the real focal line being that part of the left branch which is inside the cylinder. The asymptotes may be drawn by means of a circle, the equation of which is obtained as before.

When $a = 0$, the axis is the focal line. When a lies between 0 and $\frac{\mu r}{\mu-1}$ or -4, the curve has two branches; the branch on the right and that part of the branch on the left which is outside the cylinder are the real focal line, and that part of the branch on the left which is inside the surface is the false focal line.

When $a = -4$, the curve, XIII., has parabolic asymptotes, the equation of which is

$$y^2 = -7{\cdot}2x + 21,$$

this point being the principal focus for rays passing from right to left.

When a is negative and greater than 4 the curve has only one branch, which lies on the left of the axis, that part outside the surface being the real and that part inside the false focal line. Curve XIV. is for $a = -10$.

As the radiant-point moves off to infinity towards the right, the curve approaches and ultimately coincides with the straight line VII.

The curves shown in figs. 4 and 5 are for light falling on the convex surface of the cylinder, but by reversing them

about the axis of the cylinder they will represent the focal lines produced by light passing from left to right and falling on the concave surface of the cylinder if the corresponding values of *a* be also reversed in sign.

We shall now find the equation of the locus of the intersection of two symmetrical rays which have the greatest angle of incidence possible, viz. $\frac{\pi}{2}$. Suppose the triangle *oag*, fig. 1, to be turned down into the plane of the paper on the line *og*, as shown in fig. 2. We know that

$$oc = \sqrt{a^2 + h^2}; \quad oa = \sqrt{a^2 - r^2 + h^2}; \quad \theta = \frac{\pi}{2};$$

and that
$$\sin \varphi = \frac{1}{\mu}.$$

We also have
$$r \sin \phi = d \sin \beta,$$

$$d = \frac{r}{\mu \sin \beta} = \frac{r \cdot \sqrt{a^4 + h^2}}{\sqrt{\mu^2 - 1} \sqrt{a^2 - r^2 + h^2} - r}. \qquad . \quad (3)$$

Taking the radiant-point as origin, $x = a + d \cos \psi$, or

$$x = a + \frac{ra}{\sqrt{\mu^2 - 1} \sqrt{a^2 - r^2 + h^2} - r},$$

and putting $h = \frac{ay}{x}$, the equation reduces to

$$y^2 = \frac{x^2}{a^2(\mu^2 - 1)} \left\{ \frac{x^2 r^2 - (\mu^2 - 1)(a^2 - r^2)(x - a)^2}{(x - a)^2} \right\} \qquad (4)$$

Putting $\mu = \frac{3}{2}$, $r = 2$, the locus represented by this equation is shown plotted in fig. 6. The general nature of the curves is the same as that of the curves for small apertures shown by fig. 4. As before, we can get a relation between *a* and *h* for which the corresponding value of *d* is infinite.

In this case $d = \infty$ when $a^2 + h^2 = \frac{\mu^2 r^2}{\mu^2 - 1}$; this relation represents a circle of radius $= \frac{\mu r}{\sqrt{\mu^2 - 1}}$.

When $a = \infty$ the focal line is a straight line, XV., at a distance $\frac{r}{\sqrt{\mu^2 - 1}}$ or 1·78 from the axis. As the radiant-point moves to the right, the curve bulges out at the centre and bends towards the axis at each end, as shown by curves XVI. and XVII., for which $a = 10$ and $a = 4$.

E 2

When $a = \dfrac{\mu r}{\sqrt{\mu^2 - 1}} = 2\cdot 68$, the principal focus for rays going from right to left, the curve, XVIII., has parabolic asymptotes.

When a lies between $\dfrac{\mu r}{\sqrt{\mu^2 - 1}}$ and r, or between $\dfrac{-\mu r}{\sqrt{\mu^2 - 1}}$ and $-r$, the curve has three branches and a pair of rectilineal asymptotes. Curve XIX. is for $a = 2\cdot 4$; the false focal line is not drawn, and the branch on the left is virtual.

When a lies between r and $-r$, we cannot have an angle of incidence of $\dfrac{\pi}{2}$.

When a lies between $-r$ and $-\dfrac{\mu r}{\sqrt{\mu^2 - 1}}$ we get a curve like XIX., but reversed about the axis of y. As the radiant-point moves to the right, the focal line gradually approaches and finally coincides with the line XV. (See curve XX., which is for $a = -10$.)

When the light passes from a denser to a rarer medium we cannot obtain the equation of the locus of the intersection of symmetrical refracted rays by putting μ less than unity in equation (4), because for an angle of incidence of $\dfrac{\pi}{2}$ the rays would be totally reflected. The maximum angle of incidence will be the critical angle; we shall, therefore, find the equation of the locus for symmetrical rays having an angle of incidence the sine of which $= \frac{2}{3}$. When the triangle oag is folded down into the vertical plane on the line og, we shall get the construction shown in fig. 3, in which θ is the angle of incidence, ϕ the angle of refraction. We have

$$\sin\theta = \mu \; ; \; \beta = \theta - \alpha \; ; \; r\sin\theta = \sqrt{a^2 + h^2}\,\sin\alpha \; ; \; d = \frac{r}{\cos\beta},$$

or

$$d = \frac{r\sqrt{a^2 + h^2}}{\sqrt{1 - \mu^2}\,\sqrt{a^2 + h^2 - \mu^2 r^2} + \mu^2 r} \quad \cdot \quad \cdot \quad \cdot \quad (5)$$

$$x = a - d\cos\psi,$$

and putting $h = \dfrac{ay}{x}$ we get finally

$$y^2 = \frac{x^2}{a^2(1 - \mu^2)}\left[\frac{(\mu^2 ra - ra - \mu^2 rx)^2 - \left\{(x - a)^2(1 - \mu^2)(a^2 - \mu^2 r^2)\right\}}{(x - a)^2}\right].$$

The locus represented by this equation is shown plotted in fig. 7, r being $= 2$ and $\mu = \frac{2}{3}$.

The equation to the circle which gives the asymptotes is

$$a^2 + h^2 = \frac{\mu^2 r^2}{1 - \mu^2}.$$

When $a = \infty$ the focal line is the straight line XXI. at a distance $\frac{r}{\sqrt{1 - \mu^2}}$ or 2·68 from the axis. As the radiant-point moves to the right the focal line also moves to the right and becomes asymptotic to the axis, as shown by curve XXII. for which $a = 10$. The branch of the curve XXII. on the right is the focal line for light falling on the concave surface. The real and false focal lines coincide since the refracted ray is at right angles to the normal at the point of incidence. The parts of the curves inside the cylinder have no real existence, they are the loci of intersection supposing that total reflexion did not occur. As the radiant-point approaches the surface the real portion of the focal line becomes shorter and dwindles to zero when the radiant-point reaches the surface. Curve XXIII. is for $a = \frac{\mu r}{\sqrt{1 - \mu^2}}$ or 1·78, which is the point where the asymptote circle cuts the axis of x; the branch on the left, of which only a minute portion is real, is the focal line for light falling on the convex surface of the cylinder, and the branch on the right, which has parabolic asymptotes, is the focal line for light falling on the concave surface of the cylinder.

Curve XXIV. is for $a = 1·33$. The branch on the right and the left-hand part of the branch on the left above the axis of x is the focal line for light falling on the concave part of the cylinder, whilst the remainder of the left-hand branch is the focal line for light falling on the convex portion of the cylinder. When $a = 0$ the axis is the focal line, when a is negative the curves are got by reversing the corresponding curves for positive values of a. Thus curve XXV., which is curve XXII. reversed, is for $a = -10$.

As the radiant-point moves off to the right the focal line gradually approaches and ultimately coincides with the line XXI. from which we started.

We have now plotted the focal lines for maximum and minimum horizontal aperture in all possible cases. A curve from fig. 4 or fig. 5 and the corresponding curve from fig. 6 or fig. 7, *e. g.* curves IV. and XVII., will define the focal area, that is to say, all the light from the corresponding radiant-point will after refraction pass through the area between these curves. If we suppose ourselves at the radiant-point and facing the cylinder half the light will be bent from

left to right and the other half from right to left, but it will all pass through the central plane containing the radiant-point and the axis, in the area between these two curves.   It is easily seen that the width of the focal area on the axis of *x* will be equal to the spherical aberration of the section of the cylinder made by a horizontal plane containing the axis of *x*.

We will now consider the focal areas produced in the second symmetrical plane, viz.: the plane containing the radiant-point and normal to the axis of the cylinder.   Draw a similar construction to that shown in fig. 1 for two rays symmetrical with regard to the horizontal plane, and produce the refracted rays backwards till they intersect at the point *k* in the horizontal plane.   This point *k* will be on a straight line drawn through the radiant-point and parallel to the two normals.

Consider, first, a thin horizontal slice of the cylinder containing the axis of *x*.   For small horizontal aperture the distance $cf = \dfrac{ra}{(\mu-1)a - \mu r}$.   By small apertures I mean those for which one can neglect the spherical aberration in comparison with the length *cf*.

Let
$$cf = c, \quad \text{and} \quad ok = d'.$$

Then from the figure we have

$$\frac{c}{a+c} = \frac{r}{d'}$$

$$d' = \frac{r(a+c)}{c} = (\mu-1)(a-r).$$

For small horizontal apertures then the focal line is an arc of a circle having its centre at the radiant-point and its radius $= (\mu-1)$ times the distance of the radiant-point from the surface.   The focal line is virtual for diverging light and real for converging light.   If we take two horizontal strips of the cylinder at a distance of *h* above and below the horizontal plane the focal line formed by rays which fall on these two strips will also be a circular arc for small horizontal apertures.   Its radius d'' is obtained from the relation

$$\frac{d}{\sqrt{a^2+h^2}} = \frac{r}{d''}$$

$$d'' = \frac{r\sqrt{a^2+h^2}}{d} = \sqrt{\mu^2(a-r)^2 + (\mu^2-1)h^2} - a.$$

All the light which falls on the cylinder will therefore
virtually pass through the area in the horizontal plane
between arcs of two circles of radius d' and d'' having the
radiant-point as centre. The width of the focal area on the
axis of $x = d'' - d'$ and this width increases as the vertical
aperture h increases, and becomes infinite when h is infinite.

The focal lines or areas we have discussed are produced by
the intersections of symmetrical rays. Besides these the
refracted rays produce two caustics which are the loci of
intersections of consecutive rays. The first is the locus of
the intersections of consecutive refracted rays in the plane
containing the radiant-point and the axis of the cylinder, and
is the same as that which would be produced by rays in one
plane refracted at a plane surface dividing any two media of
different refractive indices. The second is the locus of the
intersections of consecutive refracted rays in the plane con-
taining the radiant-point and normal to the axis of the
cylinder, and is the caustic of the circle. Incident rays in
oblique sections of the cylinder do not produce caustics since
the corresponding refracted rays are not in the same plane.

If we suppose light from a radiant-point to fall on a
cylinder the radius of which gradually increases to infinity,
it will be seen that ultimately, when the curvature is zero,
the two focal areas will coincide and be reduced to a short
piece of the line through the radiant-point and normal to the
surface. Similarly, if we suppose the cylinder to become a
semi-ellipsoid which gradually becomes a hemisphere, then
the two focal areas will ultimately coincide and become a
portion of the line joining the radiant-point and the centre of
the hemisphere.

Instead of having caustics in two planes only we now have
a caustic in every plane passing through the radiant-point
and the centre of the hemisphere.

If light proceeding from or to a point a fall on the plane
or spherical surface of a plano- or sphero-cylindrical lens of
small aperture it will pass on to the cylindrical surface as if
it proceeded from a point a', a and a' being conjugate with
respect to the first surface. The focal areas produced by a
plano- or sphero-cylindrical lens are therefore, for small
apertures, identical with those produced by refraction at a
single cylindrical surface; and if we define two optical
systems as " equivalent " when they produce identical focal
areas, then we can say that a plano- or sphero-cylindrical lens
with the radiant-point at a is equivalent to a single cylin-
drical surface with the radiant-point at a'; a and a' being
conjugate with respect to the first surface.

In conclusion, I should like to call attention to a misleading statement made by Prof. S. P. Thompson, in a paper on this subject read before this Society on December 8th, 1899 [*]. He says: "In any lens having at one surface a radius of curvature r, the curvature which that surface will impress upon a plane wave is $\frac{\mu-1}{r}$; where μ is the refractive index of the material. If the lens is cylindrical, having a curvature in one meridian only, the impressed curvature will also be cylindrical."

"Let AA' be the axis of a cylindrical lens, and NN' a line normal to that axis. A plane normal to the axis intersecting the lens in NN' will have as its trace through the curved surface of the lens a line of the same curvature as the lens, viz. $\frac{1}{r}$. Let now an oblique intersecting plane be drawn through the optic axis of the system; its intersection PP' making an angle $NOP=\phi$ with the line NN'. The curvature at O of the trace of this plane, where it cuts the curved surface along PP', will be $\frac{1}{r}\cos^2\phi\ldots$ We may further consider the intersection QQ' of another oblique plane at right-angles to PP'. The curvature at O along the line QQ' will be $\frac{1}{r}\sin^2\phi\ldots$ If light were admitted through narrow parallel slits set respectively along PP' and QQ', the convergivity of the two beams respectively impressed by the lens would be $(\mu-1)\frac{\cos^2}{r}\phi$ and $(\mu-1)\frac{\sin^2}{r}\phi$. If r is expressed in metres, then these two convergivities will be expressed in dioptries...."

Now if a plane wave fall on a thin plano-cylindrical lens the emergent wave-surface, for small aperture, will be a cylinder of radius $\frac{r}{\mu-1}$. Every refracted ray will pass through a line or narrow band parallel to the axis of the cylinder and at a distance $\frac{r}{\mu-1}$ from the lens. If we suppose a card with a narrow diagonal slit to be placed in front of the lens it is obvious that a great part of the cylindrical wave-surface will be cut off, but the portion that remains will still be cylindrical, and will have the same radius. The rays that pass through the slit still pass through the line at the distance $\frac{r}{\mu-1}$, and the convergivity is the same as before.

The power of impressing convergence on a plane wave of the diagonal strip of the lens is the same as the power of the whole surface. A cylindrical lens can only properly be held to have *two* powers, and it appears to me to be a mistake to speak of the power of a cylindrical lens along a line making an angle ϕ with the axis as being $\frac{\mu-1}{r}\sin^2\phi$.

If parallel light be allowed to fall on a cylindrical lens and a ground-glass screen be placed at a distance of $\frac{r}{\mu-1}$ from the lens, then a line of images of the source of light, parallel to the axis of the cylinder, is seen on the screen. If a stop with a diagonal slit be now inserted and the screen be moved up close to the lens, a line of images out of focus, less intense than before, and parallel to the slit is seen on the screen ; as the screen is moved away this line of images rotates and gradually becomes sharper, until when at length the screen reaches the distance $\frac{r}{\mu-1}$ the line of images is parallel to the axis of the cylinder and is perfectly sharp. This shows that the slit has made no difference to the position of the focus or to the power of the lens. As the screen moves further back still, the line of images continues to rotate in the same direction and gets more and more fuzzy or out of focus.

In a paper entitled " On Astigmatic Lenses," read before this Society on November 9th, 1900 *, Mr. R. J. Sowter makes the same mistake, and speaks of the power of a cylindrical lens in a direction OR making an angle ϕ with the axis of the lens as $A \sin^2\phi$, where A is " the equatorial or focal power of the lens." He also speaks of the power of a plano-ellipsoidal lens along a direction OR making an angle ϕ with an axis of the elliptic plane face of the lens as being $= A \cos^2\phi + B \sin^2\phi$, A and B being the two powers of the lens.

My remarks apply as well to an ellipsoidal lens as to a cylindrical. When the radiant-point is on an axis of an ellipsoidal lens, the light produces two focal areas and two caustics in the planes of maximum and minimum curvature. A thin slice of the lens parallel to the direction OR will not produce caustics, and the rays which pass through it will all pass through the same focal areas as they would if the whole lens were employed. A screen placed at distances $\frac{1}{A}$ or $\frac{1}{B}$ would show the same sharp but less intense lines of images of the source of light as when the whole lens was used.

Both of these writers appear to have considered the curvature of the section of the lens in any direction to be the important element in determining the position of the focal lines instead of the principle of symmetry.

I have to thank Prof. Everett for suggesting to me the reason why my equations represent the false focal lines as well as the real; also Mr. Lyndon Bolton for tracing the curve represented by equation (2), and showing that in a particular case it has parabolic asymptotes.

VII. *Spectra of Gases and Metals at High Temperatures.* *By* JOHN TROWBRIDGE[*].

[Plate IV.]

THE spectra of metals in atmospheric air are the visible evidence of extremely complicated chemical reactions due to the metallic vapour and the gases of the atmosphere. The spectra of gases also in narrow containing vessels of glass or of quartz are modified by the walls of these vessels when the temperatures of the gases are very high ; moreover, the ordinary method of obtaining photographic spectra either of metals in air or rarefied gases, by long continued discharges produced by the Ruhmkorf coil or transformers, masks certain fundamental reactions.

It is therefore desirable to study the effect of known quantities of energy successively applied to produce spectra either of metals or gases. This can best be accomplished by charging a condenser to a known amount by a known electromotive force, and by discharging the condenser between terminals of metals either in air or in gases. If the spectra produced in this manner, by discharges varying from one to any desired number, are photographed on the same plate and treated alike in the same developer, the ground may be prepared for some generalization of the extremely complicated reactions I have mentioned. I believe that this method is a fundamental one to use if order is to be brought out of the chaos of spark-spectra.

I have applied this method in the following manner :—A storage-battery of from ten thousand to twenty thousand cells is employed to charge a condenser ·1 to ·3 microfarad. By a simple mechanical appliance the condenser is detached from the poles of the battery, and is discharged between suitable terminals. Although it is impossible to avoid a slight spark

at the moment of making contact with the receiving system, an approximately equal quantity of electricity is communicated to this system at each discharge. The method also permits of a definite control.

In order to photograph on the same plate the spectra produced by successive discharges, the photographic plate must be slid vertically from one position to another at the focus of a Rowland grating. The most convenient arrangement for the study of gaseous spectra is to employ a grating of short focus, and to inclose it in a light-tight box for compactness, and to dispense with a dark room. I have employed the method of mounting the grating in such a manner that the normal to the ruled surface passes through the slit. The camera swings on the arc of a circle described by an arm of half the radius of the grating. This arm is pivoted at a point halfway between the ruled surface and the slit.

In fig. 1, G is the grating, M the point midway between

Fig. 1.
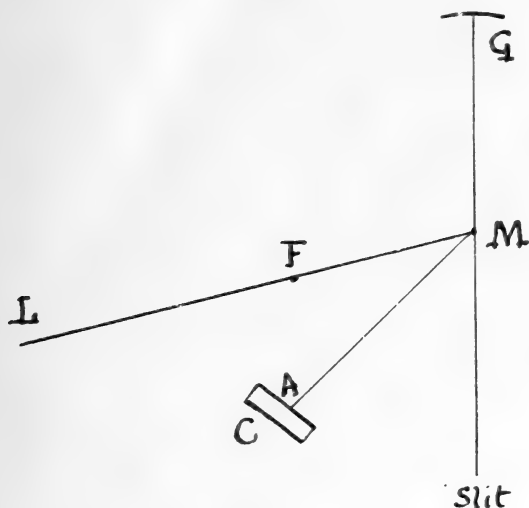
the grating and the slit, C the camera swinging on the arc described from M. Fig. 2 gives a side view and elevation of the camera; P is the photographic plate, O an opening closed by a slide operated by a lever-arm which engages with A. The plate-holder, closed by another slide S, can move up or down in parallel ways. In fig. 1 is shown a lever-arm LM with a fulcrum at F. This lever lifts another lever AM which lifts or closes the shutter A, fig. 2. The

lever LM is outside the dark box, and the operation of exposing the plate and closing the camera can be performed without opening the box.

Fig. 2.
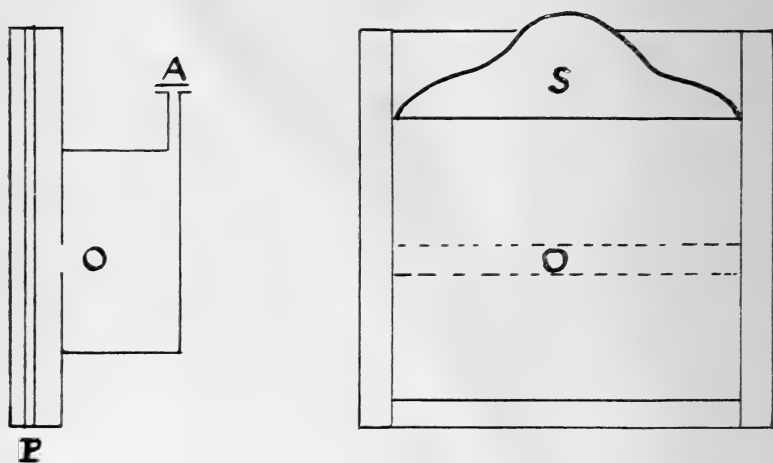
The method which Rowland used in mounting the grating is undoubtedly preferable to the above, when an accuracy greater than one-tenth of an Ångstrom unit is aimed at. The advantage, however, of the method I have employed is in its compactness, and in the possibility of working in a light room ; moreover, in gas-spectra an accuracy even to one Ångstrom unit is often respectable.

Fig. 3, Pl. IV., (the photographs are reproductions of negatives, and are not positives) is a reproduction of the spark-spectrum of calcium in the neighbourhood of the H.H lines of the solar spectrum, taken by the method of successive discharges of known amounts of electrical energy. The discharges ran from one to five. It is interesting to notice on the negatives that the photometric intensity of the lines estimated by the blackness is not directly proportional to the amount of energy. Thus the spectrum produced by four discharges is not twice as intense as that produced by two discharges. It is noticeable, also, that the calcium lines wave-lengths 3737 and 3706 are stronger on the negative than those which coincide with the H.H lines, and always appear with these lines.

Fig. 4 (Pl. IV.) is a negative of discharges running from one to four in a Geissler tube of glass which did not contain

an appreciable amount of lead in its composition. The capillary was 2 millimetres internal diameter, and 4 inches in length. The pressure of hydrogen was 1 millimetre. It is noticeable that the lines which coincide with the H.H lines are not accompanied by the lines wave-lengths 3737 and 3706; although the intensity of the lines coinciding with the H.H lines, if they were calcium lines, would require the presence of these lines.

A series of experiments were undertaken with metallic terminals 1 centimetre apart in lead glass. The diameter of the capillary varied from 1 to 2 millimetres. A spectrum similar to that of fig. 4 was obtained with the addition of certain lead lines, several of which were reversed on the side towards the ultra-violet. The portion of these lines not reversed broadened towards the red end of the spectrum; and this broadening increased with the intensity of the discharge (fig. 7). No calcium lines appeared. When cadmium terminals were used many of the cadmium lines were reversed ; and here also the bright portions of these lines were much broadened. In the case of cadmium no other lines were observed (fig. 6). The gaseous ions contributed little or nothing to the photographic effect.

When iron terminals were employed no iron lines were obtained, even when the terminals were only 3 millimetres apart ; nevertheless the lines and bands usually attributed to silicon came out with great intensity. When, however, aluminium terminals were substituted for iron terminals, aluminium lines together with the supposititious silicon lines were obtained (fig. 5). It was noticeable that the two lines coinciding with H.H lines of the solar spectrum did not appear, while the two characteristic aluminium lines between the H.H lines came out reversed. The lines corresponding with the H.H lines always appeared when a discharge of like intensity produced the spectrum of aluminium in air.

The iron of which the terminals were made was ordinary soft iron with a melting-point not far from 1100 ; while the melting-point of aluminium is between 700 and 800. If the silicon is volatilized, it is difficult to see why the iron gave no spectrum while the aluminium yielded one ; for there is not a very great difference between their melting-points.

Another series of experiments then were made with metallic terminals in quartz capillaries varying in internal diameter from 2 to 3 millimetres, the terminals being 1 centimetre apart. The same spectra were observed as are represented in fig. 4, with an absence of the lines corresponding with the solar H.H lines. This absence was noticeable, also,

when metallic terminals were 1 centimetre apart in lead-glass. Iron terminals gave no iron lines in the quartz tubes, while aluminium lines appeared when aluminium terminals were used instead of iron terminals.

When the metallic terminals were placed 3 millimetres apart in the quartz tubes, the light from the tubes was very feeble; traces of metallic spectra appeared, and the walls of the quartz capillaries were speedily covered with thin films of the metals; even at the first discharge, before there was a sensible obscuration due to the formation of the films, there was no evidence of gaseous spectra. The main discharge appeared to be carried over by the metallic vapour, and no dissociation of the gas was evident.

Measured by definite amounts of electrical energy, the rating of the intensity of spectral lines differs totally from existing eye estimates. The lines which coincide with the H.H lines in the spectra of the metals with high melting-points generally came out first on the photographic plate when the method of successive discharges was employed.

Rarefied nitrogen gave far less light than hydrogen, water vapour, or oxygen. When oxygen was employed characteristic groups of doublets were obtained similar to the A and B groups in the solar spectrum. The heads of these groups apparently coincided with the middle of the broad lines shown in fig. 4. The middle of these broad bands or broad lines coincides also with narrow lines usually attributed to silicon. Are certain lines attributed to silicon really oxygen lines? Salet and also Rowland assign the photometric intensity of 4 to the lines 4131·5 and 4126·5, 3 to the lines 3905 to 3855·7, and 10 to the line 2881. When the lines given on fig. 4 and those on fig. 7 are photographed on the same plate by the same number of discharges, and are compared in regard to intensity, the rating is completely reversed; the lines at 4131·5, 4126·5, and 3905 to 3855·7 being 10, and the line 2881 being 3 or 4.

The broadening of what have been considered metallic lines in rarefied gases I consider a most interesting phenomenon. Only the strong lines of the spark-spectrum of the metal in air seem to be reversed under the effect of powerful discharges in rarefied gases. This broadening appears to be the evidence of reactions between the vapour of the metal and the surrounding gases. In this connexion it is well to bear in mind the fact that metals continue to give off gases for a long time when submitted to powerful electrical discharges in vacuum-tubes.

These nascent gases are in condition to exhibit complicated reactions with the strongly heated metallic terminals.

My experiments lead me to doubt conclusions drawn from the apparent absence of this or that element in the spectra of stars ; for there is a strong possibility that reactions enter which may mask the presence of this gas or that metal.

The conclusions of my work thus far are as follows :—

1. The metallic lines due to terminals in rarefied hydrogen, and rarefied air, when these terminals are one centimetre apart in glass or quartz capillaries exhibit a reversed action. When this takes place it is generally coincident with the position of the line when the spectrum is taken in air, while the spectrum of the line on the least refrangible side is much broadened. This seems to indicate a gaseous product, an oxidization or hydration due to the dissociation of the air and water-vapour present.

2. Highly heated rarefied hydrogen and rarefied air passing over containing walls of glass or amorphous silica, give broad bands which apparently coincide with narrow silicon lines of far lesser intensity in air. These also I attribute to the dissociation of air and water-vapour. The brilliancy of the light produced in this reaction is far greater when there is an excess of hydrogen in the tubes than when rarefied air fills them. It is a question whether certain lines produced by metals like silicon, which volatilize with difficulty in air, are really due to the metals. I am inclined to attribute some of them to the environment; that is, to a reaction between the metal and the gases present.

3. Spark-spectra of metals appear to represent complicated reactions of gases with the metallic vapour.

4. Metallic vapours carry the main portion of an electric discharge when these terminals are within 3 millimetres of each other in rarefied hydrogen or rarefied air. The gaseous ions, if the dissociation occurs, give little light.

5. The broadening of the light accompanying the reversed lines, if unsuspected, might lead one to conclude that a shift of the bright portion had occurred.

6. Since the iron lines do not appear under what seem favourable conditions while aluminium lines appear, while in other cases gaseous lines mask metallic spectra, it seems desirable to be cautious in regard to speculations respecting types of stars.

7. Whatever may be the cause of the reversals of lines observed in narrow capillaries of glass or of quartz, it seems to me that it is a fact which should be reckoned with in photographic study of stars, especially in the case of sudden changes of light.

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VIII. *On the Atomic Weight of Radium.*
By W. Marshall Watts, *D.Sc., F.I.C.**

IN a short paper read before the Physical Society † I have recently called attention to some relationships between the spectra of certain allied elements, and the squares of their atomic weights, which do not seem to be generally known.

There appear to be two distinct kinds of connexion between the spectra of allied elements. In one class of cases, of which the family of zinc, cadmium, and mercury, and that of gallium and indium, furnish the best examples, the differences between the oscillation-frequencies of certain lines of the one element are to the differences between the oscillation-frequencies of the corresponding lines of the other element as the squares of their atomic weights; so that, if it be admitted that the lines *do* correspond, it is possible to calculate the atomic weight of the one element from that of the other by means of the spectra.

In the other class of cases, of which the families potassium, rubidium, cæsium, and calcium, strontium, and barium offer the best examples, the element of greater atomic weight has the smaller oscillation-frequency, and three elements are so related that the differences of oscillation-frequency between the elements, in comparing corresponding lines in their spectra, are proportional to the differences between the squares of the atomic weights; so that we can calculate the atomic weight of one element from the atomic weights of two other elements of the same family by means of their spectra.

These relationships are most easily seen by plotting squares of atomic weights as ordinates, and oscillation-frequencies as abscissæ : it is then seen that—in the first case—the straight lines joining corresponding points in the homologous spectra intersect on the line of zero atomic weight ; and, in the second case, that the corresponding points in the three spectra compared lie on straight lines. It is further observed that these straight lines are very nearly parallel, although they seem to converge slightly.

As an example, the following lines in the arc-spectra of barium, strontium, and calcium lie on such (nearly) parallel straight lines :—

	Barium.		Strontium.		Calcium.	
(a)	15387·0	(6r)	19387·7	(10)	21616·6	(1n)
(b)	17872·7	(4n)	22061·0	(6)	24390·0	(4n)
(c)	21951·2	(10r)	26975·9	(6n)	29735·8	(8n)
(d)	22979·0	(8r)	28176·9	(6n)	30991·1	(4n)
(e)	24200·7	(8r)	29399·4	(1n)	32228·6	(1n)

Taking the atomic weights of calcium and strontium on the oxygen scale as 40·1 and 87·6, we may then calculate the atomic weight of barium on the assumption that the points assumed to correspond in the three spectra have some physical connexion, and that they do actually lie upon straight lines. We thus get by calculation for the atomic weight of barium

	From	(a)	136·90
		(b)	136·97
		(c)	137·51
		(d)	138·06
		(e)	137·85
	Mean		137·46

the accepted value being ...... 137·4

It would seem that similar relationships hold, to a certain extent, between the spectrum of radium and the spectra of mercury, barium, and calcium, and in the present communication I have made an attempt to calculate the atomic weight of radium from these relationships. They seem to me too remarkable to be attributed to chance. Both kinds of relationship seem to exist.

The brightest lines in the spectrum of radium are 20714·8 of intensity (10), 21350·9 (14), and 26207·7 (16). There are two lines in the barium spectrum, 35892·6 (8*r*), and 36070·2 (6), which seem to correspond to the first two radium lines, and to a calcium line at 43930·4 (8*r*). There does not appear to be a second calcium line at this place, but it is situated in the extreme violet, and it is quite possible that a second line may have escaped observation.

Taking the calcium line to correspond to the less refrangible barium and radium lines, by calculating from the atomic weights Ba = 137·4 and Ca = 40·1, we get 226·32 for the atomic weight of radium.

The radium line 26207·7 lies on a straight line with mercury 31982·2 (10 *r*), and barium 42807·0 (8 *r*). Calculating from Hg = 200 and Ba = 137·4 we get 226·42 for the atomic weight of radium.

Other correspondences of the like kind seem to be

	Radium.	Mercury.	Barium.	Calcium.
(a)	17195·5	22362·7 (5)	32545·2 (6*r*)	—
(b)	22052·7	27386·6 (10*r*)	37845·2 (4)	—
(c)	21149·7	..	36987·4 (4*r*)	45423·0 (8*r*)

These give   225·21 from (a) Mercury and Barium,
             225·32 from (b)   „   „   and
             226·52 from (c) Barium and Calcium,

The connecting lines in these five cases are very nearly

parallel to the five lines for barium, strontium, and calcium given as examples, but the whole ten lines exhibit a slight convergence towards the same point.

Taking now the other mode of correspondence mentioned first, which is illustrated by the homology of the triplets (and many other lines) in the spectra of zinc and cadmium, we find the following connexion between the spectra of radium, mercury, and barium.

Lines meeting on the Line of Zero Atomic Weight at Oscillation-Frequency of about 44610.

Radium .... 26207·7 (16) and Barium .... 37940·1 (8)
Radium .... 20714·8 (10) and Barium .... 35892·6 (8n)
give 225·05 for the atomic weight of Radium.

Lines meeting on the Line of Zero Atomic Weight at Oscillation-Frequency 52450.

Radium .... 17195·5   and Mercury .... 24703·6 (6r)
           26207·7 (16)   31921·9 (8r)
give 223·47.
Radium .... 26207·7 (16) and Mercury .... 31921·9 (8r)
           27393·7   32898·8 (4n)
give 220 36.
Radium .... 17195·5   and Mercury .... 24703·6 (6r)
           27393·7   32898·8 (4n)
give 223·13.
Radium .... 21149·7   and Mercury .... 31982·2 (10r)
           26207·7 (16)   28069·2 (4n)
give 227·39.
Radium .... 26207·7 (16) and Mercury.... 31919·3 (8r)
           21350·9 (14)   28069·2 (4n)
give 224·63.

The mean of all these results is 224·89.   The atomic weight of radium as determined by Madame Curie is 225.

IX. *On the Graphical Solution of Astronomical Problems.*
By Harold Hilton [*].

IT is proposed in this paper to give a brief description of the use of the stereographic projection in the graphical solution of astronomical and other problems.   In this projection points and lines on a sphere are projected on the plane (h) of a great circle from either of its poles P and Q.   It is usual to project from P points on the sides of h remote from P, and from Q points on the side of h remote from Q; marking the points with a cross ($\times$) in the first case, and with a little circle ($\circ$) in the second.   In this way all points of the sphere are projected into points lying within the great circle s in

which h cuts the sphere. Great circles are projected into circles cutting s at the ends of a diameter, and small circles are also projected into circles; while angles are projected into equal angles. The stereographic projections required are very much more rapidly and easily sketched if they are drawn on tracing-cloth (Herr G. Wulff * recommends "Pauspapier"), and use is made of the "stereographic net." This may be described as the stereographic projection of lines of latitude and longitude on a plane through a line of longitude. Wulff * has given an accurate diagram in which every other degree is shown and a full description of this net, so that it is unnecessary to give a careful drawing here or to enter upon the details of its use. For purposes of illustration, however, a rough diagram is added in which every tenth degree is shown (fig. 1). The diameter of the net is equal to the diameter of

Fig. 1.
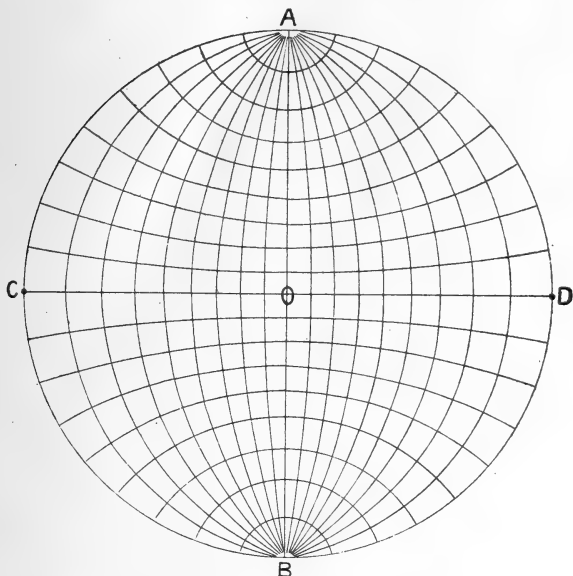
the circle s of the projection and the centre of the projection (drawn on the tracing-cloth so that the net can be seen through it) is *always* placed over the centre O of the net. Mr. S. L. Penfield † describes various kinds of stereographic nets and similar contrivances for drawing stereographic projections, but the stereographic net described by Wulff (and

* *Zeitschr. f. Kryst. u. Min.* xxxvi. 1, p. 14 (1902).
† American Journal of Science, xi. 1901.

shown in fig. 1) is sufficient for most purposes; and simplicity
and economy are gained by confining ourselves to a single
net.

Wulff's net consists of a series of arcs of coaxial circles
intersecting in two real points A and B, and a series of arcs
of coaxial circles orthogonal to them; we shall allude to these
lines as *m* and *l* respectively.

To draw a circle through two given points on the tracing-
cloth which shall be the projection of a great circle, we turn
the cloth till both points lie on the same line *m*, and then trace
this line through *. The angular distance between these
points can be at once read off on the net. To find the pole
of the projection of a great circle, we turn the net till the
projection coincides with one of the lines *m*, and then read off
the position of the pole on the net. To find the angle at
which two great circles cut, we measure the angle at which
their projections cut; either by drawing tangents at their
intersection and measuring the included angle, or perhaps
better by measuring the distance between their poles †.

It may be objected that graphical methods of solving
astronomical problems are far inferior in accuracy to methods
of calculation. To this it may be replied that for many
problems (*e. g.* those connected with the duration of twilight)
a solution correct to within 30′ or so is, from the nature of the
case, all that is required. Moreover, " most persons have
never studied spherical trigonometry, and those who have
studied it usually regard the solution of a spherical triangle
as at least a laborious, if not a difficult matter " ‡. Even a
man accustomed to numerical calculations could hardly solve
with the use of tables the problems we shall consider as
quickly as he could solve them by the aid of the stereographic
net even with very little practice. If it is required to solve
a large number of problems of the same kind, the saving of
time by the graphical method is enormously greater. In
fact, what is practically only a single problem by the graphical
method is often a whole series of distinct problems by the
usual procedure (*e. g.* " trace the changes in the duration of
twilight during the year in any given latitude "). Moreover,
Penfield has shown that with practice and care the errors in
the graphical method can be reduced considerably below 10′.
Again, " by making use of the graphical stereographic

---

* That is, if the points are both marked with a ×, or both with a ○;
if one is marked with a × and the other with a ○, we must turn the net
till the points lie one on each of two lines *m* equally inclined to AOB.

† Wulff, *loc. cit.* pp. 15, 16.

‡ Penfield, *loc. cit.* p. 121.

methods . . . a check upon the results of numerical calculations can be made. The importance of having some simple method of checking cannot be overestimated "*. Lastly, " most persons . . . use formulæ and tables, as a rule, in a mechanical way. With graphical methods . . . every operation is clearly understood . . . In the majority of cases, numerical calculations are laborious, while graphical solutions appeal to one like pictures, which, to a certain extent, tell their own story "†. On this account the stereographic methods for *educational* purposes can hardly be too highly valued.

Fedorow, Wulff, and Penfield have used the stereographic net for the purposes of crystallography; and the latter has also employed similar methods for navigation problems.

We shall proceed to illustrate its use for the answering of certain typical questions in astronomy.

(1) To convert right ascension and declination into longitude and latitude.

We suppose the celestial sphere stereographically projected on the plane of the solstitial colure ; then the poles of the equator and of the ecliptic lie on the circle *s*. Make the north pole coincide with the point A of the net, and draw the line OR (on the tracing-cloth) such that the angle ROC=*i* (the obliquity of the ecliptic) ‡. Then with the aid of the net mark in the positions of any number of stars whose right ascension and declination are known. Turn the tracing-cloth so that OR coincides with OC. The pole of the ecliptic now coincides with A, and the longitude and latitude of all the stars can at once be read off.

(2) To convert longitude and latitude into right ascension and declination.

This is solved by a precisely similar method.

(3) To find the altitude and azimuth of stars whose north polar distance and hour-angle are known.

We project onto the plane of the meridian and then proceed as in (1), replacing the pole of the ecliptic by the zenith and the obliquity of the ecliptic by the colatitude of the place of observation.

(4) Find the hour-angle and north polar distance of stars whose altitude and azimuth are known.

This is solved by the same method as (3).

(5) Given the right ascension and declination of any number of stars at the present day, draw a stereographic map (on the plane of the horizon) which shall represent the appearance of the heavens in given latitude at 4.40 P.M. at the winter solstice *x* years ago.

* Penfield, *loc. cit.* pp. 121, 122.   † Ibid.

‡ The net of course serves the purpose of a protractor.

Proceed exactly as in (1), marking in the positions of the stars, and then making OR coincide with OC. Now diminish the longitude of all the stars by $x \times 50 \cdot 2''$ by moving all the stars along the lines l through this amount. This is readily done with the help of the net: it is best to mark the new positions of the stars on another piece of tracing-cloth laid over the first (the net is readily seen through two pieces of good cloth), and then remove this first sheet, after tracing through the line OR onto the new piece of cloth. Now turn back the line OR into its original position. Find a point Z such that the angle between the circles AZB and ACB is 70°, and the arc ZA represents the colatitude; then Z is the position of a star at the zenith at the given time. Mark on the cloth a star X at the pole A, and turn the diagram till Z lies on the line OC of the net. Move all the stars along the lines l through the number of degrees represented by ZO (preferably on a new sheet of tracing-cloth as before). Let the star X be brought into the position Y by this process. Then the diagram now obtained is the one required, Y being the position of the celestial pole.

(6) What is the interval between the times of rising of a given star as observed by two men, one at the top of a mountain and the other on a plane at its base? (The effects of refraction and the ellipticity of the earth are disregarded.)

On the tracing-cloth trace through the point R coinciding with A, the line COD, and that one of the lines l which is distant $90° + \cos^{-1}\dfrac{a}{a+r}$ from A (or R), a being the radius of the earth and r the height of the mountain. Turn the diagram till the angle ROA is equal to the north polar distance of the star. Let that one of the lines l whose distance from A is equal to the colatitude of the mountain cut the two lines on the tracing-cloth in X and Y; convert the angle between the circles AXB and AYB into time.

For instance, the interval for a mountain on the earth 12,750 feet high in latitude 66° N. is found to be 28 minutes, and for a mountain in latitude 56° N. to be 18 minutes, the star's declination being 20° S. For a mountain on the moon 23,000 feet high in latitude 80° N. the interval is $95\frac{1}{2}$ lunar minutes and for a mountain in latitude 70° N. is $46\frac{1}{2}$ lunar minutes, the star's declination being 1° 25′ N. These results are not very accurate, but greater exactness could readily be obtained by the use of a larger and more finely divided net.

(7) In what direction and at what time will the first signs of twilight be observed in north latitude λ when the sun's declination is 10° S. ?

On the tracing-cloth trace through the point R coinciding with A, and that one of the lines l which is distant 108° from A (or R). Turn the diagram in the clockwise direction till the angle $ROA = 90^\circ - \lambda$, and mark on the cloth the point X in which that one of the lines l which is 100° from A intersects the line drawn on the cloth. The angle α between the circles AXB and ACB gives the time of dawn. To find the direction turn the cloth till R coincides with A again and read off the angle β between the circles AXB and ACB, then dawn is seen at an angle β from the north point of the horizon.

For instance, putting $\lambda = 50^\circ$ N., it is found by the use of Wulff's net that $\alpha = 74^\circ$ and $\beta = 84^\circ$ $15'$; the calculated values are 73° $51'$ and 84° $4'$.

(8) Trace the changes in the duration of twilight in any given north latitude λ throughout the year.

We shall neglect the change of longitude of the sun during twilight; and shall at first reckon the length of twilight from the time that the zenith distance of the sun is 108° to the time that its true zenith distance (unaffected by refraction) is 90°.

On the tracing-cloth trace through the point R coinciding with A, P coinciding with C, the line COD and that one of the lines l which is distant 108° from A (or R). Turn the diagram in the clockwise direction till the angle $ROA = 90^\circ - \lambda$. Let one of the lines l distant γ degrees from A cut the two lines traced on the cloth in X and Y; then the angle θ between the circles AXB and AYB gives the duration of twilight when the sun's north polar distance is γ (the times of dawn and sunrise are also given). We can at once find the duration for any value of γ: thus for latitude 50° N. we find the following values for the angle θ:—

$\gamma =$	68	70	72	74	76	78	80	82	84
$\theta =$	$60\frac{2}{3}$	44	39	36	$34\frac{1}{6}$	$32\frac{5}{6}$	$31\frac{1}{2}$	$30\frac{1}{2}$	30

$\gamma =$	86	88	90	92	94	96	98	100	102
$\theta =$	29	$28\frac{1}{2}$	$28\frac{1}{6}$	28	$27\frac{5}{6}$	$27\frac{5}{12}$	$27\frac{1}{6}$	$27\frac{1}{3}$	$27\frac{5}{6}$

$\gamma =$	104	106	108	110	112 degrees.
$\theta =$	$28\frac{1}{3}$	$28\frac{1}{2}$	29	$29\frac{1}{2}$	$30\frac{1}{6}$ degrees.

If we multiply the numbers in the second rows by 4, we get the length of twilight (expressed in minutes of time) corresponding to the given values of γ.

As might have been expected, it is not easy by tabulating numbers as in the example given to answer exactly the question "for what value of γ is the duration of twilight a minimum?" This problem may, however, be solved graphically with considerable accuracy; but the discussion is rather complicated since $90° \sim \gamma$ must be $\not> i$ (the obliquity of the ecliptic), and since only a limited number of the lines l cut *both* the lines traced on the cloth.

Make the angle $OPV_1 = 81°$, and let PV_1 cut RO produced in V_1; draw V_1S_1 perpendicular to OV_1 cutting AB produced in S_1. Then that one of the lines l which passes through the points of contact of tangents from S_1 to the circle ADBC corresponds to the value (γ_1) of γ for which θ is a minimum. It is readily seen from this construction that

$$\sin (\gamma_1 - 90°) = \sin \lambda \cdot \tan 9°;$$

and since the construction occupies rather a large space, it is preferable to calculate γ_1 from this formula. For $\lambda = 50°$ we have $\gamma_1 = 96° \, 58'$. When $\lambda = 81°$, $\gamma_1 = 99°$, and twilight has a minimum duration of 12 hours; twilight ceasing exactly at midnight and sunrise taking place exactly at midday. When γ has any other value the conditions of our problem cannot be satisfied; for if $\gamma > 99°$, the sun does not rise, and if $\gamma < 99°$, twilight does not cease. If $\lambda > 81°$, the conditions of our problem are never satisfied. If $\lambda < 81°$, the value γ given by the equation $\sin (\gamma - 90°) = \sin \lambda \tan 9°$ always corresponds to a minimum duration of twilight which is a true minimum satisfying the conditions; for since

$$\sin \lambda \cdot \tan 9° > \sin \{18° - (90° - \lambda)\} \text{ if } \cot \lambda > \tan 9°,$$

twilight ceases and sunrise really takes place when γ has the value given by the equation

$$\sin (\gamma - 90°) = \sin \lambda \cdot \tan 9°.$$

The geometrical construction gives only a minimum and no maxima for duration of twilight; there must be maxima, however, when γ has its greatest and smallest possible values (*i. e.* $90° \pm i$), if these values satisfy the conditions of the problem. This is the case for the greatest value of γ in all places in north latitude for which $\lambda \not> 90° - i$, and for the smallest value of γ in places for which $\lambda \not> 72° - i$.

On the diagram make the angle $OPV_2 = 9°$, and let PV_2 cut RO produced in V_2, and draw V_2S_2 perpendicular to OV_2 cutting AB (produced if necessary) in S_2. Then that one of the lines l which passes through the points of contact of tangents from S_2 to the circle ADBC, corresponds to the

value (γ_2) of γ for which the sum $(180° + \phi)$ of the angle between the circles ADB, AXB and the angle between ADB, AYB is a minimum (X, Y, as before, are the points where any one of the lines l cuts the two lines traced on the cloth). This is the value of γ for which daytime reckoned from the middle of twilight in the morning to the middle of twilight in the evening is a minimum. From the construction it follows that

$$\sin(\gamma_2 - 90°) = \sin \lambda \cdot \cot 9°;$$

hence γ_2 is real only if $\sin \lambda \not> \tan 9°$, *i. e.* if $\lambda \not> 9° 7'$ (otherwise the point S_2 lies within the circle ABCD), and moreover γ_2 does not lie between the limits $90° \pm i$ unless $\lambda \not> 3° 27'$.

If $\lambda = 2° 20'$, $\gamma_2 = 104° 53'$, we have graphically the following values of ϕ for various values of γ:—

$\gamma = 68°$	$70°$	$72°$	$74°$	$76°$	$78°$	$80°$
$\phi = 21° 70'$	$20° 30'$	$20° 18'$	$19° 59'$	$19° 38'$	$19° 14'$	$19° 0'$

$\gamma = 82°$	$84°$	$86°$	$88°$	$90°$	$92°$	$94°$
$\phi = 18° 36'$	$18° 12'$	$18° 8'$	$18° 4'$	$18° 0'$	$17° 46'$	$17° 42'$

$\gamma = 96°$	$98°$	$100°$	$102°$	$104°$	$106°$	$108°$
$\phi = 17° 38'$	$17° 34'$	$17° 31'$	$17° 28'$	$17° 28'$	$17° 27'$	$17° 26'$

$\gamma = 110°$	$112°$	$(120°)$	$(130°)$.			
$\phi = 17° 30'$	$17° 30'$	$(18° 0')$	$(19° 25')$.			

The minimum value of ϕ is not very clearly marked in the tabulated numbers as the graphical readings cannot be trusted to within $10'$ or so.

If we wish to determine the interval between dawn and the time at which the centre of the sun *appears* to be on the horizon, we trace on the cloth those lines l which are distant $90\frac{1}{2}°$ and $108°$ from A (see p. 71, lines 24 *et seq.*) instead of the lines distant $90°$ and $108°$ from A (assuming that the sun appears to be on the horizon when it is really $30'$ below). We then proceed as before, making a few slight modifications. For instance, the angles OPV_1, OPV_2 must now be made equal to $\tan^{-1} \dfrac{\cos 8\frac{3}{4}°}{\sin 9\frac{1}{4}°}$ and $\tan^{-1} \dfrac{\sin 9\frac{1}{4}°}{\cos 8\frac{3}{4}°}$ respectively.

The constructions for the points S_1 and S_2 have been based on the following considerations. Firstly, the lines l for which the difference and the sum of the angle between the circles ADB, AXB and the angle between ADB, AYB are minima must cut the lines on the cloth at equal and supplementary angles respectively. This is proved readily by geometrical inversion with respect to either of the points A or B. Secondly, if the line joining the centres of any two

circles q and q' cuts them in H, K; H', K', then any circle cutting q, q' at equal angles is orthogonal to the circle t_1 coaxial with q and q', with respect to which H and K' are inverse points; and any circle cutting q, q' at supplementary angles is orthogonal to the circle t_2 coaxial with q and q', with respect to which H and H' are inverse points. This is readily proved by inverting q and q' into concentric circles. In the present case (in which the two lines on the cloth are arcs of q and q') the circle t_1 is imaginary, but S_1V_1 is the real radical axis of t_1 and ACBD.

These eight examples will serve to show that we have in the stereographic net a useful help for obtaining approximate solutions of certain types of problems, and for checking the results obtained by more tedious, if more accurate, methods. It is also a device of the highest educational value. Moreover, these geometrical methods often suggest the solution of algebraical or trigonometrical problems; for instance, the discussion on pp. 72–73 shows that the values of γ for which

$$\cos^{-1}\left(\frac{\sin\alpha+\sin\lambda\cdot\cos\gamma}{\cos\lambda\cdot\sin\gamma}\right) \mp \cos^{-1}\left(\frac{\sin\beta+\sin\lambda\cdot\cos\gamma}{\cos\lambda\cdot\sin\gamma}\right)$$

have their minimum values are

$$\cos^{-1}\left\{-\sin\lambda\cdot\left(\sin\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2}\right)^{\pm 1}\right\}.$$

Since many astronomical problems depend on the solution of spherical triangles, we shall conclude with a description of the method of using the net for the solution of such triangles[*].

(1) Given the three sides.

The procedure is the same as that employed in solving the 7th astronomical problem.

(2) Given the three angles.

The easiest method is to find the angles of the "polar triangle" whose sides are given, and thus deduce the sides of the original triangle.

(3) Given two sides and the included angle.

Trace through P, coinciding with A, the arc ADB and that one of the lines m which makes an angle with ADB equal to the given angle. Take PQ, PR along the two traced lines equal to the two given sides (fig. 2). Join QR by a circle which is the projection of a great circle, and measure the length QR and the angles PQR and PRQ (see p. 68).

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[*] See also Penfield, *loc. cit.* p. 115.

(4) Given two angles and the included side.

We may solve the polar triangle by (3), or else proceed thus. Trace through P, coinciding with A, the arc ADB, and that one of the lines m which makes an angle with ADB

Fig. 2.
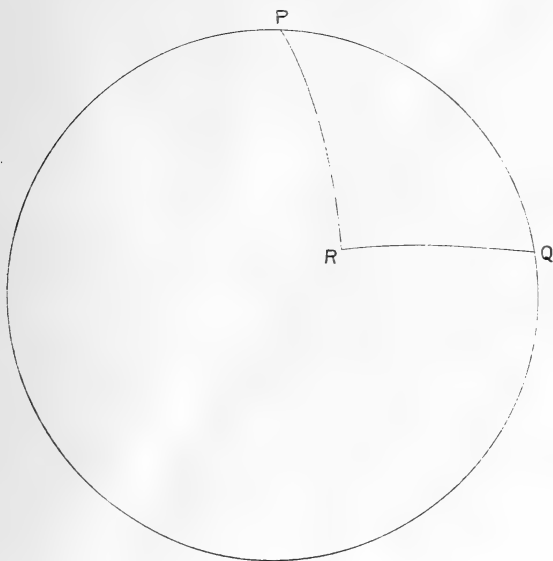
equal to one of the given angles. Take PQ along ADB equal to the given side. Turn the cloth in a counter-clockwise direction till Q coincides with A, and trace through that one of the lines m which makes an angle with ACB equal to the other given angle. Measure the triangle PQR thus obtained on the cloth.

(5) Given two sides and an angle not included.

Trace through P, coinciding with A, the arc ADB, and that one of the lines m which makes an angle with ADB equal to the given angle. Take PQ along ADB equal to one of the given sides. Turn the cloth till Q coincides with A, and find a point R (if any) in which that one of the lines l which is at a distance from A equal to the other given side cuts the line already on the cloth. Join QR and measure the triangle PQR.

The graphical solution shows at once when one solution, two solutions, or no solutions are possible (in general, evidently if QR lies between PQ and $\pi - PQ$ there is one solution, otherwise two or no solutions).

(6) Given two angles and a side not included.

This may be solved by applying (5) to the polar triangle, or we may proceed thus. Trace through the point P coinciding with A, and that one of the circles *m* which makes an angle with ADB equal to one of the given angles. On this arc take PR equal to the given side. Turn the cloth in the counter-clockwise direction until R lies on that one of the lines *m* which makes an angle with ACB equal to the other given angle. Now trace through Q coinciding with A and the arc PQ, and measure the triangle PQR.

A net is not absolutely essential for many of the constructions, but then it is necessary to draw that portion of the net which is needed for the question under discussion. Many of the above problems could be solved still more rapidly by the use of two nets of the same size, one of which is drawn on tracing-cloth or some other transparent material, and is pivoted so that it can turn freely about the common centre of the nets. This is the principle of Saxby's spherograph.

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X. *On the Absorption of Light by Mercury and its Vapour. By the* Hon. R. J. Strutt, *Fellow of Trinity College, Cambridge*[*].

IN a recent paper[†] I drew attention to the remarkable difference between the electrical properties of liquid mercury and of mercury vapour. It was found that at very high temperatures and pressures the vapour showed relatively considerable conductivity, and also distinct selective absorption of light. In the present paper I shall describe some rough experiments to compare the optical transparency of mercury vapour at the boiling-point of mercury with that of the solid (frozen) metal.

An iron gas-pipe, 3 metres in length, 2 cms. internal diameter, was closed at each end with mica diaphragms. The windows were fitted so as to be fairly air-tight by

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[*] Communicated by the Author.

[†] Phil. Mag. Nov. 1902. I may take this opportunity of recording an attempt to observe the critical phenomena of cadmium, on the same lines as those described in that paper for mercury and arsenic. A piece of clean cadmium was sealed up in a quartz tube, and the temperature raised until the tube began to yield viscously. Nothing could be seen in the part of the tube containing cadmium vapour. The vapour was quite colourless. When the tube softened, the vapour blew a hole in the tube wall, and escaped, oxidizing to a cloud of brown smoke. I cannot confirm the statement made in some chemical works to the effect that the vapour of metallic cadmium is brown.

squeezing them between iron flanges. Near one of the windows, a short side tube of iron was screwed in, at right angles to the main tube. This side tube was prolonged in a thin glass one. Under the main tube were a row of bunsen-burners, and bricks were arranged to keep in the heat from them. A small quantity of mercury was placed in the tube, which latter was kept slightly sloped, so that the mercury ran down to the end furthest from the side tube. When the tube was heated sufficiently the mercury vapour drove out the air, and rose into the vertically placed side tube. It was there condensed, and fell back into the hot tube. Thus the space between the mica windows was filled with mercury vapour. The windows were kept hot to prevent condensation of mercury upon them. In order to detect the absorption of light by the column of vapour, the light from a paraffin lamp, diffused by means of ground glass, could be observed through the tube. A comparison light was arranged by the side of the tube, the illumination being derived from the same lamp, by a reflector. The light which had come through the mercury vapour could be observed simultaneously with the light which had come from the same lamp through the air.

When the tube was cold the two lights were adjusted to equal brightness, by suitably inclining the reflectors. The tube was next heated so as to fill it with mercury vapour. It was then found that the two lights were, as before, equal in brightness, so that no absorption by 3 metres of saturated mercury vapour, at the boiling-point of mercury, could be detected. We may consider the experiment accurate to, perhaps, 10 per cent. It may be concluded that the coefficient of absorption for the light of a paraffin lamp cannot exceed ·0003.

To determine the absorption by mercury in the solid form, a very thin film must of course be used. In order to get this, I have made use of Prof. Dewar's observation that mercury can be deposited on glass as a film by condensation from the vapour, using liquid air as a cooling agent[*]. A glass bulb of 630 c. c. capacity contained mercury vapour, saturated at 20° C. Inside this, and concentric with it, was a very small bulb, into which liquid air could be poured. When this was done the mercury vapour condensed as a transparent film on the outer surface of the internal bulb. This film had an area of 11 sq. cms.

The vapour-tension of mercury at 20° C. is ·0013 mm. If we compute the density by assuming the simple law of gases to hold good for the saturated vapour (and probably this is

[*] I must express my best thanks to Prof. Dewar for giving me the opportunity of carrying out this experiment.

near enough), the vessel of 630 c. c. holds $\cdot 9 \times 10^{-5}$ grs. of mercury. This is spread into a film of 11 sq. cms. Thus the thickness of the film is 6×10^{-8}. It was estimated that this film transmitted about $\frac{1}{5}$ of the light of the paraffin lamp. The transmitted light had a neutral tint.

The coefficient of absorption of solid mercury calculated from these data comes out 3×10^7.

In order to make a fair comparison between the vapour and the solid, the greater density of the latter must be allowed for. By dividing the coefficients of absorption by the respective densities, we get numbers by which the absorptions of equal masses of material in the two states can be compared.

For the vapour the $\dfrac{\text{coefficient of absorption}}{\text{density}}$ is less than $\cdot 08$.

For the solid, this constant has a value of something like 2×10^6.

Thus, if we suppose the intensity of a beam of light to be diminished in a given ratio by passing through a film of solid mercury, then a stratum of mercury vapour containing not less (and perhaps much more) than 2×10^7 times the mass of mercury per sq. cm. would be required to produce the same effect.

XI. *Note on a New Form of Vacuum Stopcock.* *By* S. R. MILNER, *D.Sc., Lecturer in Physics, University College, Sheffield* *.

AN obvious defect of the ordinary form of mercury-sealed stopcock is that with it, although the interior of the apparatus to which it is connected is sealed off by mercury from the atmosphere, there is no mercury sealing between the two portions of the apparatus joined by the stopcock; yet this is, in the usual form of stopcock, the route by which a leak is most likely to take place. The only way of removing this defect with which I am acquainted is the somewhat inconvenient one of using a movable reservoir by which mercury can be let into or out of one of the connecting-tubes to the stopcock. To obviate the necessity for this addition in a case where special airtightness of the tap was desired, I have recently made use of a form of stopcock which interposes a mercury sealing to the entrance of air into the apparatus by *every* route along which a leak might take place.

A reference to the figure, which shows a section of the barrel x and the plug y of the stopcock in position, will make its action clear without much description. The barrel is inclined to the horizontal at an angle of about 45°, and, in the position of the plug figured, a through connexion exists between the tubes A and B. When the plug is turned through 180°, the separate portions of mercury m_1, m_2, m_3 seal off B both from the atmosphere and from A. C is a
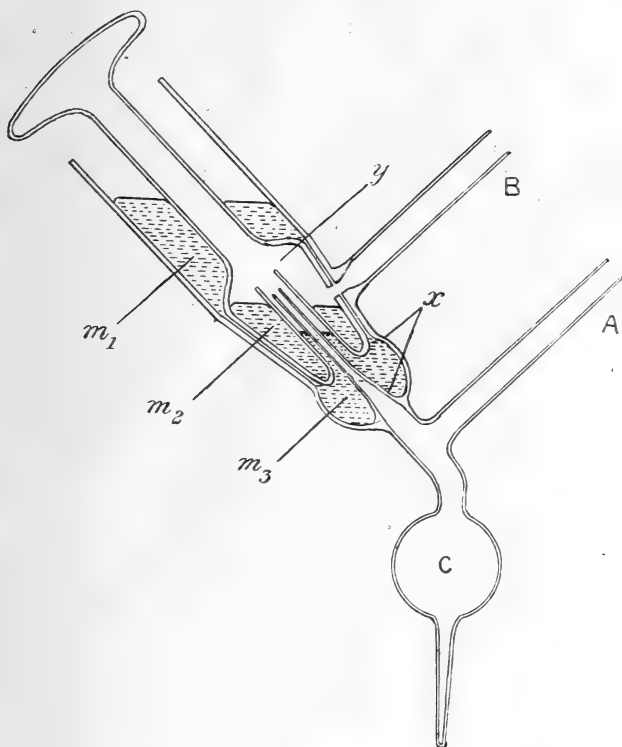
trap to catch any excess of the mercury m_3. It may be worth while to note that the stopcock should be ground fairly conical in shape, as, if there is a vacuum in A, the plug is forced into the barrel by the whole atmospheric pressure ; also, that there is a true mercury sealing only in the direction from A to B, so that the apparatus into which it is desired to prevent leakage should be connected to this latter tube.

To get some idea of the extent to which a stopcock of this

form would prevent leakage, I evacuated a tube onto which the stopcock was sealed, and, after allowing a couple of days for air condensed on the glass to escape, sealed it off from the pump at a stage at which the Crookes dark space just extended to the end of the tube (20 cms.), and a bright green fluorescence of the glass was obtained throughout. After a week's exposure to leakage from the atmosphere through the stopcock, I could detect only a trace of blue coloration near the further end of the tube, indicating a diminution of two or three centimetres only in the length of the dark space. It is possible that some at any rate of the effect was due to further escape of air from the glass (which had not been heated during the evacuation), but in any case it is evident that the leakage through the stopcock was extremely minute. When the mercury m_1 was removed, the vacuum fell off at the rate of several cms. per day.

XII. *On the Positive Ionization produced by Hot Platinum in Air at Low Pressures.* By O. W. RICHARDSON, *B.A., B.Sc., Fellow of Trinity College, Cambridge* *.

[Plate V.]

THE experiments to be described were originally undertaken in order to examine the connexion between the leak from a hot wire, when charged positively and surrounded by air, at a low pressure, and the potential and temperature of the wire respectively.

The arrangement of apparatus at first used was similar to that employed by the author in investigating the negative leak from hot conductors †. The wire, which was in the form of a spiral, was supported along the axis of a cylindrical electrode from which it was insulated. The spiral was of pure platinum wire ·1 mm. thick, and was heated by an electric current, its temperature being determined by means of its resistance. It was found that the absolute temperature could easily be kept constant to one part in a thousand, so that it did not vary more than one degree at the highest temperature investigated. The vacuum-tube containing the spiral and electrode was connected with a Töpler pump and McLeod gauge, so that the gas could be kept at any desired pressure.

The leak was measured by means of a quadrant electrometer. The wire was charged to any desired positive

potential, whilst the surrounding cylindrical electrode was connected to one of the quadrants of the electrometer, the other being earthed. In general the experiments were made at such a temperature that the currents could conveniently be measured without adding any external capacity to that of the electrometer. The temperature was always below that at which the negative leak became detectable.

The first experiments were made to determine the way in which the current from the wire varied with the applied potential, the other conditions being maintained constant. The temperature was 442° C. The pressure remained constant, and equal to ·00625 mm. of mercury. Observations were taken first with no potential on the wire, and then for every additional 40 volts up to 400 ; the values of the current *, in scale-divisions per minute, obtained in this manner are marked thus, ×, on the accompanying diagram (Pl. V. fig. 1). The observations were then repeated, the potential being continuously decreased by 40 volts down to zero; the values obtained in this way with decreasing potential-differences are marked thus, ○.

It will be seen that the ascending observations give rise to a curve which is slightly convex to the axis of current; on the other hand, the curve which represents the descending observations possesses about an equal curvature, but is concave to the current-axis. The disparity of the two series of observations is due to the falling-off of the current from the wire with time, a point which is examined in detail later. It will be seen that the time-effect can be allowed for by taking the mean of the two observations belonging to the same potential, since this would be—approximately at any rate—the current for that potential at a time half-way between the two experiments. This time is the same for every two observations, and is identical with that at which the reading for 400 volts was taken. This reasoning assumes that every observation takes the same length of time, a condition which was approximately fulfilled.

It will be seen that the current-E.M.F. curve is almost a straight line passing through the origin, so that the current is nearly proportional to the electromotive force. A repetition of the observations gave the same approximately linear relation, and showed that the fact that the mean point for 40 volts fell off the curve was due to accident, for it did not occur again.

Some experiments on the relation between the current and

* Throughout this paper 1 scale-division is equal to ·02 electrostatic units of quantity.

the E.M.F. were also made in hydrogen at a pressure of
·6 mm. In this case the apparatus was slightly different, the
spiral being parallel to, and about 1·5 cm. distant from, a flat
circular electrode to which the leak was measured. The
temperature was 692° C. The relation between the current
in scale-divisions per half minute and the potential-difference
in volts is given in the following table :—

Voltage.	Current.
40	550
80	1300
120	2600
160	3650
200	5150

It will be seen that the relation between current and P.D.
here is not linear, but that the current increases more rapidly
than the voltage with high voltages.

Further experiments were next made with the object of
testing the explanation of the discrepancy between the
ascending and descending curves shown in fig. 1. With this
aim the way in which the current under a constant voltage
varied with time was observed. The temperature was
maintained at 467° C. ; the pressure being ·00625 mm.
Throughout the observations a potential of +40 volts relative
to the cylindrical electrode was maintained on the wire. As
in the previous experiments, there was no leak when the wire
was put to earth or when it was charged negatively.

Corresponding values of the current in scale-divisions per
minute, and of the time in minutes from the commencement
of the experiments, have been plotted on squared paper (Pl. V.
fig. 2). It will be seen that the observational points fall
very approximately on the curved line shown. The form of
the curve shows that the rate of decay of the current is great
at first, but after about an hour almost vanishes. The leak
then becomes constant, and approximately equal to 13 scale-
divisions per minute. For brevity we shall call the final
value of the current the " steady leak " ; the difference
between the value of the current at any time and the steady
leak may be called the " induced " leak. The propriety of
using the term induced in this connexion will be demonstrated
later.

We can explain these results if we suppose the induced
leak to be due to some substance which gradually decomposes

under the influence of heat into at least one positive ion and possibly other products. The reaction is supposed to be monomolecular, and may be represented by the equation

$$X = nI + Y,$$
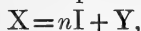
where X represents the unknown substance, n is the number of positive ions formed from each molecule of X, and Y represents the other products if they exist.

The laws of chemical dynamics then give at once

$$-\frac{dX}{dt} = kX,$$

where k is the velocity of the reaction ; so that

$$X = A_1 e^{-kt}.$$

But if the induced leak be denoted by y,

$$y = n\frac{dI}{dt} = -\frac{dX}{dt} = kX = kA_1 e^{-kt} = Ae^{-kt},$$

so that

$$\log y = B - kt.$$

(A_1, A, and B are different constants.) Evidently on this view, if we plot the logarithm of the difference between the actual current and the steady leak with the time, a straight line ought to be obtained.

The observations have been reduced in this way in fig. 3, and it will be seen that all the points fall very nearly on the straight line drawn. The irregularities are due to the fact that the wire could not be treated in exactly the same way between each two observations, and it is probable that the rate of decay depends not only on the time which has elapsed, but also on some other circumstances.

A series of experiments was now made with a new tube in which all the metal parts were of platinum, so that the whole could be carefully cleaned out by boiling in nitric acid and washing with distilled water. After this treatment, it was found that the wire gave quite as considerable a leak as one which had not been cleaned, so that the positive ionization is not due to superficial impurities soluble in nitric acid.

With this wire an attempt was made to see if the rate of decay of the variable leak depended on the temperature to which the wire was heated. The method used was to maintain the wire at a constant temperature, and take readings of the current every eight minutes. The wire was then heated to a somewhat higher temperature and the process repeated. The following table gives the results in the order in which the experiments were made. Owing to the magnitude of the

leaks capacities up to 1 microfarad had to be connected to the quadrants of the electrometer to which the leak was measured. The lengths of time during which the wire had been maintained at the temperature stated are given as well as the actual time at which the observations were taken; the currents are in scale-divisions per half minute. During the time which elapsed between any set of readings at a given temperature and the next at a higher temperature, the temperature of the wire was intermediate between the two considered.

Temperature θ.	Time wire at θ.	Actual Time.		Current.
	Minutes.	Hours.	Minutes.	
515° C.	0	1	37	15,000
	8	1	45	12,000
	16	1	53	7,000
	24	2	1	4.000
577° C.	0	2	20	31,000
	8	2	28	36,500
	16	2	36	31,000
	24	2	44	15,500
632° C.	0	3	2	280,000
	8	3	10	58,000
	16	3	18	28,000
697° C.	0	3	24	250,000
	8	3	32	64,500
751° C.	0	3	36	350,000
	8	3	44	68,500
809° C.	0	3	48	250,000
	8	3	56	53,500
	16	4	4	23,000
751° C.	0	4	8	5,000
	8	4	16	5,500
697° C.	0	4	20	950
	36	4	56	725
632° C.	0	5	1	100
577° C.	0	5	6	46
515° C.	0	5	10	41
	4	5	14	41

These results seem to indicate at first sight that at temperatures greater than 600° C. the rate of decay of the variable leak is almost constant and considerably greater

than at temperatures lower than this. The conditions at each temperature are not exactly comparable, so that this result cannot be regarded as satisfactorily established. The author hopes to make further experiments on this point in the near future.

The results do, however, clearly show not only that the leak at a given temperature is much smaller if the wire has previously been heated to a higher temperature, but also that the rate of decay is much less than before. For instance, on first heating the wire to 751° C. it gave a leak of 360,000 divs. per half minute, which fell to 68,500 in 8 minutes; whereas after heating for about 20 minutes to 809° C. the leak at 751° C. had become 5000 divs. per half minute, and suffered no appreciable diminution on heating for 8 minutes longer at the same temperature.

It is evident from these experiments that to get rid of the part of the leak which dies away rapidly, it is only necessary to heat the wire for some time at a temperature somewhat higher than that at which it is to be used subsequently. It was found, however, that the steady leak thus obtained also generally died away in time, only much more slowly; so that the distinction, at low temperatures and pressures at any rate, is one of degree rather than kind. The complexity of the phenomena generally, and especially of the rate of decay of the leak, seem to indicate that there are several distinct substances present in the metal which are capable of giving rise to positive ions.

The fact that the positive ionization from hot wires behaves as if it were due to some substance evaporating from the wire led the author to examine whether a negative electrode which had been kept near a hot wire charged positively became radioactive. The hot wire was maintained at a temperature of about 737 degrees centigrade for one hour, and was charged to $+200$ volts, the cylindrical electrode being earthed. The pressure of the gas remained constant and $= \cdot 05$ mm. During the whole of this time there was a steady current from the positive wire to the cylinder; so that it was thought that if the substance which disappeared from the hot wire were of the nature of an emanation it would be collected on the negatively-charged cylinder.

At the end of an hour the heating of the wire was discontinued and air was let into the apparatus. Experiments were then made to see if there was any leak from the cylinder to the cold wire. The results were purely negative; the current was at any rate less than 1/500 of an electrostatic unit. The experiments therefore lead to the conclusion

that a negative electrode does not become radioactive by receiving the positive discharge from hot platinum.

Further experiments showed that at low enough pressures, even at a red heat, the positive leak could be completely stopped by heating the wire continuously. It is therefore quite evident that the positive leak at low pressures cannot be regarded as a definite function of the temperature of the wire. At pressures of 1 mm. or higher, however, results were obtained which indicated that there was a leak due to the surrounding gas which increased with the temperature and pressure and did not fall away with time.

The fact that the positive leak decreased with time and vanished at low enough pressures led to the view that it was due to some foreign matter present in the wire, and that this gave rise to positive ions which were carried away by the current. On this view the positive ions would probably be carried to the negative electrode and condense there. If this were so the negative electrode would acquire the power, which the hot wire lost, of discharging positive electricity when heated. A new apparatus was therefore constructed in order to test this point.

Experiments with Two Hot Wires.

The new arrangement (Pl. V. fig. 4) consisted of two spiral electrodes (AB, CD) of fine platinum wire placed parallel to one another and about a centimetre apart. They were supported on copper rods (AA_1 &c.) which were carefully insulated with sealing-wax. By maintaining a suitable difference of potential across $A_1 B_1$ and $C_1 D_1$ the spirals could be kept at any desired temperature. A_1 was also connected to a battery so that its potential could be raised to ± 400 volts, whilst $C_1 D_1$ was connected to one quadrant of the electrometer, the other being earthed. The various sealing-wax joints are indicated by shading in the diagram. All the connexions of the electrometer were carefully protected from electrostatic induction by earthed conductors.

The method of experimenting consisted in heating spiral 1 until it gave no sensible leak with $+200$ volts at a given temperature θ. The heating current in spiral 1 was then stopped whilst spiral 2 was maintained at a high temperature for a definite interval of time. During this time spiral 2 was, of course, earthed whilst spiral 1 was kept at -200 volts. The current in the second spiral was then stopped, and after a short interval the temperature of the first spiral was raised to θ again, and the leak with $+200$ volts on the first spiral again measured. It may be permissible to state

in anticipation that in general this procedure was found to restore to the first spiral the power it originally possessed of discharging positive electrification when raised to the temperature θ.

The first experiments were made at pressures of about ·001 mm. At pressures so low as this the temperature of a wire heated by a current depends only on the strength of the current, and is independent of the pressure of the gas. In order therefore to make sure of having the wire at a constant temperature it was only necessary to observe the current through it and not to measure its resistance.

In a series of observations made on July 29th the heating current was kept constant and = ·241 ampere. The following values for the current were obtained at the times indicated. The current is in scale-divisions of the electrometer per minute. The pressures are also given (in mms.).

Current.	Time.		Pressure.
	hours.	minutes.	
46	12	34	·0008
43	12	40	·0008
30	12	47	·0012
22	12	59	·0009
21	1	7	·0008
17	1	15	·0009
16	1	23	·0005
15	1	31	·0008
13	1	40	·0007

The apparatus was now left till next day to see if the leaking power of the wire would be revived. The following observations were taken :—

Current.	Time.		Pressure.
	hours.	minutes.	
12	11	36	·0005
5	11	41	·0005
0	11	50	·0008
1	11	58	·0006
4	12	49	·C005

It is evident from these numbers that the wire is not revived to any great extent by being left exposed to air at a pressure of ·001 mm.

The wire was now allowed to cool and charged to −200

volts, whilst the second wire, which was earthed, was kept at a red heat for 15 minutes. On allowing everything to cool and stand for a short time, and again heating the first wire with ·241 ampere, the following values of the current were obtained with +200 volts (the units of current are the same as before) :—

Current (+ leak).	Heating current (amps.).
2250	·241
2810	·241
91	·224
136	·237

The activity induced by the second hot wire is in this case enormous compared with the greatest current (46) due to the activity which the hot wire originally possessed.

The above readings for the current were taken at intervals of about 6 minutes, so that it is evident that the value of the induced positive leak falls off very rapidly with the time when the wire is heated. In other words the induced activity is almost all driven off in about half-an-hour, and in some cases even less.

This effect is not due to ions which remain suspended inside the bulb, or to any effect produced on the walls of the tube by heating, for absolutely no leak was obtained when the first wire was cold. It is also evident from the above experiments, and from others which will be described later, that the effect persists for a considerable length of time.

Each time the experiments were repeated the induced activity was found to become continuously smaller and higher temperatures had to be employed to show the effect. This is probably due to the substance which causes the effect becoming diffused onto the walls of the tube and the cold parts of the electrodes, since it goes hand in hand with a falling off in the conductivity produced when the first wire alone is heated to still higher temperatures.

Three experiments were now made to see if the effect could be produced by merely charging up the first wire to a suitable potential, the second being cold and put to earth. The wire was charged to +200, 0, and −200 volts respectively and left for 40 minutes each time ; but the conductivity was again found to be zero on heating to the original temperature.

In another case the first wire was maintained at $+200$ volts while the second was heated for 5 minutes, and it was found that the induced activity, though definite, was smaller than when the first wire had been charged negatively.

Measurements were now made to compare the activity induced in these two cases, *i. e.*, when the first wire was positive and when it was negative with regard to the second, which was heated.   In order that any two experiments may give results which are quantitatively comparable it is necessary that the state of the second (hot) wire should remain the same throughout the experiments.   To ensure this the second wire was only heated for 120 seconds each time. An experiment was first made with the first wire at -200 volts, then with the wire at $+200$ volts, and so on.   The fact that the activity induced in the first and third experiments (with the first wire negative) was approximately the same shows that the state of the second wire had not greatly altered.   The induced activity here was small in all cases owing to the short time during which the second wire was heated.

The following table gives the actual numbers which were obtained.   The current which was used to heat the first wire was ·286 ampere ; in all cases this current produced no leak before the second wire was treated in the manner described above.   In each case in measuring the leak $+200$ volts was put on the first wire.   The pressure was ·001 mm.

Potential of First Wire when Second Heated.	Activity Induced in First Wire.
$+200$ volts.	5
-200 „	26
$+200$ „	11
-200 „	20

These results lead to the conclusion that the activity induced when the first wire is negative is about three times as great as when the first wire is positive.   A second series of experiments, which are more satisfactory in so far as they were taken when the second wire was in a more active state, and the effect observed was therefore much greater compared with incidental uncertainties, show that the disparity between the two cases is still greater.   In this case the pressure was ·067 mm.   One observation was taken with the first wire earthed when the second was being heated, in

addition to the cases in the previous experiments.   The results are shown in the following table :—

Potential of First Wire while Second Heated.	Activity Induced in First Wire.
-200 volts.	134
$+200$,,	27
-200 ,,	137
0 ,,	15

It will be noticed that the two numbers obtained with -200 volts only differ by two per cent.   No importance is to be attached to the difference between the values (27 and 15) obtained with $+200$ and 0 volts respectively.

The numbers on page 88 show that the activity induced by heating the second wire falls off fairly rapidly with time when the first wire is heated.   There is, however, no evidence to show that the induced activity falls off with time if the first wire is allowed to remain cold.   A definite experiment was made to test this point by allowing the first wire to remain cold for 75 minutes after the second had been heated. On now raising the temperature of the first wire to that at which it previously gave no leak with $+200$ volts, 45 divisions per minute were obtained.   This current was practically equal to that which would have been obtained if only two or three minutes had been allowed to elapse.

The next point that was investigated was to see if a wire which had lost the power of discharging positive electrification could not be revived by other methods than by charging it negatively in the neighbourhood of a second hot wire.   Of these methods the effect of air will be considered first.

The Activity induced by Air.

To test this advantage was taken of a small leak in the apparatus which caused the pressure to increase by about $\frac{1}{10}$ mm. in twelve hours.   The first wire was heated on the afternoon of August 7th by ·280 ampere, and the leak with $+200$ volts was found to be 1 division per minute at ·0023 mm. pressure.   On the following morning the pressure had risen to ·16 mm., and the leak under the same conditions as before was found to be 45 divisions per minute.   The leak induced by air in this way was found to fall off very rapidly with the time.   This is readily seen from fig. 5 (Pl. V.), where the induced

activity in scale-divisions per minute has been plotted against
the time the wire has been heated (in minutes). It will be
seen that there is practically nothing left after 5 minutes'
heating.

Another case where ·08 mm. of air was let into the appa-
ratus suddenly gave a higher rate of leak, viz. 382 divs.
per minute in the units which have so far been employed in
this paper. The actual readings and the times are given in
the following table :—

+ ve Leak ∝ .	Time.	
	min.	sec.
154	0	15
78	0	54
32	1	15
18	1	45
17	2	30

Here again it is evident that the induced activity is not very
persistent.

The apparatus was now taken down and the wire allowed
to remain exposed to air at atmospheric pressure for several
days. On reducing the pressure to ·001 mm. and keeping
the wire at + 200 volts a leak of 160 divs. per minute was
obtained even when the wire was only heated with ·228 amp.
The heating current was now raised to ·240 ampere, and the
following values of the current were obtained at the times
stated. The pressure was ·0007 mm.

+ ve Leak ∝ .	Time.	
	min.	sec.
255	0	10
19	4	15
4	8	10
2	13	10
·7	23	0

In this case also there is practically no effect left after
heating the wire to a given high temperature for some five
or six minutes.

Activity Induced by Hot Wires at Higher Pressures.

Practically all the preceding experiments with the two-wire system were made at pressures of about a thousandth of a millimetre. In order to see if the effect was obtained at higher pressures a glass tap was added to the apparatus so that any desired quantity of gas could be introduced. The induced activity was very marked at ·38 mm. ; since heating the second wire for 2 mms., the first being kept at −200 volts, produced an initial current of 348 scale-divisions per minute. This induced activity was found to die away much more slowly than that produced by air and measured at a lower pressure, as the following numbers indicate :—

+ ve Leak α .	Time.	
	min.	sec.
173		0
145	3	15
115	6	30
91	9	0
70	12	0
58	15	0
34	24	0
22	32	0
12·5	45	0
8	70	0
3·5	130	0

The logarithms of the values of the induced leak, obtained by subtracting three from the numbers in the above table, are plotted against the time in fig. 6. It will be observed that all the points fall very nearly in a straight line, showing that the transferred activity follows the same law of decay as the original activity of the wire.

The high values of the induced activity given above were now obtained every time. A repetition of the experiment, in which the second wire was only heated for one minute, gave an initial leak of 200 divisions in 15 seconds.

The pressure was now reduced to ·055 mm., and the experiments repeated. Heating the second wire for thirty seconds was now found to give an induced activity of 332 divisions per minute. The way in which the current fell off with the time is shown in the following table :—

+ve Leak ∝ .	Time.
	min. sec.
154	0 15
18	4 0
9	8 0
6	11 15
4	22 0

It appears from these numbers that the rate of falling off is greater than at higher pressures. The difference may, however, be due to a difference in the temperature of the wire in the two cases.

Activity induced by the Luminous Discharge.

In making experiments to see if the induced activity was increased by increasing the potential-difference between the two wires, it was found that an extraordinarily big effect was obtained with −400 volts on the first wire. When the experiment was repeated it was found that an ordinary vacuum-tube discharge had been passing between the two electrodes. Further experiments were therefore made with both electrodes cold in order to see if the discharge produced the effect even when the second spiral was not heated.

The first wire was heated in the usual way until it gave no leak with +200 volts and a heating-current of ·35 ampere. It was then allowed to cool, and raised to a potential of −400 volts, the second wire being earthed. This difference of potential was sufficient to produce a discharge which was allowed to pass between the two electrodes for 60 seconds. After waiting a few minutes the first wire was heated to the same temperature as that at which it had been tested before, and the leak was measured with + 200 volts on the first wire. An induced activity of 500 divisions in 30 seconds was observed.

It was evident, as in the former case, that this effect was not due to ions left in the gas by the discharge, since there was no leakage of electricity from the first wire until it was heated.

An experiment was next made to see if the induced activity would persist for 40 minutes if the wire was kept cold after passing the discharge. As before, the wire was tested and found not to leak with ·35 ampere of heating-current. It was then cooled and the discharge passed for 60 seconds.

After waiting 40 minutes the wire was heated with ·35 ampere, and charged to + 200 volts as before. The induced activity was now found to be somewhat over 500 divisions in 40 seconds, and was therefore, within experimental errors, equal to that found when the wire was tested immediately. The activity induced by the discharge therefore falls off very slowly, if at all, by simply keeping the wire cold.

In all the preceding experiments, the wire which had been made active was the negative electrode. Experiments were now made to see if any activity was induced in this wire when the discharge was passed in the opposite direction, and if so what the relative magnitude of the induced activity was in the two cases.

The results show that the activity induced on the positive electrode is only about one-fifth of that induced on the negative, but that nevertheless it is quite marked and definite.

Electrode on which Activity Induced.	Induced Activity.
+	176
−	1200
+	260

It is to be noted that the ratio of the effects in the two directions is practically equal to that found for the corresponding effects produced by the discharge from hot wires and given on pp. 87 and 88. It seems probable therefore that the two phenomena are intimately connected.

The fact that activity could be induced both by the positive leak from hot metals and by the ordinary discharge led to the view that it was due to positive ions which stuck to the electrode and formed a double layer. In this case the same effect ought to be produced whenever positive ions discharged to the surface of the metal.

With the view of testing this supposition careful experiments were made to see if collecting positive X-ray ions on a wire increased the leak from it at a given temperature at atmospheric pressure ; but no effect was obtained. A similar result was got at a low pressure. It was thought that these negative results might be due to the X-rays themselves possibly having the power to destroy the induced activity. This was the more probable, since X-rays are well-known to have a very intense action on metallic surfaces, resulting in

the production of secondary rays; but on exposing a wire, which had been made strongly active, to the rays, no diminution of the induced leak was obtained.

Conclusion.

The experiments which have been described show that the positive leak from a hot platinum wire at low pressures is not a definite function of the temperature, but that it gradually decays with time. The complexity of the phenomena seem to indicate that in general the ionization is produced by more than one substance ; but where one effect is predominant the rate of decay of the leak is proportional to its value at the time considered. At a constant temperature the induced leak thus falls away as a negative exponential function of the time.

Further, a wire which has been heated at a given temperature until it has lost the power of discharging positive electricity, may have that property restored to it by any of three agencies. These are :—(1) Exposure to air ; (2) exposure to the positive discharge from a second hot wire ; and (3) making it an electrode during the passage of a luminous discharge.

The first and third methods lead to the conclusion that the effect is due to gas. This gas must, however, be in a peculiar state since the experiments show that none of it comes off the wire negatively charged. In the case of the second and third methods the induced activity is much greater when the wire is made the negative electrode. This might indicate that it is due to the positive ions ; but this supposition is rendered untenable by the fact that Röntgen-ray ions produce no effect. The fact that the discharge induces more activity on the negative than on the positive electrode leads to the conclusion that there are more gas molecules in the state necessary to produce the effect in the neighbourhood of the negative than in that of the positive electrode.

These effects have been shown to happen between ·001 and ·3 mm. pressure, but there is no reason to believe that similar phenomena would not occur at higher pressures.

When a " revived " wire is heated at a constant temperature the induced activity is gradually dissipated. In every case the activity fell off in such a manner as it would if the rate of decrease were proportional to the activity momentarily present. In fact the phenomenon is exactly analogous to a monomolecular chemical reaction. The quantity which corresponds to the velocity of the reaction is probably a function of the pressure of the gas and of the temperature of the wire.

In air at atmospheric pressure the falling off of the conductivity with time is much slower than at low pressures. Thus Mr. H. A. Wilson* found the following numbers for the current in amperes between two coaxial platinum cylinders at the temperatures stated. The potential-difference was 800 volts and the outer tube was negative.

Temperature 800°.	900°.	1000°.	1100°.
July 6......... 8 ×10⁻⁶	40 ×10⁻⁶	120 ×10⁻⁶	400×10⁻⁶
July 10......... 2 × ,,	11 × ,,	50 × ,,	140× ,,
July 30......... ·1× ,,	·7× ,,	2·5× ,,	8× ,,

The greater persistence of the positive leak at high pressures is probably to be attributed either to the greater difficulty experienced by the induced activity in diffusing away, or by the fact that a new active layer is continually being reformed on the wire. Which of these two views is correct is a matter for future experiment to decide. In the present experiments the induced activity appeared to be more persistent at ·3 mm. than at ·06 mm.

The activity induced by the luminous discharge, and several other points which are not definitely settled in this communication, are being further investigated at present.

The experiments were carried out in the Cavendish Laboratory, and the author's best thanks are due to Professor J. J. Thomson for his kindly interest and advice during the course of the work.

XIII. *The Anomalous Dispersion, Absorption, and Surface-Colour of Nitroso-dimethyl-aniline, with a Note on the Dispersion of Toluene.* By R. W. WOOD, *Professor of Experimental Physics in the Johns Hopkins University†.*

[Plates VI.–VIII.]

THE very high dispersion of the aniline dyes and other absorbing media is due, as is well known, to the fact that the absorption-band lies within the visible spectrum. The absorbing power of these substances, for wave-lengths far removed from the centre of the absorption-band, is so great that only prisms of very small angle can be used, which puts a limit on the length of the spectrum which can be obtained with them.

In the case of the so-called transparent substances, the absorption-band lies so far down in the ultra-violet that the steepness of the dispersion-curve in the visible spectrum is not comparable with that of the substances which are said to exhibit anomalous dispersion. If, however, we push the curve down to the absorption-band, employing some photographic method, we find that it may be even steeper than the curve for cyanine in the red and orange.

I have found that nitroso-dimethyl-aniline is of peculiar interest, in that it fills in the gap existing between the aniline dyes and ordinary transparent substances. It has a band of metallic absorption in the violet, and is at the same time fairly transparent to the red, yellow, and green. The substance melts at 85° C., and can be formed into prisms between small strips of thin plate-glass. The strips should be about 2 cms. long, and are best fastened together with one of the small clamps used with rubber tubing. It is best to melt the material on the end of one of the strips, the other being warmed over the same flame, and then clamp the two together with a piece of a match between the other ends, to give the required prismatic form. A candle-flame viewed through the prism is spread out into a most remarkable spectrum fifteen or twenty times as long as one given by a glass prism of the same angle. It is instructive to have a prism of the same angle made of Canada balsam or some such substance pressed out between two similar glass strips.

In addition to its remarkable dispersion, the nitroso, as I shall call it for short, exhibits a most beautiful violet surface-colour, which can best be exhibited by employing a small cell heated by steam such as I shall describe later on.

The substance possesses in addition several other interesting features. It has, for example, in addition to its sharp and narrow band of metallic absorption in the violet, a weaker pair of bands near the end of the ultra-violet which flatten out the dispersion-curve, but do not bend it into oppositely directed branches as the stronger band does. Moreover, the substance can be vaporized without decomposition, which makes it possible to compare its optical properties in the three states—solid, liquid, and gaseous.

I shall take up in order the dispersion in the visible spectrum, the ultra-violet dispersion, the reflecting power in different parts of the spectrum, the angles of maximum polarization, and the changes in the position of the absorption-band which accompany a change of state. The various results will finally be discussed in their bearing on the electro-magnetic theory of dispersion and absorption.

Dispersion in the Visible Spectrum.

As the refractive index of the nitroso changes very rapidly with the temperature, it was necessary, in making the determinations of the dispersion, to keep the temperature of the melted substance constant. The point chosen was the solidifying point, as it simplified the experimental conditions. The prism was constructed of a pair of interferometer plates accurately plane-parallel, and was mounted on the table of a spectrometer in a small clamp-frame made especially for it. A current of hot air was directed against the prism by means of a bent glass tube, under one end of which a small gas-flame was burning. The slit of the spectrometer was illuminated with approximately monochromatic light furnished by a monochromatic illuminator built by Fuess. This extremely useful instrument is not as well known as it deserves to be, and a few words regarding it may not be out of place. It is essentially a small spectroscope with collimator and telescope at right angles. The two prisms, which are inclosed in the body of the instrument, are turned by means of a micrometer screw, from the reading of which the wave-length can at once be determined from the calibration-curve of the instrument. The eyepiece can be removed, and a draw-tube carrying an adjustable slit inserted in its place. The spectrum can be made to pass across this slit by turning the micrometer screw, and by noting the readings when known lines in the spectrum fall on the slit, the instrument can be calibrated. This is easily and quickly accomplished by means of an ingeniously arranged microscope which can be thrown into and out of position as desired. A small lens carried in a tube in front of the slit focusses the monochromatic light which issues from it on the slit of the spectrometer.

The dispersion of the nitroso was measured in the following manner :—The prism having been set at minimum deviation, and the usual adjustments made, the deviated image of the slit, illuminated in monochromatic light, was brought into the field of the telescope. The gas-flame was then moved out of position and the current of hot air stopped. As the fluid nitroso cooled, the deviation increased, the reading being taken just at the point of solidification, when the image disappeared gradually owing to the crystallization of the medium. The warm air was then turned on again, and a second reading taken in the same way. A number of prisms were used, the angles varying from 1 to 10 degrees, those of small angle being necessary when working with the bluish-green, owing to the absorption. In this way the dispersion was

measured between the extreme red and wave-length ·00050, below which point the prims refused to transmit sufficient light to make readings possible. The values obtained in this way are given in the following table.

Prism-angle 8° 7'.		Prism-angle 53'.	
λ.	*n.*	λ.	*n.*
508	2·025	497	2·140
516	1·985	500	2·114
525	1·945	506	2·074
536	1·909	513	2·020
546	1·879	577	1·826
557	1·857	647	1·754
569	1·834	669	1·743
584	1·815	696	1·723
602	1·796	713	1·718
611	1·783	730	1·713
620	1·778	749	1·709
626	1·769	763	1·697
636	1·764		
647	1·758		
659	1·750		
669	1·743		

The results are shown graphically in Plate VI., together with the dispersion-curve for bisulphide of carbon for the same region of the spectrum. The remarkable dispersive power of the nitroso is at once apparent when we compare the two curves, and contrast it with that of the bisulphide, which has the highest dispersive power of any substance in common use.

Over the region of the spectrum given above, the nitroso can be considered as a transparent substance, and the dispersion formula for transparent substances can be applied to it. From three values of n and the corresponding values of λ the value of λ′ the centre of the absorption-band can be calculated.

In the case of selenium, which has a dispersion-curve resembling that of the nitroso, the absorption appears to increase steadily from the yellow down to the extreme ultra-violet, making it impossible to determine experimentally the centre of the absorption-band which is chiefly responsible for the dispersion. Applying the formula for transparent substances to the values found for selenium, I found that if we assume a single absorption-band, its centre must be at wave-length ·00056. To account for the continued absorption as

H 2

we pass down the spectrum, we have only to assume that there is a series of bands of which the calculated one is the first member. I expected that the nitroso would behave in a similar manner, but found on examining its transmission that the absorption, which commenced at $\lambda = \cdot0005$ ended quite abruptly at about $\cdot00037$, the substance transmitting the ultra-violet almost down to the last cadmium lines. This property of the substance enabled me to prepare screens transparent only to ultra-violet light, which I have described in a previous paper (Phil. Mag. Feb. 1903).

The transparency of the nitroso on the ultra-violet side of the absorption-band is, however, very much less than on the green side, as I soon found in endeavouring to measure the dispersion by crossing a prism of the substance formed between quartz plates, with the prism of a quartz spectrograph. No trace of the spectrum on the more refrangible side of the absorption-band appeared on the photographic plate. After a number of failures, which obviously resulted from the insufficient transparency of the nitroso prism, combined with the necessarily short exposure, I abandoned this method, and made some rough determinations of the ultra-violet dispersion by observations on the angle of maximum polarization for these wave-lengths. The results, while not very accurate, gave unmistakable evidence of anomalous dispersion, the refractive index for wave-lengths below the absorption-band ranging from 1·1 to about 1·5. I was subsequently able to use the method of crossed prisms, by employing very acute prisms, kept warm by means of an electrically heated platinum wire, which made long exposures possible.

I shall first, however, consider the polarization phenomena accompanying the reflexion of light from the surface of the fluid nitroso. The substance was kept in a liquid condition by means of a small cell heated by a current of steam (fig. 1, p. 101).

On examining the light reflected at a fairly large angle with a Nicol prism, it will be found to vary from light blue to deep violet and purple, as the angle of incidence is increased, the nicol being held in such a position as to refuse transmission to the light polarized by reflexion. If a spectroscope is placed behind the nicol, a dark band will be seen crossing the spectrum, which shifts its position as the incidence angle varies. The centre of this dark band is evidently the wave-length for which the angle of incidence happens to be the angle of maximum polarization; or, in other words, the refractive index of the substance for this wave-length is the tangent of the angle of index. In the case of glass and

substances of low dispersion, the different colours are polarized at nearly the same angle, *i. e.* very little colour-effect is observed when the reflected light is examined with the nicol.

Fig. 1.
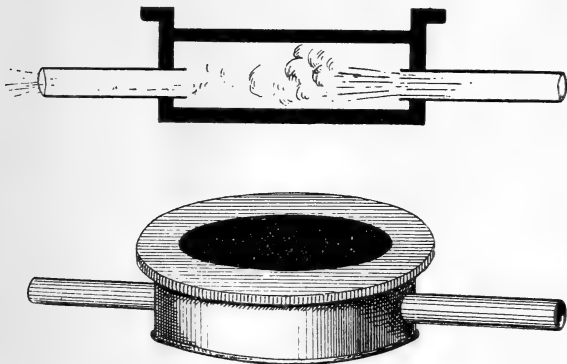
In these cases the dark band to which I have referred is so broad as to occupy practically the entire visible spectrum. In the case of a substance with as high a dispersion as that of the nitroso, the angle of maximum polarization is quite different for the different colours, so that the dispersion may be determined by observing the position of the centre of the dark band and the angle of incidence. It was found that values agreeing very closely with those obtained with the prisms could be obtained by this method in the yellow, green, and greenish-blue portions of the spectrum. On attempting to drive the band through the orange and into the red, it broadened so much that it was quite impossible to locate its centre with any precision. This was of course due to the fact that the dispersion in the red and orange is not sufficient to make the method very accurate in this part of the spectrum. The dispersion of selenium was also determined in this manner, and found to agree very well with the results obtained with prisms. To apply the method to the ultra-violet the observations were made photographically with a small quartz spectrograph made by Fuess. This instrument was furnished with a Rochon prism mounted immediately behind the quartz collimation-lens. This prism, as furnished by the maker of the instrument, refused to transmit the ultra-violet, and I found that the two halves had been cemented together with balsam, which I replaced with glycerine. Some preliminary experiments were made with selenium mirrors, as they were easier to handle than the fluid cell. The light from a cadmium

spark was reflected into the slit of the instrument at various angles of incidence, and dark bands were found running across the two polarized spectra furnished by the Rochon prism. I spent some time endeavouring to extract data from these bands, and finally came to the conclusion that either they were not due to the selective polarization by reflexion, or else that the dispersion-curve as determined for selenium by means of the interferometer was in error. On experimenting further, I found that these bands were due solely to the rotatory dispersion of the light in the quartz collimating-lens, some colours being rotated through such an angle as to be quenched in one spectrum by the Rochon, and others in such a degree as to be absent in the other. I mention this defect in the instrument, as it may be of interest to others working along similar lines. The proper design of the instrument should have called for a collimating-lens made of two thin lenses, one of right-handed, the other of left-handed quartz. I remedied the defect in my instrument by placing a plate of left-handed quartz immediately behind the lens. This plate had a thickness equal to the thickness of the lens at its centre, and abolished the bands entirely, when the lens was stopped down to a small area at the centre.

I found, however, in working with the horizontal cell of fluid nitroso, that better results were obtained by using a Nicol prism in front of the slit of the spectrograph than with the Rochon prism. The nicol was made transparent to ultra-violet light by separating the two halves, cleaning off the balsam, and substituting glycerine.

Evidence of the very low value of the refractive index on the ultra-violet side of the absorption-band was obtained before any polarization experiments were tried. The light of a cadmium spark was reflected from the pool of liquid nitroso at nearly normal incidence, and then thrown into the spectrograph by means of a quartz total-reflecting prism. A series of spectra was taken, with times of exposure varying from 2 seconds to 3 minutes. By comparison of the different spectra it was possible to form a rough estimate of the reflecting power in different parts of the spectrum. A similar series was made with a flat reflector of magnalium, which is remarkably constant in reflecting power throughout the entire spectrum. It was found that the nitroso, in the region of its absorption-band, reflected almost as strongly as magnalium, while just on the ultra-violet side of the band there was a gap in the spectrum, due to the very low reflecting power which the substance has for these waves (Plate VIII. fig. 8). The reflexion coefficient at this point was estimated at about

2 per cent. only, while a short distance further up the spectrum the coefficient is at least 80 per cent.* This indicates that the refractive index has a very low value at the point of minimum reflexion. The following table gives an idea of the reflecting power of the nitroso in the various regions of the spectrum as compared with the magnalium. The values are of course only approximate, being deduced by picking out two spectra (one from nitroso, the other from magnalium) which showed the same intensity at the given wave-length, and then comparing the times of exposure. The reflecting power is expressed as a fractional part of the reflecting power of magnalium.

λ.	Refl. Power.	λ.	Refl. Power.
217	1/2	380	1/2
240	1/2	399	1
290	1/3	467	1
330	1/6	508	1/2
343	1/12	535	1/4
350	1/40	569	

When the incidence angle is increased and the reflected light examined with a Nicol prism, a most beautiful surface-colour appeared, ranging from a brilliant blue through violet to a reddish purple. This is very easy to understand. The violet light within the region occupied by the absorption-band is metallically reflected at all angles of incidence, consequently it is always present in excess in the reflected light, and is never cut off by the nicol. As the angle of incidence is increased, the polarizing angle for red light is first reached, and the red end of the spectrum disappears, being cut off by the nicol, the reflected light appearing blue. As the angle increases more of the spectrum is removed by the nicol, and the colour changes to deep violet, the dark band advancing down the spectrum. Finally, by further increase the red appears again in full intensity, being reflected unpolarized, and mixing with the metallically reflected violet gives the brilliant reddish purple. Very similar appearances can be observed with selenium, though the colours are not as saturated and not nearly so brilliant.

Having roughed out, so to speak, the dispersion in the ultra-violet by photographing the spectrum of the reflected

* This value is much too high. It was based on the supposition that the magnalium reflected 80 per cent., which was subsequently found to be incorrect, the metal containing too little magnesium, and having a rather low reflecting power.

light at various angles of incidence through a Nicol prism, an attempt was next made to secure more reliable data by means of the method of crossed prisms. A prism of small angle, held in a clamp, was mounted with its refracting edge in a horizontal position, immediately behind the prism of the quartz spectrograph. A fine platinum wire immediately below the edge of the prism, when heated by a current of suitable strength, kept the nitroso in a fluid condition. A system of small screens was arranged so that the light could either be made to pass through the nitroso prism, or through a small clear space in the quartz plates immediately above it, thus photographing the deviated and undeviated spectra one above the other. The current strength was so adjusted as to keep the temperature of the prism as nearly as possible at the melting-point of the nitroso. It was possible in this way to give exposures of an hour or more, and obtain a photographic record of the dispersion-curve from the orange down to the extreme end of the ultra-violet. The cadmium spark was employed as a source of light, and the slit of the spectrograph was diaphragmed down to a length of less than a millimetre in order to obtain very narrow spectra. As the conditions necessary for success were determined by repeated experiments, prisms of larger angle were used, and the two spectra, which overlapped in the first experiments in the region where the refractive index had a low value, were completely separated. I found that a great improvement resulted from carefully grinding the edge of the quartz plate, which rests against the other plate, perfectly straight with fine emery. It was not until this expedient was adopted that the larger angles became possible. The deviated spectrum is quite sharp except close to the edge of the absorption-band, where absorption produces a broadening of the image by reducing the effective width of the beam of light, as is always the case with strongly absorbing prisms. It is even possible to follow the general trend of the dispersion-curve right through the band of metallic absorption, though the broadening resulting from diffraction is, in this region, so great that accurate measurements were impossible. The continuity of the curve is shown, however, better than I have ever seen it in any photograph. To deduce numerical values from the photographs, the distances between the spectra were measured with a dividing-engine at the principal cadmium lines. From these distances the actual angular deviations were calculated, making proper allowance for the fact that the plate stood at an angle—*i. e.,* the focal length of the lens of the spectrograph was very different in the different regions of the spectrum.

The angle of the prism was calculated from the deviations in the yellow and green, for which region the refractive index had been already determined with considerable accuracy. In spite of the rather crude method, the results obtained with the different prisms agreed surprisingly well.

It will be seen from the photographs, some of which are reproduced in fig. 3, Plate VIII., that the deviated and undeviated spectra come together at the ultra-violet edge of the principal absorption-band, indicating a refractive index not much above unity.

(*a*) deviated spectrum obtained with prism of very small angle; (*b*) (*c*) (*d*) deviated and undeviated spectra obtained with prisms of larger angle. The continuity of the curve through the absorption-band can be seen in (*a*) and (*b*), also in (*e*), which was enlarged from (*b*). The ultra-violet is to the left in the first four of these figures.

The prism-angles in these experiments varied from 20′ to 2°.

It was found possible to get a fairly accurate determination of the refractive index for wave-length 48 from these photographs, the mean of several determinations giving 2·28 as the value. The values found on the ultra-violet side of the band are given in the form of a curve in Plate VII. This portion of the curve is especially interesting, as the effect of the ultra-violet absorption-bands is most strikingly shown. The weak band at wave-length 25 causes a dip in the curve, which has a maximum at $\lambda = 28 \cdot 5$, after which it again rises rapidly as it approaches the heavier absorption-band situated not far from $\lambda = 20$.

On this same plate I have given the dispersion in the visible spectrum, together with the value $n = 8 \cdot 85$ for wave-length $\lambda = 43$ the centre of the absorption-band, which was determined from the constants of elliptical polarization. The trend of the curve within the absorption-band I have sketched in free-hand, as well as the general form and position of the absorption-bands.

I have already shown that the reflecting power of the nitroso for the wave-lengths immediately adjoining the absorption-band on the ultra-violet side is exceedingly small, which is not surprising when we consider that the refractive index for this region is not very different from air. If now we consider a quartz-nitroso surface instead of an air-nitroso surface, we should expect fairly strong reflexion at this point, the nitroso acting as the rarer medium, while further down the spectrum, at about wave-length 29, where the indices of the quartz and nitroso are very nearly the same, we should expect scarcely any reflexion at all.

This was found to be the case. An acute prism of quartz was ground and polished, and one face of it brought in contact with a small drop of nitroso fused on a strip of blackened brass. One end of the strip was heated in a flame, the nitroso remaining fluid as a result of the heat conduction along the strip. The object of using the blackened brass was to get rid of reflexion from the back surface of the nitroso. The light reflected from the front surface of the quartz was thrown to one side, owing to the inclination, so that everything was eliminated except the reflexion from the quartz-nitroso surface. The spectrum of this selectively reflected light is shown in fig. 5 (Plate VIII.). The minimum at wavelength 36 will be seen to have disappeared, and a new one will be found at 29 just about where we should expect it. This minimum would doubtless be more pronounced were it not for the fact that the absorption coefficient has at this point a not inconsiderable value, which will cause the bounding surface in question to have a higher reflecting power than if both media were perfectly transparent.

Absorption of Solid, Liquid, and Gaseous Nitroso.

No determinations of the dispersion of the solid nitroso have been made, owing to the difficulty of getting suitable prisms : when the fluid prisms cool off the nitroso crystallizes in aborescent forms, and no longer transmits regularly. If the prism is held close to the eye, and a lamp-flame viewed through it, enormously deviated spectra are seen, due to the formation of crystals having a much larger angle than the prism. The substance in the solid condition is double-refracting, the two spectra which every prism furnishes being extinguished in turn by a revolving Nicol prism. The absorption-band of the solid nitroso was studied by photography. A drop of fluid nitroso was pressed in a clamp between two hot plates of quartz, which were allowed to cool under pressure. In this way a very thin film of the solid substance was obtained. On photographing a spectrum through this screen, it was found that the principal absorption-band had broken up into two placed symmetrically with respect to the band shown by the fluid. One of these doubtless belongs to the ordinary, the other to the extraordinary ray. The centres of the bands are at wave-lengths 36 and 46, while the centre of the single band possessed by the liquid is at $\lambda = 43$.

In fig. 6 (Plate VIII.) we have two photographs of this double band.

The absorption of the liquid nitroso does not differ very materially from that of its solution in glycerine, which I

have described in a previous paper on screens transparent only to ultra-violet light *.

Th absorption-spectrum was photographed by pressing out a drop of the fluid substance between a quartz lens and plate, the whole being kept warm by a current of hot air. In this way a film of variable thickness was obtained which, when brought close to the slit of the quartz spectrograph, enabled a record to be made of the relative intensities, positions, and general form of the absorption-bands. One of these photographs is shown in fig. 7, Pl. VIII.

The nitroso begins to vaporize at a temperature only a little above its melting-point, and the absorption of the vapour is interesting, for unlike most absorbing vapours and gases this substance shows a broad absorption-band, similar to the liquid only shifted well down into the ultra-violet. The nitroso was vaporized in a glass tube, the ends of which were closed with quartz plates. The tube was previously exhausted, and was heated by a water-bath through which it passed. The absorption-spectrum was photographed at different temperatures ranging from 85°, the melting-point, to 100°. The absorption does not begin as a narrow line, as is usually the case with gases, but with a broad band, which increases in intensity as the density of the vapour increases. The centre of this band is at wave-length 34, while the centre of the band in the case of the liquid is at 43.

I made numerous experiments to see whether the density of the vapour could be sufficiently increased to cause a shift in the position of the band towards the red, all of which failed owing to the fact that the nitroso decomposes at temperatures above 150°. The substance was heated in strong sealed bulbs in an air-bath, but decomposition always resulted before a density sufficient to shift the absorption-band in a measurable degree had been obtained.

On the other hand, the position of the absorption-band can be shifted by increasing the density of the medium in which the nitroso-vapour is present. If a solution of nitroso in ether is heated above its critical temperature in sealed glass tubes, the centre of the absorption-band can be given almost any position between that of the vapour-band and the solution-band, by varying the amount of ether in the tube, or, in other words, by varying the density of the vapour.

I made a number of attempts to prove that the double overlapping band shown by the solid was connected with the double refraction of the nitroso crystals, by placing a nicol before the slit of the spectrograph and photographing the

absorption-spectrum of a thin crystalline film of the substance between two quartz plates. A number of spectra were taken with the nicol in different positions, a comparison of which gave unmistakable evidence that one band belonged to the ordinary, the other to the extraordinary ray. In no case did either of the bands disappear entirely, owing to the fact that they overlap, which makes it impossible to have complete transparency with the nicol in any position. No measurements have as yet been made of the values of the extinction-coefficient in different parts of the spectrum, owing to the difficulty of preparing a film thin enough to transmit light within the absorption-band. Though it is possible to get a wedge-shaped film between a lens and a flat plate which transmits all wave-lengths to a certain degree in its thinnest portions, a film of this nature is quite unsuitable for quantitative measurements of the absorption. Probably by working with solutions in glycerine of different concentrations and thicknesses, a fairly correct idea of the absorption-curve could be obtained by calculation, though this method would be open to some objections. Until the substance has been investigated in the infra-red, and until the dielectric constant has been determined, the dispersion-formula cannot be applied to it to the best advantage. At the present time, facilities for investigating these two points are not at my disposal, but I expect in the near future to investigate them. In the meantime it is instructive to apply the formula to the results which have already been obtained.

The Dispersion Formula.

For parts of the spectrum in which the extinction-coefficient has a small value, the dispersion is represented by the formula

$$n^2 = 1 + \Sigma \frac{m_k \lambda^2}{\lambda^2 - \lambda_k^2},$$

in which λ_k are the wave-lengths of the centres of the absorption-bands (nearly), and m_k constants for these bands, the summation being taken for all of the absorption-bands, whether near or far removed from the portion of the spectrum under consideration. If the bands are far removed from this region the fraction is practically equal to unity, and the constants m', m'', &c. have merely to be added or subtracted according to whether they lie on the more refrangible or less refrangible side of the spectral region under investigation, *i.e.* according as the sign of $\dfrac{\lambda^2}{\lambda^2 - \lambda'^2}$ is positive or negative.

For a medium having but a single absorption-band, the formula takes the form

$$n^2 = 1 + \frac{m'\lambda^2}{\lambda^2 - \lambda'^2}.$$

This formula can, I think, be applied to sodium vapour, judging from results recently obtained. Most other substances have, however, a second band further along in the ultra-violet, though for the region investigated experimentally this is so far removed that m'' can be considered constant. The formula then takes the form

$$n^2 = m + \frac{m'\lambda^2}{\lambda^2 - \lambda'^2},$$

in which $m = m'' + 1$. This formula has been found to represent the dispersion of most transparent substances.

I have applied this formula to the region of the spectrum for which the nitroso is most transparent, using the data obtained with the spectrometer and prisms of comparatively large angle.

The position of the centre of the absorption-band can be calculated from three observed values of the refractive index n_1, n_2, n_3 for wave-lengths λ_1, λ_2, λ_3 from the formula

$$\lambda'^2 = \frac{\lambda_3^2(\lambda_1^2 - \lambda_2^2) - \lambda_2^2(\lambda_1^2 - \lambda_3^2)C}{(\lambda_1^2 - \lambda_2^2) - (\lambda_1^2 - \lambda_3^2)C},$$

in which

$$C = \frac{n_2^2 - n_1^2}{n_3^2 - n_1^2}.$$

This formula gives $\lambda' = 43.1$ for the centre of the absorption-band, a value agreeing closely with the observed value. Further,

$$m' = \frac{(n_3^2 - n_1^2)(\lambda_1^2 - \lambda'^2)(\lambda_3^2 - \lambda'^2)}{\lambda'^2(\lambda_1^2 - \lambda_3^2)}$$

$$m = n_2^2 - \frac{m'\lambda_2^2}{\lambda_2^2 - \lambda'^2}.$$

The values found for m' vary slightly with the region of the spectrum in which the values of n and λ are chosen. This indicates the presence of one or more absorption-bands in the infra-red. The mean values found were

$$m' = 0.53$$
$$m = 2.13.$$

This should make the dielectric constant 2·66, if no infra-red

absorption-bands are present. If it had turned out that m equalled unity, the inference would have been that the absorption-band at $\lambda = 43$ was the only one present. The large value of $(m-1)$ indicates that there are bands of metallic absorption in the ultra-violet. As I have already shown, a strong absorption-band exists a little below $\lambda = 20$, and a weak one at $\lambda = 25$.

Though the formula as it stands represents fairly well the dispersion in the red, yellow, and green, it breaks down if we try to apply it to the values found in the ultra-violet by the photographic application of the method of crossed prisms. This is due to the fact that we are getting into the region in which $\dfrac{\lambda^2}{\lambda^2 - \lambda''^2}$ is no longer approximately equal to unity.

This quantity increases in magnitude as λ decreases, and the values of n will consequently be higher than those calculated on the assumption that the quantity is equal to unity. To meet this contingency we must use the formula (neglecting the weak band at $\lambda = 25$)

$$n^2 = 1 + \frac{m'\lambda^2}{(\lambda^2 - \lambda'^2)} + \frac{m''\lambda^2}{\lambda^2 - \lambda''^2}.$$

Since $m = 2 \cdot 13 = m'' + 1$, we can take $m'' = 1 \cdot 13$.

The value of the refractive index n for $\lambda = 34$, calculated from the original or simplified formula, is $n = 1 \cdot 1$, while the observed value is $1 \cdot 3$; in other words, the value is raised by the influence of the remote ultra-violet band. Using this observed value of n, it is possible to calculate the wave-length λ'' of the ultra-violet band, assuming, as above, $m'' = 1 \cdot 13$.

This was found to be $\lambda'' = 18$, a value which looks very reasonable, judging from the photographs of the absorption in this region. It is of course impossible to determine experimentally the centre of this band, since the nitroso cuts off everything below $\lambda = 20$. Whether or not a return of transparency would be found further along by employing fluorite plates and a vacuum spectrograph, it is impossible to say.

We are now in a position to calculate other values of n in the ultra-violet and compare them with the observed values.

For $\lambda = 36$ (the wave-length for which the lowest value of the refractive index was found experimentally), n calculated by the first formula is $n = 0 \cdot 92$; by second formula, $n = 1 \cdot 08$; observed, $n = 1 \cdot 05$. For $\lambda = 31$, by second formula $n = 1 \cdot 42$; observed, $n = 1 \cdot 43$.

Obviously we cannot apply the formula to that portion of

the spectrum between wave-lengths 23 and 29 until we know the value of the extinction-coefficient in this region.

I feel certain that there are absorption-bands in the infra-red, not only on account of the indications which the dispersion-formula gives, but also because when working with prisms of large angle 1 found that a fairly strong absorption occurred in the extreme visible red. It was not sufficiently intense to make measurements impossible, and whether it was due to the presence of a weak diffused band similar to the one at $\lambda = 25$, or to a strong band of metallic absorption, I was unable to determine, the observations being confined to the extreme edge. I expect to investigate this point as soon as our apparatus for the study of the infra-red region of the spectrum is in working order. I hope to be able in the near future to make a determination of the dielectric constant of the substance. With these points thoroughly investigated, a more rigorous application of the dispersion-formula to the results will be possible. I feel less hopeful of getting satisfactory data regarding the absorption, but shall at all events make the attempt.

The Dispersion of Toluine.

In examining the ultra-violet absorption of some organic solvents, I found that toluine has a fairly strong and narrow absorption-band at wave-length ·00027, in addition to the band of metallic absorption below ·00020, which is the one which chiefly influences the dispersion of the substance. It occurred to me that it would be interesting to determine the dispersion of the substance through this region, and see to what extent the curve was modified by the weaker band. So far as I was able to find, the ultra-violet dispersion of toluine has never been determined, which is not very surprising, since only very acute prisms transmit anything below the band above mentioned. The same method was employed as in the case of nitroso-dimethyl-aniline, namely, the crossed-prism method adapted to the quartz spectrograph. The dispersion was first measured in the visible spectrum with a hollow prism of 60° angle. Small prisms were then made of quartz plates with the toluine between them, the angles varying from two to six degrees. The angles were determined from the photographs of the deviated spectra obtained with the quartz spectrograph, using the data obtained with the spectrometer in the blue region of the spectrum. Fig. 9 (Pl. VIII.) shows one of the deviated spectra, the ultra-violet being to the left, and illustrates well the general form of the dispersion-curve in this region. Figs. 10 and 11 show the deviated and undeviated spectra, the one above the other,

obtained with prisms of two different angles. It is at once apparent that the absorption-band at wave-length ·00027 modifies the curve to no small degree. In the original negatives, a very slight trace of oppositely curved branches such as are always present at the edges of strong absorption-bands is discernible. The band is quite narrow and fairly strong, yet it does not seem to modify the curve to any great degree on the red side, at least the curve has nothing like the steepness which it has in the remote ultra-violet. On the other side of the band, however, the curve is seen to be depressed to a considerable distance. The influence of the band seems therefore to be unsymmetrical. This is to be ascribed to the fact, I believe, that the absorption on the ultra-violet side of this band is quite strong, as will be seen from fig. 11.

The plates were measured in the same manner as the nitroso plates, and a table of the refractive indices is given below.

λ.	n.	λ.	n.
2266	1·885	2681	1·640
2288	1·8503	2750	1·628
2314	1·821	2980	1·595
2322	1·808	3250	1·570
2330	1·807	3400	1·554
2372	1·767	3659	1·542
2470	1·709	3995	1·526
2527	1·679	4799	1·507
2570	1·649		

The curve is shown graphically on Plate VI.

Applying the dispersion formula to the results, we find, neglecting the band at ·00027 for the centre of the band of metallic absorption in the ultra-violet, the value ·000182, which is not inconsistent with the photographic records. For the other constants we find $m = 1·37$ and $m' = ·77$.

The sum of these two constants should give us the dielectric constant of the substance if no other bands of absorption are present.

The sum $m + m' = 2·14$, while the dielectric constant is 2·36 (determination by Palaz), from which it seems probable that there is an absorption-band in the infra-red.

The formula expresses the dispersion very fairly except for a narrow range immediately on the more refrangible side of the band at wave-length ·00027. Adding another term to the formula will not help matters much in this region, in my opinion, owing to the indefinite nature of the absorption.

The substance seems to be worthy of a more complete investigation.

XIV. *The Preparation and Properties of an Intensely Radioactive Gas from Metallic Mercury.* *By* Hon. R. J. STRUTT, *Fellow of Trinity College, Cambridge* *.

SOME time back † I described a series of experiments on the leakage of positive electricity from hot metals. These experiments were conducted at a pressure of 1 cm. of mercury, in order to avoid the leakage of electricity which always takes place in a vessel containing air even at ordinary temperatures, and which is now known to be due to a feeble radioactivity of the walls. On some occasions it was found that after heating the apparatus for some hours, allowing it to cool, and admitting air to atmospheric pressure, the leakage became very much greater than the normal. This effect disappeared when the air in the vessel was replaced by fresh air; so that it appeared clear that it must be due to a radioactive gas in the vessel. It was concluded (too hastily, as the event has proved) that some traces of radium salt had accidently got into the vessel, and that the leakage was due to the radium emanation.

In a recent paper ‡ I showed that different samples of the same material possessed different radioactivities. In order to clear up this point, I prepared some cylinders of copper §. These were annealed and put in the testing apparatus. It was found that they gave a considerable rate of leak, which was much diminished by washing out with fresh air. The explanation seemed to be that a radioactive gas was liberated from the copper by annealing, and that it gradually diffused from the inner parts of the metal sheet to the air contained in the cylinder. This conclusion was in agreement with the observations of Prof. McLennan and Mr. Burton ‖, who concluded that ordinary metals gave out a slight radioactive emanation, similar to that due to radium and thorium.

I next tried drawing air through a glass tube heated just below redness, and containing scrap-copper. This air was found to give a leakage three or four times the normal.

It seemed likely to be better to bubble air through a fluid

* Communicated by the Author.
† Phil. Mag. July 1902.
‡ Nature, Feb. 19, 1903; Phil. Mag. June 1903.
§ It was intended to try the effect of purifying the copper electrolytically, but my attention has for the time been diverted from this experiment.
‖ Phil. Mag. June 1903.

metal than merely to bring it in contact with a solid one. Air bubbled through hot mercury (at about 300° C.) was accordingly tried next. This was found to give a large effect, and was therefore selected for study first. To obtain a strongly active gas the following procedure was adopted.

A mercurial air-pump was used to circulate the air continuously through a gently sloping tube containing mercury heated to 300° C., in which this bubbling took place. The air, after passing through this tube, was drawn through a stopcock into the pump-head. It passed out from the blow-off of the pump to a vessel connected with the electroscope, in which the electrical leakage could be tested. Thence it passed to the bubbler again. The pressure in the testing-vessel was always atmospheric while an electrical test was being made, the bubbling being stopped meanwhile.

As the circulation proceeded the rate of leak steadily increased, until after some days it reached the value of 250 scale-divisions per hour. The leak with untreated air was only 2 scale-divisions; so that it had been increased 125-fold.

In this experiment the mercury was heated, as I have said; but this is not essential.

Active air can be obtained by bubbling air through cold mercury; I am inclined to think that this is not quite so effective. But no exact quantitative comparisons have yet been made. It has been found that after prolonged bubbling the mercury becomes temporarily exhausted, and that it recovers its power on standing. The full description of these experiments is reserved for a future paper.

A part of the most active sample of gas was placed in a testing-vessel, and allowed to remain some hours. It was then pumped out and replaced by ordinary air. There was found to be still a considerable leak; it amounted, immediately after admission of the air, to about $\frac{1}{6}$ that which the radioactive air had given. This leak fell in about 20 minutes to half its original value [*]. There can be little doubt that this activity left in the vessel after the removal of the active air is of the same nature as the "induced radioactivity" discovered by Rutherford in the cases of radium and thorium. This conclusion is further supported by a series of accurate experiments made in order to observe the rate of decay of the activity of the radioactive gas itself. A portion of this gas was inclosed airtight in a vessel, and the rate of leak observed from time to time.

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[*] The law of decay has not yet been accurately determined.

The results were as follows :—

Time (days)...	·04	·19	·31	·44	·94	1·44	1·94	2·44	2·94
Leak............	116	140	138	134	118	112	94·5	94·1	78·7

Time (days)...	3·46	3·94	4·46	4·96	5·44	6·04	7·01	7·96	8·95
Leak............	69·3	66·3	60·6	51·7	45·7	40·6	31·6	26·0	21·5

Curve of Decay of Mercury Emanation.
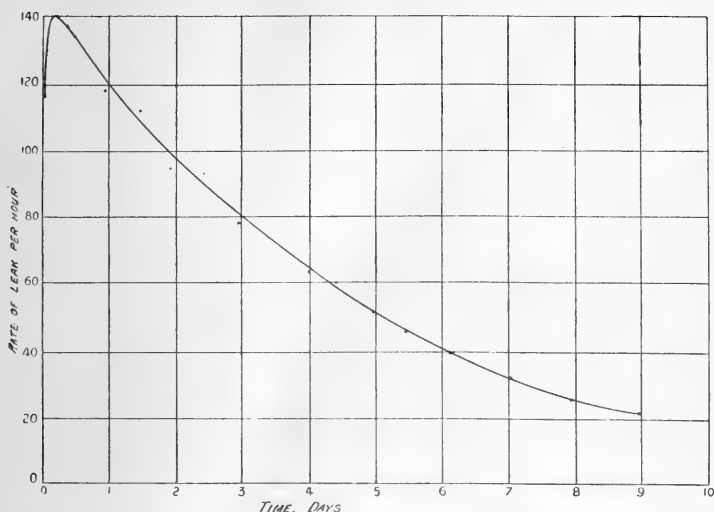
The results are graphically exhibited on the diagram. It will be noticed that the leak rises to a maximum value, about $\frac{1}{6}$ part greater than its initial value, after ·15 day; it then diminishes in geometrical progression with the time. Taking successive $\frac{1}{2}$-day intervals on the smoothed curve, after the first $\frac{1}{2}$ day, we find for the successive ratios of initial to final leak :—

1·12	1·11	1·10	1·10	1·11	1·11	1·10	1·12	1·12
1·12	1·14	1·14	1·11	1·11	1·11	1·10	1·10	

So that the geometrical law is closely followed. The leak falls to $\frac{1}{2}$ value in 3·18 days. This is not very different from the value for the radium emanation. Rutherford gives for it 3·7 days, Curie 4·0 days. The initial increase of leak exactly corresponds to Rutherford's observation on the radium emanation [*], and may be referred to the same explanation.

Many important points remain to be investigated, but

* Phil. Mag. vol. v. p. 442.

the conclusions hitherto reached may be summarized as follows :—

(1) A radioactive gas or emanation can be obtained by drawing air over hot copper, or by bubbling it through hot or cold mercury.

(2) By repeated circulation through mercury very considerable activity can be obtained, activity of quite a different order from that of metals as ordinarily observed.

(3) The mercury emanation deposits radioactive matter on the walls of the vessel containing it. This deposit remains after blowing out the gas, and possesses at first perhaps $\frac{1}{6}$ the activity of the latter. This induced activity falls to half value in 20 minutes.

(4) The emanation itself decays in activity according to an exponential law, falling to half value in 3·18 days (24 hours each).

XV. *Remarks on the Theory of Ionization by Collision of Ions with Neutral Molecules.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,

A. SCHUSTER[*] (1884) suggested the view that electric particles, ions, produced new ions out of neutral gas molecules by collision. Ph. Lenard[†] (1894) found that cathode-rays gave conductivity to a gas through which they passed ; some years afterwards these rays were recognized as negatively charged particles. J. J. Thomson[‡] and J. S. Townsend[§] have (1900) used that view of ionization by collision of ions ; the first explained by its aid some phenomena of the glowing discharge, the second the important observation made independently by himself that with low gas-pressure and secondary ionization the current, after having attained saturation, increases again if the potential-difference of the electrodes is increased. This observation is already contained in Stoletow's[∥] researches on the photo-

[*] A. Schuster, Proc. Roy. Soc. xxxvii. p. 336 (1884).

[†] Ph. Lenard, Wied. *Ann.* li. p. 240 (1894).

[‡] J. J. Thomson, Nature, lxi. March 1900, p. 458; Phil. Mag. (5) l. p. 279 (1900), (6) i. p. 368 (1901).

[§] J. S. Townsend, Nature, lxii. August 1900, p. 340 ; Phil. Mag. (6) i. p. 198 (1901), iii. p. 557 (1902) ; Townsend and Kirkby, Phil. Mag. (6) i. p. 630 (1901).

[∥] A. Stoletow, *Journ. de Phys.* (2) ix. p. 468 (1890).

electric current. In several papers * I also have explained a series of phenomena already known on the ground of the hypothesis of ionization by ionic shock.

A recently published paper of Townsend † contains the following remark :—" In one of his papers on this subject **J.** Stark (*Ann. d. Phys.* vii. p. 437, 1902) refers to my work, and states that the velocities which I gave in my first paper are too small. According to Stark a negative ion must travel between two points differing in potential by 50 volts, in order to acquire sufficient velocity to produce new ions on collision, because there is a fall of potential of 50 volts near the anode when a continuous discharge takes place in air. Even if it be granted that the fall of potential at the electrode is to be explained by this property of a moving negative ion, the phenomenon gives no reason for supposing that new ions are not generated by collision when the velocity of the negative ion is less than that corresponding to 50 volts."

It is necessary to remark that Townsend is under a misapprehension with regard to my views.

Firstly, I represent indeed the view that an ion must freely pass a certain minimum of potential-difference between two points, in order to acquire sufficient velocity to produce new ions by collision at the end of its free path. But I have not founded this view on the phenomenon that there is existing at the anode a drop of potential of 20 to 40 volts, when a continuous discharge takes place.

Secondly, I have said clearly that this drop of potential at the anode is *not* equal to the ionizing potential of the negative ion (ionizing potential = potential-difference to be freely passed by an ion to produce new ions by collision).

Thirdly, I distinguish between an ionization by ion shock in the interior of a gas passed by an electric current and an ionization at the boundary between a gas and a metal. By some catalytic action of the metal the ionizing potential is in the second case smaller than in the first. The experiments of Townsend refer to the second case, as I said clearly. It is only for the first case that I have assigned 50 volts as an upper limit of the ionizing potential of the negative ion.

The words to which Townsend refers are in English as follows :—"As we have seen above, the ionizing potential

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* J. Stark, *Physik. Zeitschrift,* ii. p. 17 (1900), iii. p. 504 (1902) ; *Ann. d. Phys.* iv. p. 402 (1901), vii. pp. 417, 919 (1902), viii. p. 829 (1902).
† J. S. Townsend, Phil. Mag. (6) v. p. 397 (1903).

of the negative ions within the gas passed by a current is smaller than 50 volts. As the ionizing potential of the positive ions is smaller near a metal than in the interior of the gas, it is to be expected that also the ionizing potential of the negative ions is smaller near a metal, in consequence of its catalytic aid, than in the interior of the gas. This is indeed the case ; even a few volts of freely passed potential-difference enable the negative ion to ionize the gas at the anode. J. S. Townsend (Phil. Mag. (5) l. p. 572, 1900) ionized a rarefied gas between two neighbouring electrodes by Röntgen rays and determined the potential-difference or electrode-potential at which the current-intensity leaving the stage of saturation began to increase again. This potential-difference was the ionizing potential to be freely passed by the negative ions in order to be able to ionize the gas at the surface of the anode. Townsend assigns 5 volts as the value of that ionizing potential for air near brass. It is possible that this is too small."

" In the glow current and in the negative-point current the anode is generally covered with a luminous layer, which arises from the ionization by the ions flying towards the anode. If a solid body, for instance a rod, is brought immediately before the anode, the rod throws a shadow on the anode by screening off the negative ions flying towards its projection (C. Skinner, Phil. Mag. [5] l. p. 573, 1900). It is not allowable to equate, even approximately, the anode drop of the glow current to the ionizing potential of the negative ions, because in that case this is of the order of magnitude of the inner electromotive force in the boundary layer."

When I put in doubt Townsend's value of 5 volts I was not led by my theoretical views ; but Townsend seemed to me to have interpreted his measurements without necessity in favour of a too small value. According to the later measurements of P. J. Kirkby * the ionizing potential of the negative ions for air towards a metal lies between 25 and 12 volts ; Lenard † has found 11 volts.

In a paper also recently published J. S. Townsend ‡ introduces, in order to explain an observation made by himself, the view that positive ions also are able to produce new ions by collision. This view has been already before represented

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* P. J. Kirkby, Phil. Mag. (6) iii. p. 212 (1902).

† Ph. Lenard, *Ann. d. Phys.* viii. p. 193 (1902).

‡ J. S. Townsend, Electrician, l. p. 971 (1903).

by **J. J.** Thomson * and by me †. I have shown that the normal cathode drop of potential in the glowing discharge (glow current) is equal to the ionizing potential of the positive ion relative to the cathode metal. Further, J. A. McClelland ‡ has already made an observation which is similar to the phenomenon studied by Townsend, but more simple. There are produced by heating positive and negative ions at the surface of a wire which is placed in the axis of a metal cylinder; the positive or negative ions are driven by an electric field through the rarefied gas to the inner side of the cylinder; if the freely passed potential-difference is large enough, they are able to ionize by collision the gas at the inner surface of the cylinder. I have shown (*Ann. d. Phys.* viii. p. 826, 1902) that in a set of measurements of McClelland it is the positive ion which produces new ions by collision.

I remain, Gentlemen,
Yours very truly,
Göttingen, 17 April, 1903. **J. Stark.**

XVI. *On the Aberration of the Concave Grating, when used as an Objective Spectroscope.* By F. L. O. Wadsworth§.

THE question of the aberration of concave gratings was first investigated by Rowland ‖ and Glazebrook ¶. The results obtained by them refer only to the case in which the ruled surface is spherical in form, and is so mounted that both it and the source of light lie on the circumference of a circle whose diameter is ρ, the radius of curvature of the spherical surface. When so ruled and mounted the aberration is symmetrical in form, and is so small that for gratings of the angular aperture usually employed we may use resolving-powers as high as 4,000,000 before the definition begins to be affected. Since the resolving-powers ordinarily employed in the largest concave-grating spectroscopes are only from one to five per cent. of this limiting value, the effect of aberration can be safely neglected, even when the angular aperture of the grating is increased from two to three times.

The same conclusion, however, does not hold when the method of mounting the grating is changed. In 1895 the writer suggested and investigated several new methods of using the concave grating as a direct objective spectroscope, with a parallel incident beam of light. At that time a number of mountings were designed and constructed for the purpose of testing the proposed new methods, and the results obtained were very satisfactory [*]. The grating used in these tests was, however, a comparatively small one of only about 3 cm. aperture, and 152 cm. radius of curvature. Used in the ordinary way (Rowland mounting) its semi-angular aperture measured at the centre of curvature would therefore be about $\beta = \cdot 01$. The effect of aberration on gratings of larger angular aperture than this was computed, and it was concluded at that time [†] that semi-angular apertures as large as $\beta = \cdot 05$ could be used if the resolving-power did not exceed 30,000 units. Since this paper was written the concave grating has been used in the way described by Poor and Mitchell at Johns Hopkins, and later by Mitchell at Yerkes Observatory, in the study of stellar spectra ; and by Frost, Mohler and Daniel, Jewell and Humphreys, and the writer, in photographing the flash spectrum during a solar eclipse. The gratings used by Poor and Mitchell [‡] were 100 cm. radius of curvature and about 5 cm. and 14·6 cm. aperture, ruled respectively with 5905 and 2842 lines per cm. The resolving-powers in the first order spectra were therefore about 30,000 and 41,500, and the semi-angular apertures about $\beta = \cdot 025$ and $\beta = \cdot 073$ respectively. The results obtained on stellar spectra were considered very good.

The results obtained by eclipse observers have all been much less satisfactory. Frost [§] used a concave grating of 4·4 cm. aperture and 150 cm. radius of curvature, ruled with 5684 lines per cm. The resolving-power in the first spectrum was therefore about 25,000, and the semi-angular aperture about $\beta = \cdot 015$. The field obtained was small, but this is ascribed by Prof. Frost to be in part due to the fact that flat plates were used instead of curved plates. The grating used by Mohler and Daniel [‖] had an aperture of 10 cm., a radius of curvature of 305 cm. and a ruling of 5684 lines per cm. In the first order the resolving-power was about 57,000 and the semi-angular aperture, β, was about 0·016. The definition

[*] Astrophysical Journal, vol. iii. pp. 54–60, Jan. 1896.
[†] Ibid. pp. 60–61.
[‡] Ibid. vii. p. 157, March 1898; and vol. x. p. 29, June 1899.
[§] Ibid. xii. p. 85; p. 307, December 1900.
[‖] Ibid. vol. xii. p. 361.

was unsatisfactory. The authors of the paper suggest that this might have resulted from any one or more of the following causes:—(1) an error of setting the plane of dispersion in the correct position-angle, (2) errors of following, (3) errors of focussing, (4) astigmatism. The grating used by Jewell and Humphreys* at the Sumatra eclipse was still larger than that used by Mohler and Daniel, having a ruled surface of 20 cm. aperture and a radius of curvature of 915 cm. The ruling was defective on one side, and in actual use only 0·6 of the whole aperture was employed, so that the effective semi-angular aperture β was about 0·0065. It is stated that in this case also the definition was unsatisfactory, the cause ascribed being an error of focussing.

The grating used by the writer in his eclipse work at Union Springs in May 1900 was one ruled especially by Prof. Rowland for the Allegheny Observatory. It has an aperture of 13·2 cm., a radius of curvature of 175·3 cm., and an unusually long ruling, 8 cm. The number of lines per cm. is 1421. Hence the resolving-power in the first-order spectrum is about 18,750, and the semi-angular aperture at the centre of curvature is about 0·04. This grating gives good definition at the centre of curvature, but when it was used as an objective spectroscope at Union Springs the definition was greatly impaired, both visually and photographically, so much so that I do not consider the spectrum plates taken with the instrument mounted in the manner there used good enough for accurate measurement of the wave-lengths of the spectral lines. In this case the lack of definition cannot be ascribed to errors of focussing or mal-adjustment of the apparatus, and I have therefore been led to investigate more rigorously than was at first done the effect of aberration of a concave grating at its principal focus. The results of this investigation show very clearly that the field and range of action of the concave grating, when used as an objective spectroscope with parallel incident light, is very much more restricted than when it is mounted in the usual manner (Rowland method) ; and that in order to avoid injuring the definition in the former case the angles of incidence and diffraction must never exceed certain limiting values, which are rapidly varying functions both of the resolving-power and the angular aperture of the grating.

We will first consider the case in which the ruled surface is spherical, the usual form adopted for concave gratings. For a parallel incident beam of light the optical conditions

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* Ibid. vol. xvi. p. 313, June 1902.

in the primary plane are shown in fig. 1, in which $i=$ the angle $pac=$ the angle of incidence; $\theta=$ angle $cao'=$ angle of diffraction; $u=ao'=$ focal distance from grating to primary

Fig. 1.

image; $\rho=ca=$ radius of curvature of the spherical surface, and $\beta=$ angle $bca=$ semi-angular aperture of surface at the centre of curvature.

In investigating the aberration at the primary focal point o' we shall follow, with some slight modifications, the method adopted by Lord Rayleigh in his paper "Aberration of Lenses"[*], and also followed by Glazebrook in the paper already referred to. Let ah, $b'h'$ be instantaneous sections of the plane wave-front incident at the central point a, and the extreme left-hand edge of the grating aperture. Then the relative retardations of the central with reference to the extreme edge rays at the focal point o' will be

$$Z_1=u-(bo'+bh) \text{ for the right-hand half,}$$
$$Z_2=u-(b'o'-ah') \quad ,, \quad \text{left} \quad ,,$$

From the geometry of the triangles bao', bah, and bac we find for bo' and bh

$$bo'=\sqrt{u^2+4\rho^2\sin^2\frac{\beta}{2}-4\rho u\sin\frac{\beta}{2}\sin\left(\frac{\beta}{2}+\theta\right)}$$

$$=\sqrt{(u-\rho\sin\beta\sin\theta)^2-\rho\sin^2\beta\cos\theta(u-\rho\cos\theta)+\frac{\rho}{4}\sin^4\beta(\rho-u\cos\theta)}, \quad (1)$$

[*] Phil. Mag. vol. ix. p. 40, Jan. 1880.

and
$$bh = 2\rho \sin\frac{\beta}{2} \sin\left(i - \frac{\beta}{2}\right)$$

$$= \rho \sin\beta \sin i - \frac{\rho}{2}\sin^2\beta \cos i - \frac{\rho}{8}\sin^4\beta \cos i. \quad (2)$$

Expanding (1) by the binomial theorem in terms containing successive powers of β, and neglecting as before those powers higher than the fourth, we obtain

$$\left. \begin{aligned}
bo' =\ & u - \rho \sin\beta \sin\theta \\
& - \frac{\rho}{2u}\sin^2\beta \cos\theta(u - \rho\cos\theta) \\
& - \frac{\rho^2}{2u^2}\sin^3\beta \sin\theta \cos\theta(u - \rho\cos\theta) \\
& - \frac{\rho^3}{2u^3}\sin^4\beta \sin^2\theta \cos\theta(u - \rho\cos\theta) \\
& + \frac{\rho}{8u}\sin^4\beta(\rho - u\cos\theta) \\
& - \frac{\rho^2}{8u^3}\sin^4\beta \cos^2\theta(u - \rho\cos\theta)^2 \\
& + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{aligned} \right\} \ . \quad . \quad . \quad . \quad (3)$$

Combining this value with that for bh from (2), and subtracting the same from u, we find for Z_1

$$\left. \begin{aligned}
Z_1 =\ & \rho \sin\beta(\sin\theta - \sin i) && = \mathrm{M} \\
& + \frac{\rho}{2}\sin^2\beta\left(\cos\theta + \cos i - \frac{\rho}{u}\cos^2\theta\right) && = \mathrm{N} \\
& + \frac{\rho^2}{2u^2}\sin^3\beta\left[\sin\theta\cos\theta(u - \rho\cos\theta)\right] && = \mathrm{O} \\
& + \frac{\rho^2}{2u^2}\sin^4\beta\left(\tfrac{3}{2}\rho\sin^2\theta\cos\theta - \tfrac{1}{2}\rho\cos\theta - \frac{u}{4}\sin^2\theta\right. \\
& \qquad \left. - \frac{\rho^2}{u}\sin^2\theta\cos^2\theta + \frac{\rho^2}{4u}\cos^4\theta\right) && = \mathrm{P} \\
& + \frac{\rho}{8}\sin^4\beta(\cos\theta + \cos i) && = \mathrm{Q}
\end{aligned} \right\} \quad (4)$$

The expression for Z_1 can also be derived at once from Glazebrook's paper by putting u in equation (3) equal to ∞, and substituting for ϕ, ψ, and ω the corresponding values i, θ, and β.

For the left-hand half of the surface we have similarly
$$Z_2 = \mathrm{M} + \mathrm{N} - \mathrm{O} + \mathrm{P} + \mathrm{Q}, \quad . \quad . \quad . \quad . \quad (5)$$

which is the same as the expression for Z_1, with the exception of the sign of the term O.

From the general theory of the grating* we always have for any given spectral image

$$\rho \sin \beta (\sin \theta - \sin i) = mn\lambda.$$

Also for parallel incident light†

$$u(\cos \theta + \cos i) - \rho \cos^2\theta = 0. \quad . \quad . \quad . \quad (6)$$

The first two terms of equations (4) and (5) are therefore always satisfied when the spectrum is in focus. The remaining terms express the aberration at the primary focal point o' for any given values of θ, i, and β.

From (6) we have for u

$$u = \frac{\rho \cos^2\theta}{\cos^r\theta + \cos i}. \quad . \quad . \quad . \quad . \quad (7)$$

Substituting this value of u in the terms O, P, and Q of equations (4) and (5) and reducing, we obtain as the general expression for the primary aberration of a concave grating used with parallel incident light

$$Z_1 = -\frac{\rho}{2} \sin^3\beta \sin \theta \cos i \frac{(\cos i + \cos \theta)}{\cos^2\theta}$$

$$+ \frac{\rho}{8} \sin^4\beta \frac{(\cos i + \cos \theta)}{\cos^2\theta} [\cos^2 i - (\cos \theta + 2 \cos i)^2 \tan g^2\theta], \quad (8)$$

and for Z_2 the same expression, save for the opposite sign of the first term in $\sin^3\beta$. The corresponding general expression for the primary aberration of a concave grating mounted in the usual Rowland manner, as deduced by Glazebrook and Rayleigh, is

$$Z_0 = (u + u' = \text{central ray}) - (\text{lateral ray})‡$$

$$= -\tfrac{1}{8}\rho \sin^4\beta (\sin \theta \tan g \, \theta + \sin i \tan g \, i). \quad . \quad (9)$$

Comparing (8) and (9) we see that in general the effect of aberration is not only larger, but is of a more prejudicial character, when the grating is used as an objective spectroscope than when it is used as originally proposed by Rowland. In the former case the aberration is unsymmetrical on account of the term in $\sin^3\beta$; in the latter case this term is eliminated

* See, for example, Rowland, Phil. Mag. vol. xvi. (1883).

† See Astrophysical Journal, vol. iii. p. 55.

‡ In Glazebrook and Rayleigh's papers the aberration is expressed as the difference, lateral ray — central ray, and is therefore of the opposite sign.

by the conditions of the mounting. It will also be noted that the signs of the terms in $\sin^4\beta$ are opposite in the two cases. This is a point of special significance, the importance of which will be developed later.

The general equation (8) reduces to a simpler form in several special cases corresponding to different forms of mounting. These will now be considered.

(A) Let the grating be mounted so that the eyepiece or the centre of the photographic plate lies on the axis of the grating, *i. e.* on the normal to the centre of the ruled surface. This is the form of mounting described and illustrated on pp. 54–57, and on Plates VII. and IX. of my previous paper, and is the one that has been most commonly used in subsequent work. In this case $\theta = 0$ at the centre of the field, and we have for the aberration

$$Z_1' = Z_2' = Z_A = + \frac{\rho}{8} \sin^4\beta \cos^2 i (1 + \cos i). \quad . \quad (10)$$

For the usual Rowland mounting in which this condition is also fulfilled we have similarly from (9)

$$Z_0 = - \frac{\rho}{8} \sin^4\beta \frac{\sin^2 i}{\cos i}. \quad . \quad . \quad . \quad . \quad (11)$$

Equations (10) and (11) show that with this form of objective grating mounting the aberration decreases with increasing angles of incidence, and becomes vanishingly small for values of i near $90°$. With the Rowland mounting the reverse is the case.

Since the factor $\frac{\rho}{8} \sin^4\beta$ is common to both (10) and (11) the relative amount of aberration for any given grating with the two mountings will be given by the two trigonometrical functions

$$f_1(i) = \cos^2 i (1 + \cos i) = a \quad . \quad . \quad . \quad . \quad (12)$$

$$f_2(i) = \frac{\sin^2 i}{\cos i} = a_0. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For convenience the values of these coefficients, a and a_0, have been computed for values of i from $0°$ to $90°$, and are tabulated in Table I. columns 2 and 3.

(B) In using the grating photographically we cannot satisfy the condition $\theta = 0$ for all parts of the field simultaneously. If we bend the plate to conform to the focal

curve [as given by (6)]*, and place it with its centre on the
optical axis, the aberration at any other point of the field
near the axis, *i. e.* for small values of θ, will be

$$Z_B = \mp \frac{\rho}{2} \sin^3\beta \, \tang \theta \cos i (1 + \cos i)$$

$$+ \frac{\rho}{8} \sin^4\beta (1 + \cos i) [\cos^2 i - (1 + 2\cos i)^2 \tang^2\theta] \sec^2\theta. \quad (14)$$

For the Rowland mounting it will be similarly

$$Z_0 = - \frac{\rho}{8} \sin^4\beta \left[\frac{\sin^2 i}{\cos i} + \frac{\sin^2\theta}{\cos\theta} \right]. \quad . \quad . \quad . \quad (15)$$

This expression (14) may be put in the form

$$Z_B = \mp \frac{\rho}{2} \sin^3\beta \frac{\tang\theta}{\cos i} (1 + \cos i) \cos^2 i$$

$$+ \frac{\rho}{8} \sin^4\beta (1 + \cos i) \cos^2 i \left[1 - \left(\frac{1 + 2\cos i}{\cos i} \right) \tang^2\theta \right] \sec^2\theta, \quad (16)$$

in which the same functions of i appear as in (10). The only
new function is

$$f_3(i) = \left(\frac{1 + 2\cos i}{\cos i} \right)^2 = \left(2 + \frac{1}{\cos i} \right)^2 = b. \quad . \quad . \quad (17)$$

The values of $f_1(i) \div \cos i = a_1$ and of $f_3(i) = b$ have been
computed for the same values of i assumed before, and are
given in columns 4 and 5 of Table I.

(C) Let the grating be mounted so that the angles of in-
cidence and diffraction are equal but opposite in sign, *i. e.*
so that $\theta = -i$. This is the form of mounting discussed and
illustrated on pp. 58, 59, and on Plate VII. fig. 9 of my
former paper. As there shown, the central spectral image
in the field always lies on a circle whose centre is on the axis
of the grating, and whose radius is $\frac{\rho}{4}$. The mechanical
mounting which fulfils this condition is very simple, but it

* For small values of θ the equation of the curve may be written

$$u = \frac{\rho}{1 + \cos i} \cos \theta, \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

which is the equation of a circle with its centre on the axis of the grating
and a radius equal to one-half u. The statement made by Poor and
Mitchell in their paper (Astrophysical Journal, vol. vii. p. 159) that
" The spectra .. are brought to a focus on a circle whose radius is u," is
an error probably due to an oversight in considering $\cos^2\theta$ to be of the
same order as $\cos \theta$.

has not yet been used, to my knowledge, except in the small trial instrument there illustrated.

In this case we have for the aberration in the central image

$$Z_C = \pm \rho \sin^3\beta \sin i + \frac{\rho}{4} \sin^4\beta(1 - 9 \tan^2 i) \cos i. \quad . \quad (18)$$

The values of the function

$$f_4(i) = (1 - 9 \tan^2 i) \cos i = \cos i - 9a_0 = c \quad . \quad . \quad (19)$$

are tabulated in column 6, Table I.

(D) When the grating is mounted as just considered, and is used for photographic work, we have to consider not only the line at the centre of the field $\theta = -i$, but also lines at a distance κ each side of the centre. For any given value of i we have for θ_D

$$\theta_D = -i \pm \kappa. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The aberration in this case is most easily computed directly from the general equation (8) by substituting for θ and i the values determined by the relation (20).

(E) Let the grating be mounted so that the incident light is always normal to the surface. This case has been investigated for the Rowland mounting[*], but not for the O.S. mounting. In the latter we have from (7) for $i = 0$

$$u = \frac{\rho}{1 + \cos \theta} \cos^2\theta, \quad . \quad . \quad . \quad . \quad (21)$$

which is the equation of the curve on which *all* focal images lie, *i. e.* the focal curves for both central and lateral images coincide.

The aberration for any image on the focal curve is

$$Z_E = \mp \frac{\rho}{2} \sin^3\beta \sin \theta\left(\frac{1 + \cos \theta}{\cos^2\theta}\right)$$

$$+ \frac{\rho}{8} \sin^4\beta \frac{1 + \cos \theta}{\cos^2\theta} [1 - (2 + \cos \theta)^2 \tan^2\theta]. \quad . \quad (22)$$

The last term of (22) may be put in the form

$$\frac{\rho}{8} \sin^4\beta \frac{1 + \cos \theta}{\cos^2\theta} \sin \theta \left[\frac{\cos^2\theta}{\sin \theta} - 1\frac{1 + \cos \theta}{\cos^2\theta} \sin \theta\right].$$

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[*] "Fixed Arm Concave Grating Spectroscopes," Astrophysical Journal, vol. ii. p. 370. Forms of mounting fulfilling this condition are shown on plates xiii. (for $i = 0$) and xv.

The functions

$$f_5(\theta) = \frac{\cos^2\theta}{\sin\theta} = d, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$\left.\begin{array}{l} f_6(\theta) = \dfrac{1+\cos\theta}{\cos^2\theta} = e \\[2mm] f_7(\theta) = e\sin\theta \quad = e_1 \end{array}\right\} \quad . \quad . \quad . \quad (24)$$

which appear in (21) and (22) have been computed for the same values of θ as have already been used for i in $f_1(i)\ldots\ldots f_4(i)$. The results are given in columns 7, 8, and 9 of Table I.

(F) When θ is very small expressions (21) and (22) reduce respectively to

$$u = \frac{\rho}{2}\cos\theta, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and $$Z_{\mathrm{F}} = \mp\rho\sin^3\beta\,\tang\theta$$
$$+ \frac{\rho}{4}\sin^4\beta\sec^2\theta[1-9\,\tang^2\theta]. \quad . \quad . \quad (26)$$

These expressions may also obtained from $(6\,a)$* and (14) by putting $i=0$. It is very nearly the same as (18) for small values of i. When the angle of incidence is nearly normal to the grating Cases B and C may therefore be considered as practically identical with Case F.

(G) Finally, let $\theta = +i$. This is the case of specular reflexion. The aberration at the central point in the field, *i. e.* in the undeviated image of the source of light, will be given by the expression

$$Z_{\mathrm{G}} = \mp\rho\sin^3\beta\sin\theta + \frac{\rho}{4}\sin^4\beta(1-9\,\tang^2\theta)\cos\theta, \quad . \quad (27)$$

which is identical with (10) with the exception of the reversed sign of the first term. For the special case $\theta = i = 0$ we have

$$Z_{\mathrm{G}} = \frac{\rho}{4}\sin^4\beta = \frac{u}{2}\sin^4\beta, \quad . \quad . \quad . \quad (28)$$

the well-known expression for the aberration at the principal focus of a spherical mirror.

(H) When the spectral image is situated at a distance κ from the undeviated image the aberration will be found, as in Case D, by substituting in (8) the value

$$\theta = i \pm \kappa.$$

* See p. 126, footnote.

Making use of the quantities a, a_0, a_1, b, b_1, c, e, and e_1, already computed, and putting the quantity

$$\tfrac{1}{8} \rho \sin^4\beta = \mathrm{K},$$

a constant for any given grating, we may write the expressions $\mathrm{Z}_0 \ldots \mathrm{Z}_\mathrm{H}$ as follows :—

$$\mathrm{Z}_0 = -\mathrm{K}a_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11a)$$

$$\mathrm{Z}_\mathrm{A} = +\mathrm{K}a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

$$\mathrm{Z}_{\mathrm{B}_0} = -\mathrm{K}(a_0 + \sin^2\kappa) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

$$\mathrm{Z}_\mathrm{B} = \mathrm{K}\left[-4a_1 \frac{\tang\kappa}{\sin\beta} + a(1 - b\,\tang^2\kappa)\sec^2\kappa \right] . \quad (16a)$$

$$\mathrm{Z}_\mathrm{C} = \mathrm{K}\left(+8\frac{\sin i}{\sin\beta} + 2c \right) . \quad . \quad . \quad . \quad . \quad . \quad (18a)$$

$$\mathrm{Z}_\mathrm{E} = \mathrm{K}\left[-\frac{4e_1}{\sin\beta} + e_1(a_0 - 4e_1) \right] \quad . \quad . \quad . \quad . \quad (22a)$$

$$\mathrm{Z}_\mathrm{F} = \mathrm{K}\left(-\frac{8\tang\kappa}{\sin\beta} + \frac{2c}{\cos^2\kappa} \right) \quad . \quad . \quad . \quad . \quad . \quad (26a)$$

$$\mathrm{Z}_\mathrm{G} = \mathrm{K}\left(-8\frac{\sin\theta}{\sin\beta} + 2c \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (27a)$$

For Cases D and H, in which we must use the general equation (8), we may write

$$\mathrm{Z} = \mathrm{K}\left\{ \begin{array}{l} -\dfrac{4\sin\theta\cos i}{\sin\beta} \cdot \dfrac{\cos i + \cos\theta}{\cos^2\theta} \\[2mm] +\dfrac{\cos i + \cos\theta}{\cos^2\theta}\left[\cos^2 i - (\cos\theta + 2\cos i)^2\,\tang^2\theta \right] \end{array} \right\} \quad (8a)$$

By means of the general formulæ $(8a) \ldots (27a)$ and the aid of Table I. we can readily compute the aberration of the concave grating for any form of mounting, and for any given point in the spectral field. In order to compare the efficiency of the same grating mounted in the different ways above discussed, the aberrations $\mathrm{Z}_\mathrm{A} \ldots \mathrm{Z}_\mathrm{H}$ have been computed for twelve different points in the spectral field for a grating of the semi-angular aperture ($\beta = \cdot 01$) generally used. Four of these points correspond to the images formed on the axis of the grating for $i = 5°$, $i = 15°$, $i = 30°$, and $i = 60°$, of mounting A (or A_0), and are designated by the symbols S_0', S_0'', S_0''', and $\mathrm{S}_0^{\mathrm{iv}}$ respectively. Eight others correspond to the spectral images which lie $5°$ on each side ($\kappa = \pm 5°$) of the

first four, and these are designated by S'_+, S'_-, S''_+, S''_-, S'''_+, S'''_-, and S^{iv}_+, S^{iv}_- respectively.

TABLE I.

i.	$\dfrac{a}{(1+\cos i)\cos^2 i}$	$\dfrac{a_0}{\sin^2 i}\cdot = \dfrac{}{\cos i}$	$\dfrac{a_1}{(1+\cos i)\cos i}$	$b\left(2+\dfrac{1}{\cos i}\right)^2$	$c\dfrac{(1-9\tan^2 i)}{\cos^2 i}\times$	$d\cdot\dfrac{\cos^2 i}{\sin i}$	$e\dfrac{1+\cos i}{\cos^2 i}$	$e_1\sin i$.
0 ...	2·000	0·000	2·000	9·000	1·000	∞	2·000	·000
5 ...	1·981	·0076	1·988	9·023	+·928	11·35	2·011	·1754
10 ...	1·925	·0306	1·954	9·093	+·709	5·58	2·046	·355
15 ...	1·834	·069	1·899	9·213	+·341	3·61	2·107	·545
20 ...	1·713	·125	1·823	9·389	−·181	2·58	2·192	·750
25 ...	1·566	·197	1·728	9·631	−·867	1·94	2·320	·981
30 ...	1·399	·289	1·616	9·952	−1·732	1·50	2·488	1·244
35 ...	1·220	·402	1·490	10·373	−2·794	1·17	2·711	1·555
40 ...	1·036	·539	1·353	10·93	−4·086	·913	3·010	1·935
45 ...	·854	·707	1·207	11·66	−5·657	·707	3·414	2·414
50 ...	·678	·913	1·056	12·64	−7·573	·539	3·976	3·045
55 ...	·518	1·170	·902	14·01	−9·955	·402	4·783	3·919
60 ...	·375	1·500	·750	16·00	−13·000	·289	6·001	5·196
65 ...	·254	1·944	·601	19·06	−17·07	·197	7·965	7·218
70 ...	·157	2·582	·459	24·24	−22·89	·124	11·47	10·78
75 ...	·084	3·515	·326	34·39	−32·87	·067	18·79	18·15
80 ...	·035	5·585	·203	60·20	−50·26	·031	38·92	38·33
85 ...	·008	11·386	·095	181·5	−102·4	·008	143·2	142·6
90 ...	·000	∞	·000	∞	∞	·000	∞	∞

In order that the points S' to S^{iv} shall correspond to the same spectral images for each of the various mountings we must choose the values of θ and i so that we always have

$$(\sin\theta - \sin i) = \text{constant} \quad . \quad . \quad . \quad (29)$$

for any given point S. We therefore have

For the points .. S' S'' S''' S^{iv}

Case A—A_0 ... $i_A = 5°$ $15°$ $30°$ $60°$

$\theta_A = 0°$ $0°$ $0°$ $0°$

Case E_0 .... $\theta_{E_0} = 5°$ $15°$ $30°$ $60°$

$i_{E_0} = 0°$ $0°$ $0°$ $0°$

For Case C...... $\theta = -i$

$$\therefore (\sin\theta_A - \sin i_A) = -2\sin i_C,$$

and we therefore have for i_C [to be used in (18a)]

$$\left.\begin{array}{c} i_C \\ -\theta_C \end{array}\right\} \quad 2° \; 30', \quad 7° \; 26', \quad 14° \; 29', \quad 25° \; 40'.$$

For the lateral images $S_\pm$ to the right and left of the central ones we have

For $B_0 - B$

	$S'_\pm$	$S''_\pm$	$S'''_\pm$	$S^{IV}_\pm$
$i_B =$	$5°$	$15°$	$30°$	$60°$
$\theta = \kappa = \pm$	$5°$	$\pm 5°$	$\pm 5°$	$\pm 5°$

For Case D we must have

$$\sin i_A \mp \sin 5° = \sin i_C + [\sin (i_0 \mp \kappa) = \sin (-\theta_D)],$$

from which we get for θ_D to be used in (8a)

	$S'_\pm$	$S''_\pm$	$S'''_\pm$	$S^{iv}_\pm$
$i_D = i_C =$	$2° 30'$	$7° 26'$	$14° 29'$	$25° 40'$
$\kappa_+ = \theta_{+D} =$	$2° 30'$	$-(2° 25')$,	$-(9° 22')$,	$-(31° 21')$.

For the lateral images of Case E we must have

$$\sin i_A \mp \sin 5° = \sin (\theta + \kappa) = \sin \theta_E,$$

and therefore for θ_E to be used in (8a)

	S'	S''	S'''	S^{iv}
for $\theta_E =$	$0°$	$9° 53'$	$24° 23'$	$51° 9'$
	$10° 2'$	$20° 15'$	$35° 58'$	$72° 24'$
$i_E =$	$0°$	$0°$	$0°$	$0°$

The values of S'_+ and S'_0 are likewise the values of the aberration at the centre and edges of a $10°$ field for Case F, which therefore requires no separate consideration.

Finally, for Case H we have a number of identical fields, S' ...... S^{iv}, each corresponding to a particular value of i_A. Only one set of these is considered, *i. e.* the set S' for which $i = 5°$. The centres of these will lie at points θ_H defined by the relation

$$\sin 5° = \sin i_H - \sin \theta_H.$$

Hence for $i_H =$	$5°$	$15°$	$30°$	$60°$
$\theta_H =$	$0°$	$9° 53'$	$24° 23'$	$51° 9'$

Also the two edges of this set S' will lie at points

$$\theta_H = \sin^{-1}(\sin i_H - \sin 5° \pm \sin 5°),$$

$\therefore$ for $i_H =$	$5°$	$15°$	$30°$	$60°$
$\theta_{+H} =$	$5°$	$15°$	$30°$	$60°$
$\theta_{-H} =$	$-5°$	$+ 4° 51'$	$19° 1'$	$43° 46'$

K 2

Each of these fields lies to the left of the geometrical (undeviated) image of the source which always corresponds to S_+. The first of these, S', for $i_H = 5°$ evidently corresponds exactly to the corresponding one for Case A–B. For each field lying to the left there is a corresponding field lying to the right, which it is also of interest to consider in this special case. The centres of these right-hand fields lie also at points defined by the relation

$$\theta_{H_0} = \sin^{-1}(\sin i_H + \sin 5°),$$

or for $\quad i_H = \ 5° \qquad\qquad 15° \qquad\qquad 30° \qquad\qquad 60°$
$\quad$ (right) $\theta_H = 10° \ 2' \qquad 20° \ 15' \qquad 35° \ 58' \qquad 72° \ 24'$

The edges of the right-hand fields similarly lie at points

$$\theta_H = \sin^{-1}(\sin i_H + \sin 5° \mp \sin 5°),$$

or for different values of i_H

$\quad$ (right) $\theta_{+H} = \ 4° \qquad\qquad 15° \qquad\qquad 30° \qquad\qquad 60°$
$\quad\qquad\ \ \theta_{-H} = 15° \ 10' \qquad 25° \ 40' \qquad 42° \ 24' \quad \text{—}$

For θ_{+H} the aberrations may be computed by ($27a$); for all other points they must be computed by the general formula (8).

The values of the relative aberrations $Z_A/K \ldots\ldots Z_H/K$ for a grating of semi-angular aperture $\beta = \cdot01$, and for the different fields $S' \ldots\ldots S^{iv}$ above considered, are tabulated in Table II. For each point there are in general two values of Z, Z_1, and Z_2, as already defined, which are tabulated separately, although in most cases we are concerned only with the maximum (numerical) values of Z, which for positive values of β (right hand of grating) is always Z_2 except for a few cases which may be noted in the table.

Comparing the relative aberrations in any field for these different cases we see that the most advantageous type of mounting for an objective concave-grating spectroscope is A, for which $\theta = 0$ for S_0, the centre of the field. This form is particularly advantageous for large values of $Nm\lambda$, *i. e.* for large angles of diffraction, because, as shown clearly both by (10) and (16) and by the table, the aberration both at the centre of the field and at the edges *decreases* with increasing values of i. With all of the other mountings, and with the Rowland mounting as well, the aberration *increases* with increasing values of i.

Comparing the aberrations of the concave grating when used as an objective spectroscope with the aberrations of the

TABLE II.

Objective Grating Spectroscope (O.S.) Mounting.

Field	Rowland A₀—B₀ mounting Z_0/K	Case A—B Z_1/K	Case A—B Z_2/K	Case C—D Z_1/K	Case C—D Z_2/K	Case E (F special) θ_K	Case E Z_1/K	Case E Z_2/K	S' for i_H	Case H Left Z_1/K	Case H Left Z_2/K	Case H Right Z_1/K	Case H Right Z_2/K
S'_0 {S+	−·02	−67·7	+71·4	+107·5	−104·1	0° 0'	−137·9	+140·8		−67·8	+71·5	−67·8	+71·5
S'_0 {S₀	−·01	+1·98	+1·98	+36·8	−32·9	5° 0'	−68·2	+71·8	= 5°	+1·98	+1·98	−140	+143
S'_0 {S−	−·02	+71·4	−67·7	−32·9	+36·8	10° 2'	−140·1	+143·0		+71·5	−67·8	−218	+220
S_0'' {S+	−·08	−64·7	+68·2	+178·6	−176·3	9° 53'	−137·9	+140·8		−206	+208	−206	+208
S_0'' {S₀	−·07	+1·83	+1·83	+105·2	−101·9	15° 0'	−217	+219	15°	−132	+135	−290	+289
S_0'' {S−	−·08	+68·2	−64·7	+35·3	−31·5	20° 15'	−305	+304		−62·8	+66·3	−387	+383
S_0''' {S+	−·30	−55·3	+57·9	+128	−125	24° 23'	−384	+381		−404	+397	−404	+397
S_0''' {S₀	−·29	+1·40	+1·40	+201	−199	30° 0'	−502	+493	30°	−308	+305	−527	+513
S_0''' {S−	−·30	+57·9	−55·3	+281	−282	35° 58'	−659	+642		−229	+229	−700	+675
S_0^{iv} {S+	−1·51	+26·6	−25·9	+447	−456	51° 9'	−1349	+1267		−719	+667	−719	+667
S_0^{iv} {S₀	−1·50	+·38	+·38	+345	−348	60° 0'	−2185	+1972	60°	−457	+435	−1819	+1528
S_0^{iv} {S−	−1·51	−25·9	+26·6	+260	−261	72° 24'	−6196	+4712		−330	+318	−700	+675

Case H, S' only (G special).

same grating as mounted and used by Rowland, we have the following results.

In the field S' :—

(1) For mounting A–B the aberration at the centre of the field is nearly 200 times as great as with the Rowland mounting; at the edges of the 10° field the aberrations (maximum) are about thirty-five hundred times larger.

(2) For mounting C–D the aberration at the centre of the field (maximum) is about thirty-seven hundred times as great as with A_0 (R. mounting). At one edge of the 10° field it is about sixteen hundred times larger ; at the other it is over five thousand times larger.

(3) With mounting E the aberration at the centre of the field is seven thousand times larger than with the R. mounting. At one edge it is only twenty-five times larger, but at the other it is again about seven thousand times as great.

(4) With mounting H the aberrations at the centres of the various fields S' vary from the same values already given for the A–B type to a maximum value nearly *two hundred thousand* times as great as that for the Rowland mounting.

For the fields S'', S''', and S^{iv} the comparisons are more favourable to the O.S. types of mounting, both for the centres and for the edges of the fields. In the field S'' for example the ratios

$$\frac{Z_{1-2} \text{ (O.S. mountings)}}{Z_0 \text{ (Rowland mountings)}} = Q$$

run as follows

(1) For the A–B type Q_0'' (centre) $\cong$ 26·0

                        $Q_{\pm\kappa}''$ (edges) $\cong$ 830·0

(2) For the C–D type Q_0'' (max.) $\cong$ 1502·0

                        $Q_{+\kappa}''$ (,,) $\cong$ 441·0

                        $Q_{-\kappa}''$ (,,) $\cong$ 2233·0

(3) For the E type Q_0'' (max.) $\cong$ 3132·0

                        $Q_{+\kappa}''$ (,,) $\cong$ 1760·0

                        $Q_{-\kappa}''$ (,,) $\cong$ 3810·0

For the extreme field S^{iv} the ratios run

(1) A–B type
$$Q_0^{iv} \cong 0.25$$
$$Q_{\pm\kappa}^{iv} \cong 18.7$$

(2) C–D type
$$Q_0^{iv} \cong 232.0$$
$$Q_{+\kappa}^{iv} \cong 174.0$$
$$Q_{-\kappa}^{iv} \cong 304.0$$

(3) E type
$$Q_0^{iv} \cong 1456.0$$
$$Q_{+\kappa}^{iv} \cong 893.0$$
$$Q_{-\kappa}^{iv} \cong 4100.0$$

The more favourable showing for the O.S. types of mounting for large angles of diffraction is due, in the case of the A–B type, to a real improvement in definition already noted ; in the other case the improvement is only apparent and comparative, and not real. For type C–D the total aberrations are about ten times as large for field S^{iv} as for field S', and for type E, about thirty times as large for S^{iv} as for S'.

These comparisons, particularly those for S' and S'', the fields most frequently used with the O.S. types, show very clearly the great inferiority of the concave-grating objective spectroscope, as an optical instrument, to the ordinary form of concave-grating spectroscope, and are a sufficiently complete explanation of the disappointing results that have been secured by its use in solar eclipse work.

Poor and Mitchell considered the star-spectrum plates which they obtained (using the A–B and the F types of mounting) satisfactory, but it must be remembered that such plates afford no such refined tests of definition as plates of solar spectra. Their results are considered more at length later.

Although we are thus forced to the general conclusion that the concave objective grating spectroscope is unsuited to the class of work that can be performed by the same gratings when used on Rowland mountings, the advantages of the former type of instruments are so great in certain cases that it becomes desirable to investigate the question as to whether there is not some way of reducing the aberration in these instruments.

In dealing with a general problem of this nature, there are always three expedients that may be adopted with a greater or less degree of success :—1st, altering the focus of the instrument : 2nd, diaphragming the optical surface of the instrument or limiting the field ; 3rd, altering the form of the reflected wave-front, generally by altering the form of the optical surface itself. Let us take the two most favourable types of O.S. mounting, the A–B and the F forms, and see what can be accomplished in the way of improving the definition by any one or a combination of these means.

1st. Effect of alterations of focus. By examining the Table we find that in every type of O.S. mounting save one, A, the numerical values of Z_1 and Z_2, the aberrations of the two halves of the grating-surface, are different. Since these aberrations (with the single exception noted), are in large part due to the unsymmetrical action of the lateral elements b, b', fig. 1, and are of opposite sign, it follows that a slight improvement in definition may be secured by shifting the focal point, O', by a small quantity du such that

$$du = \tfrac{1}{2}[\text{longitudinal aberration due to } Z_1$$
$$- \text{longitudinal aberration due to } Z_2] \quad . \quad . \quad (31)$$

$$= \frac{K}{2}(Z_1 - Z_2)\frac{3}{\beta_0^2} \;^*$$

$$= \frac{1}{16}\rho \sin^4 \beta (Z_1 - Z_2)\frac{3u^2}{(\rho \sin \beta \cos \theta)^2}$$

$$\simeq 0 \cdot 002 \rho (Z_1 - Z_2)\frac{\cos^2 \theta}{\cos \theta + \cos i}. \quad . \quad . \quad . \quad . \quad (31a)$$

in which the values of Z_1 and Z_2 are to be taken without regard to sign. In most cases the numerical differences, $Z_1 - Z_2$, are so small compared with $Z_{(max.)}$ that no very great improvement can be effected by departing from the focal curve defined by (6). In any case the departures will be very small. Thus in case A–B the numerical differences $Z_1 - Z_2$ at the edges of the field are always less than $0 \cdot 04$. Hence we have for the maximum values of du ($i = 60°$).

$$du_{max.} \simeq 0 \cdot 00005 \rho$$

or, for a Rowland grating of the usual radius of curvature,

$$du_{max.} = \tfrac{1}{3} \text{ mm.}$$

Although no great improvement can be effected by a

readjustment of the focus u in the objective type of mounting, it is interesting to note in passing that since the aberration at the centre of the field ($\theta = 0$) is of the opposite sign for the regular Rowland mounting and the A type of O.S. mounting, there is a certain value of u, which we will call u', intertermediate between the values ρ (for the Rowland mounting) and the value

$$u = \frac{\rho}{1 + \cos \theta}$$

for the A–O.S. mounting, for which the aberration will vanish. We can find this value if we desire by equating the last term of Glazebrook's general equation (3)* to zero. This together with the second general equation for the focal curve of the grating for this case †, *i. e.*,

$$u' = \frac{\rho}{1 + \cos i - \dfrac{\rho}{\nu}\cos^2 i} \quad . \quad . \quad . \quad (32)$$

gives us two equations from which to determine u' and ν, the two conjugate spectral focal distances, so as to satisfy the given condition of zero aberration at the point u' for any given value of i. This solution is not of general, but may become of special interest in some individual cases.

2nd. Effect of diaphragming a portion of the optical surface or limiting the field of the instrument.

Several effects may be here discriminated, but they may all be best investigated by considering the character of the aberration in the lateral fields more in detail. This may be most conveniently done by expressing the aberration at a point $S_{\pm \kappa}$ in terms of the aberration Z_0 at the centre S_0. For cases A–B and F we see from equations (10), (16), and (17)

$$Z_B = Z_A \left[-4 \tan g \, \kappa \frac{1}{\sin \beta \cos i} + \sec^2 \kappa (1 - b \tan g^2 \kappa) \right] \quad (33)$$

or

$$\frac{Z_B}{Z_A} = R.$$

By aid of the Tables already computed for b and $\dfrac{1}{\cos i}$ (Table I.) the values of the ratio R has been determined for seven intermediate points ($\kappa = 5'$ to $\kappa = 3°$) in each of the

* *Loc. cit.* p. 379.
† See Astrophysical Journal, vol. iii. p. 55.

fields S′, S″, S‴, and S^{iv} of cases A–B, and field S of case F.
This has been done for three gratings, one having the same
angular aperture ($\beta_1 = \cdot 01$) as already used, the others having
angular apertures of $\beta_2 = \cdot 025$ and $\beta_3 = \cdot 05$.   The results are
given in Table III.

TABLE III.—Values of R_1 and R_2.

For $\beta = 0.01$.

κ.	$i=0°$ R_1.	R_2.	$i=5°$ R_1.	R_2.	$i=15°$ R_1.	R_2.	$i=30°$ R_1.	R_2.	$i=60°$ R_1.	R_2.
0° 0′	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00
5′	+ 0·44	+ 1·56	+ 0·44	+ 1·56	+ 0·42	+ 1·58	+ 0·35	+ 1·65	− 0·12	+ 2·12
10′	− 0·16	+ 2·16	− 0·16	+ 2·16	− 0·20	+ 2·20	− 0·34	+ 2·34	− 1·32	+ 3·32
15′	− 0·77	+ 2·76	− 0·77	+ 2·77	− 0·82	+ 2·82	− 1·03	+ 3·03	− 2·52	+ 4·52
30′	− 2·48	+ 4·48	− 2·49	+ 4·49	− 2·60	+ 4·60	− 3·02	+ 5·02	− 5·96	+ 7·96
1°	− 5·96	+ 7·96	− 5·99	+ 7·98	− 6·21	+ 8·20	− 7·04	+ 9·04	− 12·92	+ 14·92
2°	− 12·97	+ 14·95	− 13·03	+ 15·01	− 13·46	+ 15·44	− 15·13	+ 17·11	− 26·94	+ 28·90
3°	− 19·98	+ 21·94	− 20·07	+ 22·02	− 20·72	+ 22·67	− 23·23	+ 25·18	− 40·96	+ 42·88
5°	− 34·06	+ 35·94	− 34·20	+ 36·08	− 35·29	+ 37·16	− 39·49	+ 41·36	− 69·11	+ 70·88

For $\beta = 0.025$.

κ.	$i=0°$ R_1.	R_2.	$i=5°$ R_1.	R_2.	$i=15°$ R_1.	R_2.	$i=30°$ R_1.	R_2.	$i=60°$ R_1.	R_2.
0° 0′	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00
5′	+ 0·78	+ 1·22	+ 0·77	+ 1·23	+ 0·77	+ 1·23	+ 0·74	+ 1·26	+ 0·55	+ 1·45
10′	+ 0·54	+ 1·46	+ 0·53	+ 1·47	+ 0·52	+ 1·48	+ 0·46	+ 1·54	+ 0·07	+ 1·93
15′	+ 0·30	+ 1·70	+ 0·29	+ 1·71	+ 0·27	+ 1·73	+ 0·19	+ 1·81	− 0·41	+ 2·41
30′	− 0·39	+ 2·40	− 0·40	+ 2·40	− 0·44	+ 2·44	− 0·61	+ 2·61	− 1·78	+ 3·78
1°	− 1·79	+ 3·78	− 1·80	+ 3·79	− 1·86	+ 3·88	− 2·22	+ 4·21		
2°	− 4·59	+ 6·59	− 4·62	+ 6·60	− 4·82	+ 6·80	− 5·46	+ 7·44	− 10·19	+ 12·15
3°	− 9·41	+ 9·36	− 7·44	+ 9·39	− 7·70	+ 9·66	− 8·71	+ 10·66	− 15·81	+ 17·73
5°	− 13·06	+ 14·94	− 13·12	+ 15·00	− 13·56	+ 15·43	− 15·24	+ 17·10	− 27·12	+ 28·89

For $\beta = 0.05$.

κ.	$i=0°$ R_1.	R_2.	$i=5°$ R_1.	R_2.	$i=15°$ R_1.	R_2.	$i=30°$ R_1.	R_2.	$i=60°$ R_1.	R_2.
0° 0′	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00	+ 1·00
5′	+ 0·89	+ 1·11	+ 0·89	+ 1·11	+ 0·88	+ 1·12	+ 0·87	+ 1·12	+ 0·78	+ 1·22
10′	+ 0·77	+ 1·23	+ 0·77	+ 1·23	+ 0·76	+ 1·24	+ 0·73	+ 1·27	+ 0·54	+ 1·46
15′	+ 0·65	+ 1·35	+ 0·65	+ 1·35	+ 0·64	+ 1·36	+ 0·59	+ 1·41	+ 0·30	+ 1·70
30′	+ 0·30	+ 1·70	+ 0·30	+ 1·70	+ 0·28	+ 1·72	+ 0·20	+ 1·80	− 0·39	+ 2·39
1° 0′	− 0·40	+ 2·39	− 0·40	+ 2·40	− 0·44	+ 2·44	− 0·61	+ 2·60	− 1·79	+ 3·78
2°	− 1·80	+ 3·78	− 1·81	+ 3·79	− 1·90	+ 3·88	− 2·23	+ 4·21	− 4·60	+ 6·57
3°	− 3·21	+ 5·17	− 3·23	+ 5·19	− 3·36	+ 5·32	− 3·86	+ 5·82	− 7·42	+ 9·34
5°	− 6·06	+ 7·94	− 6·09	+ 7·97	− 6·31	+ 8·18	− 7·15	+ 9·01	− 13·12	+ 14·89

As before there are two values of R for each value of Z_B, corresponding to the two sides, right and left, of the grating-surface. These are designated as before by R_1 and R_2. For positive values of κ, R_1 is negative and R_2 is positive. For negative values of κ these quantities are simply interchanged, *i. e.*

$$R_1 \text{ (for } -\kappa) = R_2 \text{ (for } +\kappa)$$
$$R_2 \text{ (}\text{,,}\text{ }\text{,,}\text{)} = R_1 \text{ (}\text{,,}\text{ }\text{,,}\text{).}$$

In general, in considering the prejudical effect of aberration when the whole surface of the grating is used we must take the maximum numerical values of Z and R. But inspection of equation (33) and of the values in Table III. shows that for each *half* of the grating-surface there is a point κ_0 in the field for which the aberration for that half is zero. In the case of the right-hand half (for which the aberration is Z_1) of grating β_1 this point lies between $\kappa = +10'$ and $\kappa = +5'$ for values of i from $0°$ to $30°$, and between $\kappa = +5'$ and $\kappa = 0$ for $i = 60°$. For gratings β_2 and β_3 the point κ_0 is farther from the centre of the field.

The exact location of the point κ_0 for any grating and any value of i is easily found by solving the equation

$$Z_B = 0,$$

or from (16a) or (33)

$$\pm \frac{4}{\sin \beta \cos i} \cdot \text{tang } \kappa = \frac{1 - b \text{ tang}^2 \kappa}{\cos^2 \kappa}. \quad . \quad . \quad (34)$$

Put $\dfrac{4}{\sin \beta \cos i} = f$. Then since κ is by assumption always small, we may readily obtain from (34)

$$\text{tang } \kappa = \mp f/2b \left(1 \mp \sqrt{1 + \frac{b}{4f^2}} \right)$$
$$= \pm \tfrac{1}{4} \sin \beta \cos i \left(1 - \frac{b}{4f^2} \right). \quad . \quad . \quad (35)$$

For grating $\beta_1 = \cdot 01$ we find for κ_0

$$\kappa_0 = \pm (8'\ 40'') \text{ for } i = 0°,$$
$$\kappa_0 = \pm (7'\ 30'') \text{ ,, } i = 30°,$$
$$\kappa_0 = \pm (4'\ 20'') \text{ ,, } i = 60°.$$

For $\beta_3 = \cdot 05$ we similarly find

$$\kappa_0 = (38'\ 40'') \text{ for } i = 0°,$$
$$\kappa_0 = 37' + \text{ ,, } i = 30°,$$
$$\text{and} \qquad \kappa_0 = 21'\ 30'' \text{ ,, } i = 60°.$$

If therefore we diaphragm off the left-hand half of the grating-surface, the spectral image at the point $+\kappa_0$ will be practically free from aberration; if we diaphragm off the right-hand half, the spectral image at the point $-\kappa_0$ will likewise be sharply defined. With gratings of the usual angular apertures we can therefore always secure sharp definition with the A–B–F types of O.S. mounting on any given spectral line λ_0 by suitably choosing the angle i such that

$$\rho \sin \beta \{\sin[\tan g^{-1}(\pm \sin \beta \cos i)] - \sin i\} = mn\lambda_0, \quad . \quad (36)$$

and then covering the right- or the left-hand half of the ruled surface.

For gratings of large angular aperture, such as β_3, this device gives less satisfactory results. The aberration of the centre and extreme elements, a, b, or a, b' (fig. 1) is of course eliminated at the point κ_0, defined by (34) and (35), but the aberration for the intermediate points (which correspond to different values of β) remains. In the case of grating β_3 for example, for which $\kappa_0 = \pm 37'$ at $i = 30°$, the aberration due to the element intermediate between elements a and b will be

$$\tfrac{1}{8}\rho \sin^4 \beta_2 [-6{\cdot}46 \tan g\, 37' + {\cdot}035\,(1 - 9{\cdot}95 \tan g^2 37')\sec^2 37']$$
$$= \tfrac{1}{8}\rho \sin^4 \beta_1 (2{\cdot}5)^3 (-3{\cdot}4) \eqsim 53 \text{ K} ;$$

i. e. the aberration of the intermediate element of the grating β_3 is nearly as great as that of the extreme elements of the grating β_1 at the limit of the 5° field (see Table II. $S''_{+\kappa}$) and over thirty times as great as that of the entire surface of β_1 at the centre of the field S_0''.

Examination of the formulæ (16), (16a), (26), (26a), and (33), or of Table III., shows that for very small values of κ, *i. e.*, very near the axis of the grating, the symmetrical aberration, which is expressed by the second term of these equations, is of the greater numerical value; for larger values, the unsymmetrical aberration expressed by the first term becomes by far the most important. Since the first varies in absolute amount as $\sin^4\beta$ and the second only as $\sin^3\beta$ it at once follows that the *relative* differences over a given field, $\pm\kappa$, become *smaller* as the value of β increases, *i. e.* while the absolute amount of aberration is greater for all parts of the field, the definition, or rather lack of definition, is more uniform.

In considering the effect of a given aberration on the performance of an optical instrument, it is usually assumed

that the " definition " (*sic*) begins to deteriorate when the
aberration exceeds one fourth of the wave-length of light.
Definition is itself rather of an indefinite term in a physical
sense, although it has a well-founded physiological basis.
Of this point it will be necessary to speak later.   Assuming
for the present the limit assumed above, we can determine on
that basis the maximum angular aperture $\beta_{max.}$ and the
maximum field $\kappa_{max.}$ that can be utilized in any given case by
the equation

$$Z \leqq \tfrac{1}{4}\lambda. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

For the case A of the O.S. spectroscope this equation
becomes

$$\rho \sin^4 \beta = \frac{2\lambda}{a}$$

or

$$\sin^3\beta = 2\frac{\lambda}{Wa} = 4\frac{\alpha_0}{a}, \qquad . \quad . \quad . \quad (38)$$

where W is the semi-ruled surface of the grating, and α_0 the
angular resolving-power of a telescope of an aperture equal
to that of the grating.

Equation (38) determines the maximum permissible semi-
angular aperture $\beta_{max.}$ which can be used for any given value
of i.   This varies, as will be seen, inversely as the cube root
of the *linear* aperture, 2W, of the grating, and also inversely
as the coefficient a.   For a grating of the usual linear
aperture $W \cong 72$ mm.   In the position $i = 15°$ we have for
$\beta_{max.}$ in the visual region of the spectrum ($\lambda = 5500$ tenth-
metres)

$$\beta_{max.} = \sqrt[3]{\frac{\cdot 0011}{1\cdot 834 \times 72}} = \cdot 020.$$

For the same grating used in the position $i = 60°$

$$\beta_{max.} = \cdot 034, \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

or since $60°$ is about the maximum permissible angle of
diffraction, (39) is about the maximum permissible semi-
angular aperture of a spherical grating of 15 cm. aperture
used as an objective spectroscope.

Equation (37) also enables us to determine for any given
grating (whose semi-angular aperture is *less* than $\beta_{max.}$), the
maximum permissible field of good definition κ_m.   To find
this we put

$$Z_{B(max.)} = + \frac{\lambda}{4}$$

since the maximum aberration is always positive.

From (33) with the same notation above used we get

$$4a_1 \tan \kappa_m + a \sin \beta (1 - b \tan^2 \kappa_m) = 2 \frac{\lambda}{W \sin^2 \beta},$$

whence

$$\tan \kappa_{\pm m} = \pm \frac{1}{2a_1} \Big(\frac{\lambda}{W \sin^2 \beta} - \frac{a \sin \beta}{2} \Big) \Big[1 + \frac{1}{8} \frac{ab}{a_1^2}$$
$$\Big(\frac{\lambda}{W \sin^2 \beta} - \frac{a \sin \beta}{2} \Big) \Big]. \quad \cdot \quad \cdot \quad (40)$$

The usual Rowland grating is of from 70 to 75 millimetres semi-aperture and 650 to 660 centimetres radius of curvature. For such a grating $\beta \cong \beta_1 = \cdot011+$, and we find for κ_m

$$\begin{rcases} \kappa_{+m} \cong 0^\circ \cdot 8 \cong 50' \text{ for } i = 15^\circ \\ \kappa_{+m} = 2^\circ\ 20' \quad ,, \quad i = 60^\circ \end{rcases} \cdot \quad \cdot \quad \cdot \quad (41)$$

As the value of β increases the available field κ_{+m} of good definition decreases and becomes zero, *i. e.* is confined to the central line of the field, at the point

$$\frac{\lambda}{W \sin^2 \beta} = \frac{a \sin \beta}{2}$$

which gives us the same condition as expressed in (38).

The only way in which we can increase the value of κ_m and β_m is to decrease W. The maximum permissible angular aperture varies inversely as the cube root of the linear aperture, while the maximum field varies very nearly inversely as the first power of the linear aperture. The maximum *spectral* field of good definition varies therefore very nearly as the inverse square of the angular aperture.

3rd. Effect of changing form of diffracted wave-front. The results of the preceding investigations show that the ordinary Rowland concave grating, whose ruled surface is a portion of a sphere, has an extremely restricted field of good definition when used as an objective spectroscope, and that although this field may be somewhat increased by slight readjustments of foci and in some cases by diaphragming off a portion of the ruled surface, the improvement resulting from such changes is small. It remains now to determine whether any better results can be secured by altering the form of the ruled surface or the character of the ruling. To do this we must first derive a general expression similar to that for Z, equation (4), which will express the relative retardation of the extreme ray bo' with reference to the central ray ao' when the light is incident on a surface of any form. This may be done in one of two ways.

First, by determining the value of the differential coefficient dZ/dr as a function of β.

Thus, in fig. 2, let ab be the spherical surface which we

Fig 2.
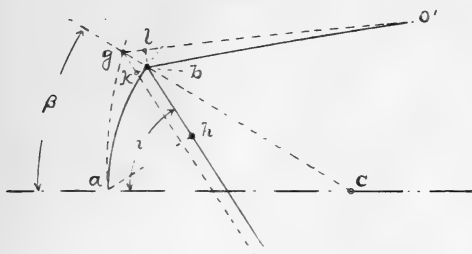
have already considered, and let ag be the new surface for which we wish to determine the aberration. If the central elements, a, of the two surfaces coincide, the *variation* in the difference of path between the central and outer rays will evidently be

$$\Delta Z = (hb + bo') - (hg + go') = kg + gl$$
$$= bg(\cos \lfloor cgk + \cos \lfloor cgl)$$
$$= dr \left[\cos(i-\beta) + \cos\left(\tan g^{-1} \frac{u \sin \theta - \rho \sin \beta}{u \cos \theta - 2\rho \sin^2 \dfrac{\beta}{2}} + \beta \right) \right]. \quad (42)$$

This expression (42) can, after some transformations, be developed in terms of successive powers of β like (3). If then we add the corresponding terms of these two developments together, we shall obtain the expression for the aberration Z' of any surface whose equation, referred to the centre of curvature of the central element, is

$$r = \rho + dr$$
$$= \text{const.} + f(\beta, \theta, i). \quad . \quad . \quad . \quad (43)$$

By making the term in $\sin^3 \beta$ in the general equation for Z' equal to zero we can determine the form of the function $f(\beta, \theta, i)$, *i.e.* the form of the grating-surface which will eliminate the unsymmetrical aberration ; by a similar operation with the term in $\sin^4 \beta$ we can likewise eliminate the symmetrical aberration : it will not, however, be possible in general to satisfy the general conditions for eliminating both kinds of aberration simultaneously.

The method above indicated of determining the form of

grating-surfaces which shall have certain predetermined amounts of aberration is a very useful and general one, and is of special value to the optician because it gives at once the variation dr of the required surface from the true spherical surface ; the latter being the usual basis of operations in the final correction and testing of optical work. The mathematical transformations and developments required in the expansion of (42) are, however, rather tedious and cumbersome, and for that reason will be omitted from this paper. A more simple and elegant, although somewhat less general, solution of the problem is found by expressing the difference in path, $ao' - (bh + hb)$, in terms of the rectangular coordinates of the points b and o', instead of polar coordinates as before. This second method is the one adopted by Plummer *, who has recently extended the investigations of Glazebrook and Rowland to the case of gratings ruled on surfaces other than spherical. Assuming the origin of coordinates at a, the centre of the grating, and expressing the coordinates of the point b by x, y, and of o' by x', y', we have in general

$$ao' - bo' = u - \sqrt{(u \cos \theta - x)^2 + (u \sin \theta - y)^2}. \quad (44)$$

Similarly we have for hb (for parallel incident light),

$$hb = y \sin i - x \cos i,$$

and therefore for any grating whose surface is represented by the equation $x = f(y)$ we have

$$Z' = f(y) \cos i - y \sin i + u$$
$$- \sqrt{u^2 + y^2 + [f(y)]^2 - 2uy \sin \theta - 2uf(y) \cos \theta}. \quad (45)$$

If we assume that the grating-surface may be represented by the equation

$$x = f(y) = ly^2 + my^3 + ny^4,$$

equation (45) may be at once reduced to the form

$$Z' = y(\sin \theta - \sin i)$$
$$+ y^2 \left[l(\cos i + \cos \theta) - \frac{\cos^2 \theta}{2u} \right]$$
$$+ y^3 \left[m(\cos i + \cos \theta) + \frac{l \sin \theta \cos \theta}{u} - \frac{\sin \theta \cos^2 \theta}{2u^2} \right]$$
$$+ y^4 \left[n(\cos i + \cos \theta) + \frac{m}{u} \sin \theta \cos \theta - \frac{l^2}{2u} \right.$$
$$\left. + \left(\frac{\cos^2 \theta}{8u^3} - \frac{l \cos \theta}{4u^2} \right) (1 - 2lu \cos \theta - 5 \sin^2 \theta) \right]^\dagger. \quad (46)$$

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* Dr. Plummer's paper, "Note on the Concave Grating" appeared in the Astrophysical Journal, vol. xvi. Sept. 1902, while the writer was engaged in this investigation.

† This expression may be readily deduced from that given by Plummer by putting $\phi = 0$, $\psi = i$, and $u' = \infty$.

To satisfy the condition that there shall be a spectral image formed at the point o' we must have as before

$$u = \frac{\cos^2 \theta}{2l\,(\cos i + \cos \theta)}. \quad \cdots \quad (47)$$

Substituting this value of u in (46), and reducing, we obtain for the aberration of the right-hand side, ab, of the surface

$$Z_1' = y^3 \left(\frac{\cos i + \cos \theta}{\cos^2 \theta}\right) (m \cos^2 \theta - 2l^2 \sin \theta \cos i)$$

$$+ y^4 \left(\frac{\cos i + \cos \theta}{\cos^2 \theta}\right)\{n \cos^2 \theta + 2lm \sin \theta \cos \theta - l^3 \left[\sin^2 i\right.$$

$$\left. + 4 \tan^2 \theta \cos i\,(\cos i + \cos \theta)\right]\}, \quad \cdots \quad (48)$$

and for Z_2' a similar expression, save that the signs of the terms in l^2 and lm are reversed.

When the grating-surface is spherical, as previously assumed, its equation with reference to the point a is

$$\left.\begin{aligned} x &= \frac{y^2}{2\rho} + \frac{y^4}{8\rho^3} \\ l &= \frac{1}{2\rho}, \quad m = 0, \text{ and } n = \frac{1}{8\rho^3} \end{aligned}\right\}, \quad \cdots \quad (49)$$

or

and (48) reduces at once to the form

$$-\frac{1}{2}\frac{y^3}{\rho^2} \sin \theta \cos i \frac{(\cos i + \cos \theta)}{\cos^2 \theta}$$

$$+ \frac{1}{8}\frac{y^4}{\rho^3}\frac{(\cos i + \cos \theta)}{\cos^2 \theta} \left[\cos^2 i - (\cos \theta + 2 \cos i)^2 \tan^2 \theta\right], \quad (50)$$

which is identical with (8).

The only other form of surface that has thus far been used for gratings is the parabola of revolution. The equation of the curve ab in this case is simply

$$\left.\begin{aligned} x &= \frac{y^2}{2\rho}, \\ \textit{i. e.,} \qquad m &= 0 \text{ and } n = 0. \end{aligned}\right\} \quad \cdots \quad (51)$$

It has been shown qualitatively by Rowland and Rayleigh, and quantitatively by Plummer, that the parabolic surface is distinctly *inferior* to a spherical surface for a grating as mounted and used according to the Rowland method. Let us see whether this is true for the O. S. type of mountings.

For these we have from (48) and (51)

$$Z_1' = \frac{1}{2} \frac{y^3}{\rho^2} \frac{\cos\theta + \cos i}{\cos^2\theta} \sin\theta \cos i$$
$$- \frac{1}{8} \frac{y^4}{\rho^3} \frac{\cos\theta + \cos i}{\cos^2\theta} [\sin^2 i + 4 \tan^2\theta (\cos\theta + \cos i) \cos i]. \quad (52)$$

Comparing this with (50) or (8), we see that the unsymmetrical aberration is just the same in amount and sign for the two surfaces (spherical and parabolic). The symmetrical aberration is, however, *less* for the parabolic surface. This may be more readily seen by putting the second term of (52) in the form

$$\frac{1}{8} \frac{y^4}{\rho^3} \frac{\cos\theta + \cos i}{\cos^2\theta} [\cos^2 i - (\cos\theta + 2\cos i)^2 \tan^2\theta - \cos^2\theta]. \quad (53)$$

The parabolic surface is therefore in general *better* than a spherical surface for a concave grating when the O. S. type of mounting is used. The degree of improvement effected by the change in the form of surface depends on the type of mounting used. For all types for which θ has any appreciable value, the first term of (52) will, as we have already seen, be *large* compared with the second term, and the improvement in definition will be slight. On the other hand, for those images for which θ is zero or very small, the improvement may be very considerable.

As before, the type of mounting which will be most advantageous will be the type A–B, for which $\theta = 0$ for the centre of the field. For these types we have for the parabolic grating

$$Z_A = \frac{1}{8} \frac{y^4}{\rho^3} (1 + \cos i)(\cos^2 i - 1)$$
$$= K[a - (1 + \cos i)] = -Ka \tan^2 i. \quad . \quad . \quad (54)$$

Comparing this with (10) and (10 a), we see that the alteration in the form of the surface has not only decreased the aberration but has altered its sign. The relative values of $\frac{Z_A}{K}$ and $\frac{Z_A'}{K}$ for the different fields, S_0, S_0', S_0'', S_0''', and S_0^{iv}, used in previous comparisons, run as follows :—

For	S_0	S_0'	S_0''	S_0'''	S_0^{iv}	
$\frac{Z_A}{K} =$	2·00	+1·98	+1·83	+1·40	+ ·38	$= a$
$\frac{Z_A'}{K} =$	0·00	− ·015	−0·13	−0·47	−1·17	$= -a \tan^2 i,$
also $\frac{Z_{A_0}}{K} =$	0·00	− ·01	− ·07	− ·29	−1·5	$= a_0.$

For the very centre of the field, therefore, the definition with the parabolic grating is much better than with the spherical grating for small angles of incidence, but inferior to it for large angles of incidence. It is very nearly the same throughout as with a spherical grating on the Rowland mounting.

Substituting for a the value $a \tan g^2 i$ in (38), we find for the limiting value of β,

$$\beta_{\text{max.}} = \sqrt[3]{\frac{2\lambda}{W a \tan g^2 i}}.$$

Hence, for the grating previously considered ($W = 73$ mm.),

$$\left.\begin{array}{ll}
\beta'_{\text{max.}} = 0{\cdot}100, & \text{for } i = 5° \\
\beta'_{\text{max.}} = 0{\cdot}063, & \text{,, } i = 10° \\
\beta'_{\text{max.}} = 0{\cdot}049, & \text{,, } i = 15° \\
\beta'_{\text{max.}} = 0{\cdot}024, & \text{,, } i = 60°
\end{array}\right\} \quad . \quad . \quad (56)$$

which are likewise very near the limiting values for $\beta_{\text{max.}}$ for the Rowland mounting.

When we consider the effect of the unsymmetrical aberration in parts of the field away from the axis of the grating the results are much less satisfactory. The general expression for Z'_B for small values of $\kappa (= \theta)$ becomes

$$Z'_B = -\frac{1}{2}\frac{y^3}{\rho^2}\tan g\,\kappa \cos i\,(1 + \cos i)$$
$$+ \frac{1}{8}\frac{y^4}{\rho^3}(i + \cos i)\{[\cos^2 i - (1 + 2\cos i)^2\tan g^2\kappa]\sec^2\kappa - 1\}. \quad (57)$$

As in the case of (33), this may be put in the form

$$Z'_B = Z_A\left[-\frac{4\tan g\,\kappa}{\cos i \sin \beta} + \sec^2\kappa\,(1 - b\tan g^2\kappa) - \frac{1}{\cos^2 i}\right]$$

$$= \frac{Z'_A}{\tan g^2 i}\left(\frac{1}{\cos^2 i} - R\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

and therefore

$$R' = \frac{Z'_B}{Z'_A} = \left(\frac{1}{\cos^2 i} - R\right)\cot^2 i. \quad . \quad . \quad (59)$$

In order to compare directly the definition of the parabolic and spherical gratings at a given point in the field, it is also desirable to find the ratio between the maximum values of R and R'. Since the maximum value of Z_B is always positive (for positive values of κ), and the maximum value of Z'_B is

always negative (for the same values), we have

$$R'' = \frac{Z'_B}{Z_B} = \frac{R_2 - \dfrac{1}{\cos^2 i}}{R_2}. \quad . \quad . \quad . \quad (60)$$

By the aid of the preceding Table III., which gives the values of R for a spherical grating, we can readily obtain the values of R' for a parabolic grating, and the values of R'' giving the ratio of the aberrations of the two. These values of R' and R'' have been computed for the fields S, S', and S'' for a grating $\beta_3 = \cdot05$, and for fields S''' and S^{iv} for a grating $\beta_2 = \cdot025$. They are tabulated in Table IV. Comparing the values of R' of this Table with those of R of the preceding Table, we see that for values of i less than 45° the *variation* in the amount of the aberration at different parts of the field is *greater* with the parabolic than with the spherical grating, but that the absolute amount, as shown by R'', is everywhere *less*. Above 45° the reverse of both propositions is true. Hence, for large values of i the parabolic grating is inferior in definition to the spherical grating, although the field is *more uniform*; for small values of i the definition of the parabolic grating is much the *better* at the centre of the field, but the field is far *less* uniform.

A further inspection of the values of R'' for these small values of i shows that for values of κ greater than 1° the parabolic grating is not much superior to the spherical grating in definition. The smaller the value of β, the less pronounced is this difference. Hence, for points any distance from the axis of the grating, there is not much to choose between the two.

The maximum field of good definition that can be obtained with any given grating is found as before by putting

$$Z'_{B\,(max.)} = -\frac{\lambda}{4}. \quad . \quad . \quad . \quad . \quad (61)$$

Since in this case the maximum aberration is always negative, this gives for the value of $\kappa_{max.,}$

$$\tang \kappa_{\pm m} = \pm \frac{1}{2a_1}\left(\frac{\lambda}{W\sin^2\beta} - \frac{a\sin\beta}{2}\tang^2 i\right)\left[1 - \frac{1}{8}\frac{ab\sin\beta}{a_1{}^2}\right.$$
$$\left.\left(\frac{\lambda}{W\sin^2\beta} - \frac{a\sin\beta}{2}\tang^2 i\right)\right]. \quad . \quad . \quad . \quad (62)$$

For $i = 0$ (field S) this becomes simply

$$\tang \kappa = \pm \frac{\lambda}{4W\sin^2\beta}\left[1 - \frac{9}{16}\frac{\lambda}{W\sin\beta}\right]. \quad . \quad (63)$$

TABLE IV.

Grating. κ	S . (i=0°)			S' . (i=5°) (β₃=0·05)			S'' . (i=15°)			S''' . (i=30°) (β₂=0·025)			Sⁱᵛ . (i=60°)		
	R₁'*	R₂'*	R''	R₁'*	R₂'*	R''	R₁'*	R₂'*	R''	R₁'*	R₂'*	R''	R₁'*	R₂'*	R''
0° 0'	+1·00	+1·00	±0·000	+ 1·00	+ 1·00	+ ·008	+ 1·00	+ 1·00	- 0·072	+ 1·00	+ 1·00	- 0·333	+ 1·00	+ 1·00	- 3·00
5'	…	…	-0·101	+ 1·57	- 13·6	- 0·108	+ 2·62	- 0·61	- 0·168	+ 1·78	+ 0·22	- 0·470	+ 1·15	+ 0·85	- 2·38
10'	…	…	-0·188	+ 31·5	- 29·4	- 0·195	+ 4·35	- 2·34	- 0·252	+ 2·61	- 0·61	- 0·566	+ 1·31	+ 0·69	- 2·04
15'	…	…	-0·261	+ 47·1	- 45·1	- 0·267	+ 6·07	- 4·07	- 0·320	+ 3·44	- 1·44	- 0·633	+ 1·47	+ 0·53	- 1·83
30'	±∞	±∞	-0·411	+ 92·5	- 90·1	- 0·417	+ 11·1	- 9·03	- 0·461	+ 5·82	- 3·82	- 0·745	+ 1·93	+ 0·07	- 1·53
1°	…	…	-0·583	+ 184	- 181·0	- 0·589	+ 21·1	- 19·0	- 0·622	+ 10·66	- 8·64	- 0·842	+ 2·88	- 0·85	- 1·30
2°	…	…	-0·710	+368	-364	- 0·745	+ 41·4	- 39·1	- 0·767	+ 20·4	- 18·3	- 0·910	+ 4·72	- 2·71	- 1·16
3°	…	…	-0·813	+553	-546	- 0·818	+ 61·8	- 59·2	- 0·834	+ 30·1	- 28·0	- 0·944	+ 6·60	- 4·57	- 1·11
5°	…	…	-0·890	+927	-909	- 0·894	+103	-99·1	- 0·901	+49·7	-47·3	- 0·961	+10·4	-8·29	- 1·09

* The sign of Z_0' is negative. The signs of Z_B' are therefore opposite to those of R'.

Since the second term of the quantity within the brackets is always small compared to unity for any practicable values of β, it may in general be neglected. The only difference therefore between the expression (62) for a parabolic grating and the expression (40) for a spherical grating is in the presence of the factor $\tan^2 i$ as coefficient of the term in $\sin \beta$. This will increase the value of κ for values of i less than 45°, and decrease it for values above 45°. Thus, for the same grating before considered ($W \simeq 73$ mm., $\beta = \cdot011$), we find for κ_m

$$\left.\begin{array}{ll}
\kappa_m = \pm 54', & \text{for } i=0 \\
\kappa_m = \pm 56', & \text{,, } i=15° \\
\kappa_m = \pm 2° \ 10', & \text{,, } i=60°
\end{array}\right\} \quad \cdot \quad \cdot \quad \cdot \quad (64)$$

Comparing these results with those of (40), we see that the effect of parabolizing the surface of this given grating is to increase the field by only about 15 per cent. for small values of i, and decrease it about the same amount for large values of i.

For larger angular apertures the field of good definition is reduced, as we have already seen, in the inverse ratio of the square of the aperture. Thus for a grating of angular aperture $\beta_3 = \cdot05$ the field of good definition is only a little over 5' (2' 35'' each side of the axis) for normal incidence, and it would become zero for a value of i a little less than 15° [see (55)].

For any given angular aperture β the value of κ_m varies also very nearly inversely as the *linear* aperture 2W. Thus, if we decrease the size (aperture and focal length) of the grating one half, we shall double the field. Or, keeping the field the same, we may increase the maximum permissible value of β_{max}. This will vary for any given value of i inversely as the cube root of the linear aperture [see (55)], the field being confined to the image on the axis of the grating. For any smaller values of i the field will be increased, and since κ_m decreases only as the square of the angular aperture, it follows that by decreasing the linear aperture we may simultaneously increase the value of β_{max}. for large values of i, and the values of κ_m for small values of i. Thus, if we decrease the linear aperture of the grating considered in (54) and (64) to, say, 2 cm. ($W = 10$ mm.), we may increase permissible angular aperture 2β to nearly $0 \cdot 10$ ($\beta \simeq \cdot05$) for $i = 60°$, and increase the angular field to about $0° \cdot 6$ ($\kappa_m = 19'$ to $\kappa_m = 17'$) for $i = 0$ to $i = 30°$.

It will also be instructive to compare the preceding results for the limiting values of κ_m and β_m from another point of

view, *i. e.*, from the standpoint of a given resolving-power. As the writer has previously shown *, the resolving-power of a grating of any form may be stated without reference to the ruling of the surface, in terms only of its aperture and total spectral deviation of the diffracted ray. So expressed, we have

$$r = \frac{2W}{\lambda} (\sin i - \sin \theta), \quad . \quad . \quad . \quad (65)$$

or for the A type of mounting above considered,

$$r = \frac{2W}{\lambda} \sin i. \quad . \quad . \quad . \quad . \quad (66)$$

Substituting this in (54) we get

$$\beta_{\max.} = \sqrt[3]{\frac{4 \sin i}{r a \tang^2 i}} = \sqrt[3]{\frac{1}{r} \frac{4}{(1 + \cos i) \sin i}}. \quad (67)$$

The general statement made in the previous paper (Astrophysical Journal, vol. iv. p. 61) is therefore rigorously correct only for certain values of *i*. It is, however, well within the truth for the maximum values of *i* and *r* there mentioned, viz., $i = 10°$ and $r = 30,000$. For these values of *i* and *r* we get for $\beta_{\max.}$

$$\beta_{\max.} = \sqrt[3]{\cdot 000387} \simeq \cdot 073,$$

or for the angular aperture at the principal focus o',

$$\beta_0 \simeq 2\beta_{\max.} (1 + \cos i) \simeq \frac{1}{3 \cdot 5} \quad (\text{for } i < 15°),$$

which is even larger than the limit $\beta_0 = \frac{1}{5}$ there imposed. The maximum linear aperture, however, must in this case not exceed 95 mm. ($W_{\max.} = 48$ mm.), if *r* is taken as 30,000 for $\lambda = 5500$ tenth-metres, as before.

Conversely, if we use a maximum angular aperture of 1/5 at the principal focus, $\beta \simeq \cdot 05$, and we can, without injury to the definition, use a maximum resolving-power $r_{\max.}$ equal to

$$\frac{4}{\sin^3 \beta (1 + \cos i) \sin i} \quad . \quad . \quad . \quad . \quad (68)$$

$$= \frac{3200}{(1 + \cos i) \sin i}, \quad . \quad . \quad . \quad (68\,a)$$

or for $i = 10°$, as in the case just considered,

$$r_{\max.} \simeq 93,000,$$

* Phil. Mag. vol. xliii. p. 319.

and the maximum linear aperture for wave-length $\lambda = 5500$ tenth-metres is correspondingly

$$2W_{max.} = 294 \text{ mm.,}$$

from which it appears that if we use angular apertures as large as 1/5 only, we can use linear apertures as large as previously assumed, *i. e.* 30 cm., and resolving-powers at least three times larger (90,000 as against 30,000), without injury to definition, *provided*, first, we use only small angles of incidence, and provided, second, we confine ourselves to points *very close* to the axis of the grating. My preceding statement was incomplete, in not more carefully pointing out these restrictions.

It is the violation of the above conditions, particularly the second, that has caused trouble in the experimental work referred to at the beginning of this paper. If we express the limiting value of κ in terms of the resolving-power, we shall have from (62) and (66) (neglecting the small variation in r with κ),

$$\tan \kappa_{max.} = \frac{\sin i}{r a_1 \sin^2 \beta} - \frac{1}{4} a_0 \sin \beta, \quad . \quad . \quad (69)$$

For any given value of i the field of good definition will depend on the value of r. For $r = r_{max.}$, as defined by (68), $\kappa = 0$; *i. e.*, the only well-defined line is that at the centre of the field. *To obtain field we must sacrifice resolving-power or reduce the angular aperture.*

If we make $r = 30,000$ and retain the same values of β and i as already used, we find for κ

$$\kappa_{max.} = \cdot 0008 \backsimeq 3'.$$

In order to increase the total field of good definition to $1°$, we must decrease r to about 4000 units, *i. e.* to the resolving-power of a single prism-spectroscope of about 4 cm. aperture.

If we put (66) in the form

$$\tan \kappa_{max.} = \tan i \left(\frac{1}{r \sin^2 \beta \, (1 + \cos i)} - \frac{1}{4} \sin i \sin \beta \right) \quad (69\,a)$$

we see that for a given resolving-power the maximum field of good definition increases very nearly proportionally to $\tan i$. Hence, if large fields are necessary or desirable, large angles of incidence are advantageous, a point that is also clearly brought out, although not in such definite proportional form, by the individual examples already considered in (40) and (64). These examples also show that in such cases spherical gratings are *better* than parabolic gratings. But if large angles of i are used, large values of β are not

permissible with either spherical or concave gratings [see (38), (55), (56), and (67)],—although in this case also the spherical grating has a decided advantage,—unless again we use very small resolving-powers.

We are thus confronted by two difficulties in the use of this form of objective spectroscope, and it seems impossible to overcome both simultaneously, except by sacrificing resolving-power *. This general conclusion is quite in harmony with the experimental results referred to at the beginning of the paper, with the single exception of those obtained by Mitchell. These require some further consideration. As already stated, Mitchell used a grating having a linear aperture of 146 mm. and a semi-angular aperture, β, of ·073. He used the B and F types of mounting, and worked with plates about 127 mm. long. The value of i for the B mounting was about 15°, and the value of u [from (6 a)] would therefore be about 508 mm. The plates (bent to a radius of about 254 mm.) would cover a field of over 14°, or 7° on each side of the axis of the grating. The semi-angular aperture β here exceeds the maximum limit imposed by the usual criterion of good definition for a grating of this size by more than 50 per cent. [see (56)]. Hence, judged by this standard, there would be no point in the field for which the definition would be good. Mitchell in his paper, however, states that the plates showed "excellent definition," presumably over the entire field. Either the preceding conclusions are radically wrong, or Mr. Mitchell's standard of "excellent definition" is decidedly different from that imposed by (37). As I shall attempt to show, the latter is not an impossible or even improper explanation of the discrepancy. Definition is, as has already been stated, a function of psychological and physiological phenomena, as well as of strictly physical conditions. As such it is strictly relative, and depends largely for its quantitative estimation on the basis of comparison employed. The latter again differs, or should differ, with the instrument used. If this basis of comparison is less exacting than it should be, then an aberration amounting to very much more than a quarter

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* Certain special solutions of the general equations (42), (43), and (48) have been found which theoretically enable us to greatly reduce the value of Z'. No such solution has yet been found, however, that does not require us to satisfy certain mechanical and optical conditions of construction and mounting that are either impossible or highly impracticable. Some of these solutions, and some other points of interest that have been developed in connexion with them, will be given in another paper, "On the most Efficient Form of Surface for Concave Gratings and Concave Mirrors for Spectroscopes," now in course of preparation.

of a wave-period may pass unnoticed. On this point Lord
Rayleigh * very pertinently remarks :—" The best results will
be obtained with an aperture . . . giving an extreme aberra-
tion of from a quarter to half a period . . . with an increased
aperture," (with proportionally increased aberration), " aber-
ration *is not so much a direct cause of deterioration as an
obstacle to the attainment of that improved definition which
should accompany the increase of aperture.*" In other words,
there will be no discrepancy between Mitchell's standard
of " excellent definition " over a 7° field and the conditions
of equation (37), if the resolving-power of his instrument did
not exceed 210 units †. *On the same basis of comparison,*
the definition would be equally " excellent " with an instru-
ment of the resolving-power actually used (about 85,000),
although the aberration would be enormously increased. In
this connexion it should be noted that the character of the
spectra photographed was such that it would be natural to
apply a less rigorous standard of definition to the results than
would be applied, for example, to solar-spectrum plates. As
the writer has previously shown ‡, a resolving-power of not
more than 2000 is required to obtain as good definition on
bright-line hydrogen-spectra as is obtained with a power of
400,000 on fine-line absorption-spectra from the sun.

Fortunately there is a much more definite and impersonal
standard of performance of an optical instrument than that
of definition. This is the standard of accuracy or metro-
logical power, which has recently been investigated in some
detail by the writer §. As has been there shown, the limiting
metrological power ϵ of an instrument is about one-fifteenth
as great as the limit of resolving-power α, or the limit of
accuracy A is

$$A \simeq 15r.$$

For the grating used by Mitchell the value of A is about
1,270,000, and the limit of metrological power ϵ is therefore
about 0·004 tenth-metre. This should be the mean or
" brute " error (not the probable error) of an individual
setting on a spectrum-line, with a grating of this resolving-
power used under the best optical conditions.

* " Influence of Aberration," Phil. Mag. vol. viii. p. 403, Nov. 1879.
† Obtained by putting $\kappa=7°$ and $\beta=·073$ in (69), and solving for r.
‡ Phil. Mag. vol. xliii. p. 336.
§ " On the Optical Conditions required to secure Maximum Accuracy
of Measurement in the Use of the Telescope and Spectroscope," Astro-
physical Journal, vol. xvi. p. 267 ; vol. xvii. pp. 1 and 100, Dec. 1902
and Jan.–Mar. 1903.

An examination of the two tables of wave-length determinations made by Mitchell with his instrument shows that the mean error of measurement was very much greater than this. In the case of the wave-length of the series of hydrogen-lines in the spectrum of Rigel, the mean error of *three* determinations on each of the twenty-two lines (H_β to head of series) as compared with the theoretical wave-lengths calculated from Balmer's law is about 0·120 tenth-metre. No data are given from which the individual errors of setting on each line can be determined, but they must necessarily have been *greater* than this. In the case of the wave-length measurements of the lines in the nebula of Orion the mean error as determined by a comparison of the two series of measurements is about 0·480 tenth-metre. Here also each determination tabulated is presumably (although it is not so stated) the mean of several settings, and if so, the mean error of individual settings would again be much greater than that given. Even without any allowance for such an increase, the errors of setting as they appear are respectively about thirty times and one hundred and twenty times what they would be if the full optical power of the grating were being realized.

In the case of Table II. of Mitchell's paper, there is further evidence that the grating was not giving the results which one of this aperture should attain under proper optical conditions. There is a group of three well-separated lines in the nebula of Orion which, according to Wright, have wave-lengths $\lambda = 3970\cdot24$; $3967\cdot69$; and $3965\cdot1$; and intensities of 5, 3, and 1 respectively. These lines were clearly resolved and accurately measured to about 0·01 tenth-metre by Wright with the Mills spectrograph, which has a theoretical resolving-power in this region of the spectrum practically equal to that of Mitchell's instrument. On Mitchell's list only one of these lines—the brighter one—appears at all, and the error of measurement on this amounts to between 0·6 and 0·7 tenth-metre.

These comparisons of Mr. Mitchell's work are instituted, not because of any wish to criticize the work itself, *per se*, but only to show that the general conclusions reached in this paper by the mathematical analysis are not incompatible with his experimental results. There now seems to be no question but that both Dr. Poor and Mr. Mitchell, in common with Mohler and Daniel and Jewell, have considerably overrated the advantages of the direct concave-grating spectroscope, or, as the writer prefers to call it, the concave objective-grating spectroscope. I am not myself in a position to

criticize these views very strongly, for the reason that, as pointed out at the beginning of the paper, I appear to be the first to have published the general theory of the instrument and to have advocated its use. I can only say that since then my own views have been decidedly modified by a further experimental and mathematical investigation of the capabilities of the instrument. After the publication of Poor and Mitchell's first paper, I pointed out some of the objections to the instrument as used by them in photographing stellar spectra, on the score of its low light efficiency as compared with other forms of stellar spectrographs, particularly the transmission (wire) objective grating *. It was not until later, in the use of the instrument in solar eclipse work, that its limitations in the directions of resolving-power and photographic field were clearly recognized. This led me, as already stated, to the present investigation. It now appears from this that perhaps the most serious objection of all to the instrument is in its lack of accuracy. This point has been very briefly touched upon in this paper and in the one on "Conditions of Accuracy, &c." † already referred to, but will be more fully developed in a paper dealing with a discussion of Frost's and Humphrey's results on the flash-spectra ‡.

In conclusion we may quote with a considerable degree of pertinency to the present question the remark of Schuster with reference to the prism-spectroscope. He says (' Enc. Brit.' vol. xxii. p. 373) :—" The dispersion and magnifying-power of a prism can be considerably altered by a change in its position, and a knowledge of this fact is of great value to an experienced observer. The use of a prism in a position different from that of minimum deviation is, however, *a luxury which only those acquainted with the laws of optics can indulge in with safety.*" Change " prism " to grating, and " minimum deviation " to Rowland's mounting, and the applicability of Schuster's statement to the case we have been discussing is very evident.

Allegheny Observatory,
    1901–1902.

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* See " Note on the Use of the Grating in Stellar Spectroscopic Work," Astrophysical Journal, vol. vii. p. 198; also " The Objective Spectroscope," *ibid.* vol. iv. p. 75 *et seq.*

† Astrophysical Journal, vol. xvi. pp. 286–288. On p. 288 there is a typographical error which is corrected in the March number of the Journal. The values of θ_{max} as given in the December number are ten times too large, *i. e.*, in place of $2^\circ\ 20'$ and 6° the values of θ_{max} should be $0^\circ\cdot22$ and $0^\circ\cdot6$.

‡ " On Measurements of Wave-Length with the Concave Grating Objective Spectroscope," Astrophysical Journal, vol. xviii. p. 1, July 1903.

XVII. *On the Pressure of Radiation.   By* T. H. HAVELOCK,
*B.A., M.Sc., Scholar of St. John's College and Isaac
Newton Student in the University of Cambridge* [*].

SINCE Maxwell's suggestion of the existence of a pressure
due to radiation [†], interesting theoretical applications
have been made in the thermodynamics of radiation [‡] ; and
more recently the pressure has been demonstrated experi-
mentally by Lebedew [§].

These circumstances lead to a consideration of the theory
of the pressure, and this naturally depends upon the view
taken of the mechanical action in the electric field. This has
been supposed by some [||] to be expressible explicitly in terms
of stress, but in the following work we use the analysis given
by Larmor [¶]. We consider first the simpler case of waves
along a stretched string, and then proceed to electrical
radiation. The leading idea is that the pressure of the
waves, or at least the part which can be measured as
mechanical action, is the average effect of a bodily force
integrated through an absorbing medium ; in other words,
the pressure is primarily due to absorption of energy, and
this operates by introducing a difference of phase between
the electric and magnetic factors in the expression for the
mechanical force.

Waves along a String.

§ 1. We begin by considering the transverse vibrations
of a stretched string, unlimited in length along the axis of x;
W being the mean constant tension, y the displacement and θ
the inclination to Ox of an element of the string. If F be
the force per unit length along Ox on an element δx, we
have

$$F = \frac{\partial}{\partial x}(W \cos \theta) = \frac{\partial}{\partial x}\left[W \cdot \frac{1}{\left\{ 1 + \left(\frac{\partial y}{\partial x}\right)^2 \right\}^{\frac{1}{2}}} \right]$$

$$= W \left\{ -\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x^2} + \text{terms of higher order} \right\} \quad . \quad (1)$$

[*] Communicated by Prof. J. J. Thomson, F.R.S.
[†] Maxwell, El. and Mag. vol. ii. § 792.
[‡] Rayleigh, Phil. Mag. 1902, vol. iii. p. 338.
[§] Lebedew, *Ann. der Physik*, 1901, vol. vi. p. 433.
[||] Goldhammer, *Ann. der Phys.* 1901, vol. iv. p. 834.
[¶] Larmor, Phil. Trans. 1897, A 190, p. 253.

Now the potential energy of the vibrations is given by

$$V = \tfrac{1}{2}W \int \left(\frac{\partial y}{\partial x}\right)^2 dx. \quad . \quad . \quad . \quad . \quad (2)$$

Thus, denoting by V_x the density of potential energy at any point, we have

$$F = -\frac{\partial}{\partial x}(V_x). \quad . \quad . \quad . \quad . \quad (3)$$

And the mean extra force on any length of the string due to the vibrations is equal to the difference of the mean values of V_x at the ends.

Hence for a simple progressive wave along the string there is on the average no extra force, and for simple waves in both directions there is a mean force which alternates in direction along the string.

But consider the case in which there is attenuation in a progressive wave; so that

$$y = Ce^{-\alpha x} \cos{(pt - \beta x)}. \quad . \quad . \quad . \quad (4)$$

The mean value of V_x now is

$$\tfrac{1}{4}(\alpha^2 + \beta^2)W \cdot C^2 \cdot e^{-2\alpha x}. \quad . \quad . \quad . \quad (5)$$

Hence in this case there is a mean force along the string in the direction of propagation of the waves.

§ 2. We may consider in detail the following example :—
From $x = -\infty$ to $x = 0$, there is no frictional dissipation of energy, and the equation of motion of the string is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}. \quad . \quad . \quad . \quad . \quad (6)$$

But from $x = 0$ to $x = \infty$ there is a frictional term and the equation is of the form

$$\frac{\partial^2 y}{\partial t^2} + K \frac{\partial y}{\partial t} = a'^2 \frac{\partial^2 y}{\partial x^2}. \quad . \quad . \quad . \quad (7)$$

Then if there is a progressive wave along the first part, there will be reflexion and absorption at $x = 0$. Suitable solutions of (6) and (7) for the two regions are

$$y = A \cos p\left(t - \frac{x}{a}\right) + B \cos\left\{ p\left(t + \frac{x}{a}\right) + \delta \right\}. \quad . \quad (8)$$

$$y = Ce^{-\alpha x} \cos\{(pt - \beta x) + \delta'\}, \quad . \quad . \quad . \quad . \quad (9)$$

where the first term in (8) represents the incident wave, the

second term the reflected wave with a change of phase δ, and (9) represents the transmitted wave.

The conditions at $x=0$ are continuity of y and $\dfrac{\partial y}{\partial x}$; these give

$$\left.\begin{array}{l} A + B\cos\delta = C\cos\delta' \\ B\sin\delta = C\sin\delta' \\ A - B\cos\delta = C(\alpha'\sin\delta' + \beta'\cos\delta') \\ B\sin\delta = C(\alpha'\cos\delta' - \beta'\sin\delta') \end{array}\right\}, \quad \cdot \quad (10)$$

where

$$(\alpha', \beta') = \frac{a}{p}(\alpha, \beta).$$

These equations lead to

$$\tan\delta = \frac{2a'}{1 - a'^2 - \beta'^2}; \quad \tan\delta' = \frac{a'}{1 + \beta'}, \quad \cdot \quad \cdot \quad (11)$$

$$\left(\frac{B}{A}\right)^2 = \frac{a'^2 + \beta'^2 + 1 - 2\beta'}{a'^2 + \beta'^2 + 1 + 2\beta'}. \quad \cdot \quad \cdot \quad \cdot \quad (12)$$

Now we require the total extra force on the part of the string from 0 to ∞. V_x is continuous, and we integrate from a point just to the left of the origin up to a point at an infinite distance to the right; at the latter limit V_x vanishes, hence the total force is given by the mean density at $x=0$ of the potential energy of the vibrations given by (8). We have

$$\text{Mean density of } V \text{ at } x = \tfrac{1}{2}W \,.\, \text{Mean}\left(\frac{\partial y}{\partial x}\right)^2$$

$$= \tfrac{1}{2}W \,.\, \frac{p^2}{a^2} \,.\, \text{Mean}\left[A\sin p\left(t - \frac{x}{a}\right) - B\sin\left\{p\left(t + \frac{x}{a}\right) + \delta\right\}\right]^2$$

$$= \tfrac{1}{4}W \,.\, \frac{p^2}{a^2}\left[A^2 + B^2 - 2AB\cos\left(2\frac{p}{a}x + \delta\right)\right]. \quad \cdot \quad \cdot \quad \cdot \quad (13)$$

Thus

$$\text{Mean density of } V \text{ at } 0 = \tfrac{1}{4}W\frac{p^2}{a^2}(A^2 + B^2 - 2AB\cos\delta). \quad (14)$$

Also the mean density of V along the string, that is, with respect to x from $-\infty$ to 0, is from (13) equal to

$$\tfrac{1}{4}W\frac{p^2}{a^2}(A^2 + B^2);$$

and this is equal to $\tfrac{1}{2}E$, where E is the mean density of the total energy, potential and kinetic.

Thus from (14) we have

$$F = \tfrac{1}{2} \cdot \frac{A^2 + B^2 - 2AB\cos\delta}{A^2 + B^2} \cdot E, \quad . \quad . \quad (15)$$

and from equations (10) we find

$$F = \frac{a'^2 + \beta'^2}{1 + a'^2 + \beta'^2} E. \quad . \quad . \quad . \quad . \quad (16)$$

We may regard this force as the pressure at O of the vibrations in the string to the left of the origin, though on the above analysis it is in reality an integrated bodily force on the string to the right of the origin.

§ 3. There is a limiting case to which we may approach by supposing a' to increase indefinitely; the limits to which the different quantities approximate are

$$\delta = 0 \; ; \; \delta' = 0 \; ; \; A + B = C = 0 \; ; \; \text{and} \; F = E \quad . \quad . \quad (17)$$

This would be the case of perfect reflexion at a fixed point O, and we see that the pressure would then be equal to the mean density of the energy; but this is only an ideal limit which cannot actually be reached.

Electric Waves.

§ 4. To proceed now to electrical radiation, we use the circuital relations in the form

$$\left.\begin{aligned} \frac{4\pi}{c}(u, v, w) &= \text{Curl}\,(a, \beta, \gamma) \\ -\frac{1}{c}\frac{\partial}{\partial t}(a, \beta, \gamma) &= \text{Curl}\,(X, Y, Z) \end{aligned}\right\}, \quad . \quad . \quad (18)$$

where (a, β, γ) is the magnetic force in electromagnetic units, (X, Y, Z) is the electric force measured electrostatically, and c is the velocity of propagation of effects in the free æther.

In general the total current is given by

$$(u, v, w) = \frac{\epsilon}{4\pi}\frac{\partial}{\partial t}(X, Y, Z), \quad . \quad . \quad . \quad (19)$$

where

$$\epsilon = n^2(1 - ik)^2, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

n being the refractive index and k the coefficient of absorption.

Consider plane-polarized waves propagated along Ox, so that we have

$$\frac{\epsilon}{c}\frac{\partial Y}{\partial t} = -\frac{\partial\gamma}{\partial x}; \quad -\frac{1}{c}\frac{\partial\gamma}{\partial t} = \frac{\partial Y}{\partial x}. \quad . \quad . \quad (21)$$

In this case the mechanical bodily force is along Ox, and is given by

$$F = \frac{1}{c} v' \gamma, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (22)*$$

where

$$v' = \text{true current}$$

$$= v - \frac{1}{4\pi} \frac{\partial Y}{\partial t}.$$

Thus

$$F = -\frac{1}{4\pi} \gamma \frac{\partial \gamma}{\partial x} - \frac{1}{4\pi c} \gamma \frac{\partial Y}{\partial t}.$$

And if all the quantities vary as $e^{i\kappa ct}$,

$$F = -\frac{1}{4\pi} \gamma \frac{\partial \gamma}{\partial x} - \frac{1}{4\pi \kappa^2 c^2} \frac{\partial^2 Y}{\partial x \partial t} \frac{\partial Y}{\partial t}$$

$$= -\frac{\partial}{\partial x} \left[\frac{\gamma^2}{8\pi} + \frac{1}{8\pi \kappa^2 c^2} \left(\frac{\partial Y}{\partial t} \right)^2 \right]. \quad \cdot \quad \cdot \quad \cdot \quad (23)$$

Now if all the interfaces are perpendicular to Ox, γ and Y are continuous throughout, and we see that the total mean force per unit area upon any plate is equal to the difference at the faces of the mean value of

$$\frac{\gamma^2}{8\pi} + \frac{1}{8\pi \kappa^2 c^2} \left(\frac{\partial Y}{\partial t} \right)^2. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (24)$$

Consider an interface between transparent media ; then if through the transition layer we have

$$\int_1^2 dx = i ; \qquad \int_1^2 \epsilon \, dx = p ;$$

the boundary conditions become

$$\left. \begin{array}{l} Y_1 = Y_2 - \dfrac{l}{c} \dfrac{\partial \gamma_2}{\partial t} \\[2mm] \gamma_1 = \gamma_2 + \dfrac{p}{c} \dfrac{\partial Y_2}{\partial t} \end{array} \right\}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (25)$$

and the differences are only of the order of the thickness of the transition layer. Thus the only case in which there is a mean bodily force along Ox which sums up to a finite quantity is when there is absorption of energy and consequent attenuation of the waves. Consider an absorbing medium bounded on the left by free æther and extending on the right to

* Larmor, *loc. cit.*

infinity; γ and Y will vanish at the latter limit, and we see from (24) that we may regard the mechanical force upon the medium as a pressure upon the surface equal to the mean value there of

$$\frac{1}{8\pi}\gamma^2 + \frac{1}{8\pi}Y^2,$$

that is, equal to the mean density of the energy of the vibrations in front of the surface.

§ 5. This case may be worth considering directly. We have in the free æther to the left of the origin

$$\left.\begin{array}{l} Y = A\cos\kappa(ct-x) + B\cos\{\kappa(ct+x) + \delta\} \\ \gamma = A\cos\kappa(ct-x) - B\cos\{\kappa(ct+x) + \delta\} \end{array}\right\}, \quad . \quad (26)$$

representing the incident and reflected waves.

And to the right of the origin we have

$$\left.\begin{array}{l} Y = Ce^{-nk\kappa x}\cos\{\kappa(ct-nx) + \delta'\} \\ \gamma = nCe^{-nk\kappa x}[\cos\{\kappa(ct-nx) + \delta'\} + k\sin\{\kappa(ct-nx) + \delta'\}] \end{array}\right\} \quad (27)$$

representing the transmitted wave.

The surface conditions at $x = 0$ are continuity of the tangential components of æthereal displacement and magnetic force; these give

$$\left.\begin{array}{l} A + B\cos\delta = C\cos\delta', \\ B\sin\delta = C\sin\delta', \\ A - B\cos\delta = nC(\cos\delta' + k\sin\delta'), \\ B\sin\delta = nC(k\cos\delta' - \sin\delta'). \end{array}\right\} \quad . \quad . \quad (28)$$

And these lead to

$$\tan\delta = \frac{2nk}{1 - n^2(1 + k^2)}; \qquad \tan\delta' = \frac{nk}{1 + n};$$

$$\left(\frac{B}{A}\right)^2 = \frac{n^2(1 + k^2) + 1 - 2n}{n^2(1 + k^2) + 1 + 2n}.$$

Also the true current v' is given by

$$v' = \text{Real part of } \frac{\epsilon - 1}{4\pi}\frac{\partial Y}{\partial t}$$

$$= \text{Real part of } \frac{n^2(1 - ik)^2 - 1}{4\pi}i\kappa cCe^{i\kappa\{ct - xn(1-ik)\}}$$

$$= \frac{\kappa c}{4\pi}Ce^{-nk\kappa x}[2n^2k\cos\{\kappa(ct-nx) + \delta'\} - \{n^2(1-k^2) - 1\}\sin\{\kappa(ct-nx) + \delta'\}]. \quad . \quad (29)$$

Thus from (22), (27), and (29) we find that the mechanical bodily force has a mean value given by

$$\frac{1}{2}\frac{nk\kappa}{4\pi}\{1 + n^2(1 + k^2)\}C^2 e^{-2nk\kappa x}. \quad . \quad . \quad (30)$$

Integrating from $x = 0$ to $x = \infty$, we see that the total force per unit area is given by

$$F = \frac{1}{16\pi}\{1 + n^2(1 + k^2)\}C^2. \quad . \quad . \quad . \quad (31)$$

If **E** be the mean density of total energy, electric and magnetic, of the vibrations to the left of the surface, we have

$$E = \frac{1}{8\pi}(A^2 + B^2). \quad . \quad . \quad . \quad . \quad (32)$$

And from equations (28) we find

$$F = E. \quad . \quad . \quad . \quad . \quad . \quad (33)$$

§ 6. Regarding then, as we have done, the pressure of radiation as an integrated mechanical effect which only sums up to a finite quantity on the average when there is absorption, we find that this pressure is equal to the mean density of energy in front of the surface whatever be the values of n and k for the medium provided the latter is not zero. If the medium is perfectly transparent, the forcive is wholly periodic, and has no effect on the average. The result also holds for a plate of any medium if the thickness and the value of k are so related that the vibrations are practically extinct before reaching the second surface; and, of course, it holds for the limiting case of perfect reflexion to which we may approximate by supposing k to increase indefinitely.

§ 7. By considering only the average effect of the forcive to be measurable as mechanical pressure, we avoid some difficulties in the thermodynamic applications; for regarding the pressure in front of a perfectly reflecting wall as periodic with double the period of the radiation, Wien * has suggested a violation of the Second Law by supposing the wall to vibrate with the period of the pressure. In any case this exception to the Law would be of the same order as Maxwell's suggested violation.

§ 8. It is interesting to compare the result in (33) with that given in (16). In the case of the vibrating string there is only one medium; but in electrical radiation we have two media to consider, the continuous æther and the material

* Wien, Wied. *Ann.* lii. p. 150 (1894).

substance constituted of systems of point charges. If in the latter case instead of (22) we had taken the mechanical force to be given by $v\gamma/c$, where v is the total current, we should have obtained a different result ; in fact, working this out, we find instead of (33)

$$F = \frac{n^2(1+k^2)}{1+n^2(1+k^2)} \, E, \quad \cdots \cdots \quad (34)$$

which is quite analogous to (16).

§ 9. We shall consider now the general case of oblique incidence, the plane of separation being the plane of yz. Then in the absorbing medium to the right of this plane there is a bodily force along Ox given by

$$F = \frac{1}{c}\left(v - \frac{\partial g}{\partial t}\right)\gamma - \frac{1}{c}\left(w - \frac{\partial h}{\partial t}\right)\beta + f'\frac{\partial X}{\partial x} + g'\frac{\partial X}{\partial y} + h'\frac{\partial X}{\partial z} \quad (35)*$$

where

$$\left.\begin{array}{l}
(f, g, h) = \dfrac{1}{4\pi}(X, Y, Z) \\[2mm]
(f', g', h') = \dfrac{\epsilon - 1}{4\pi}(X, Y, Z)
\end{array}\right\} \quad \cdots \cdots \quad (36)$$

There is also a surface traction toward the absorbing medium given by $-2\pi n^2$, where n is the normal component of the material polarization $(f', g', h')*$.

Using the general circuital relations (18) and (19), we find that for transverse waves in the medium F reduces to

$$F = -\frac{1}{8\pi}\frac{\partial}{\partial x}(\alpha^2 + \beta^2 + \gamma^2) - \frac{1}{8\pi\kappa^2 c^2}\frac{\partial}{\partial x}\left\{\left(\frac{\partial X}{\partial t}\right)^2 + \left(\frac{\partial Y}{\partial t}\right)^2 + \left(\frac{\partial Z}{\partial t}\right)^2\right\}. \quad (37)$$

Then, as before, integrating F along Ox from a point just inside the surface to a point at which the disturbance vanishes, and adding the surface traction, we see that the mechanical action upon the medium along Ox may be expressed as a pressure on the surface given by

$$p = \frac{1}{8\pi}\left[\alpha^2 + \beta^2 + \gamma^2 + X^2 + Y^2 + Z^2 - (\epsilon - 1)^2 X^2\right] \quad . \quad (38)$$

where the mean values of the terms are taken just inside the surface. Now α, β, γ, ϵX, Y, and Z are continuous at the surface ; hence

$$p = \frac{1}{8\pi}\left[\alpha^2 + \beta^2 + \gamma^2 + Y^2 + Z^2 + \frac{2-\epsilon}{\epsilon}X^2\right] \quad \cdots \quad (39)$$

where the values of the forces are those just outside the surface.

* Larmor, *loc. cit.*

It is interesting to verify by this method the case of perfect reflexion ; for then p approaches the value given by

$$p = \frac{1}{8\pi} \left[\alpha^2 + \beta^2 + \gamma^2 + Y^2 + Z^2 - X^2 \right]. \quad . \quad . \quad . \quad (40)$$

Let the angle of incidence be θ, and let the amplitude of the incident and reflected waves be given by

$$\left. \begin{array}{ll} X_1 = A_1 \sin \theta ; & X_2 = A_2 \sin \theta \\ Y_1 = A_1 \cos \theta ; & Y_2 = -A_2 \cos \theta \\ Z_1 = B_1 ; & Z_2 = B_2 \end{array} \right\} . \quad . \quad . \quad . \quad (41)$$

Then the corresponding magnetic forces are

$$\left. \begin{array}{ll} \alpha_1 = -B_1 \sin \theta ; & \alpha_2 = -B_2 \sin \theta \\ \beta_1 = -B_1 \cos \theta ; & \beta_2 = B_2 \cos \theta \\ \gamma_1 = A_1 : & \gamma_2 = A_2 \end{array} \right\} . \quad . \quad . \quad (42)$$

At the surface Y and Z are zero ; giving

$$A_1 - A_2 = 0 ; \quad B_1 + B_2 = 0. \quad . \quad . \quad . \quad . \quad (43)$$

Hence from (40)

$$p = \frac{1}{4\pi} (A_1{}^2 + B_1{}^2) \cos^2 \theta$$

$$= 2E_1 \cos^2 \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where E_1 is the mean density of total incident energy.

For radiant energy incident equally in all directions, we have

$$\text{Total incident energy} = \int E_1 dw = 2\pi E_1.$$

$$\text{Total pressure} \quad = \int 2E_1 \cos^2 \theta \, dw$$

$$= \int_0^{\frac{\pi}{2}} 2E_1 \cos^2 \theta \, . \, 2\pi \sin \theta \, d\theta$$

$$= \tfrac{2}{3} \, . \, 2\pi E_1.$$

Thus the total pressure is two-thirds of the whole incident energy ; and as there is perfect reflexion, this is equal to one-third of the total energy, incident and reflected, in front of the surface. And this is the form which is generally used in thermodynamic applications.

XVIII. *The Influence of Stiffness on the form of a Suspended Wire or Tape.* By RICHARD C. MACLAURIN, *M.A., LL.M., Fellow of St. John's College, Cambridge, Professor of Mathematics, Wellington, N.Z.*[*]

SOME of the greatest improvements in modern surveying are due to the substitution of a steel tape or wire for the old surveyor's chain. The newer instrument can, with proper precautions, be made an exceedingly accurate measurer of distances. So minute have been the corrections applied in some recent surveys that it has been questioned whether we may, with propriety, any longer regard the form of the curve in which the " chain " hangs as a catenary. It is true that the surveyor's tape is so thin as to be very flexible, but for some purposes there are advantages in using a circular wire, which is, of course, more rigid than the tape for the same weight. It may be thought that, at any rate for the circular wire, the hypothesis of perfect flexibility (on which the investigation of the form of the ordinary catenary rests) may introduce an error comparable with those for which corrections are applied in the best modern surveys. The object of this paper is to settle the matter by investigating the correction that must be applied when the rigidity of the wire or tape is taken into account.

Writing down the equations of equilibrium of a small element of the chain in the usual way, we have :—

$$\frac{d\mathrm{T}}{ds} - \mathrm{U}\frac{d\psi}{ds} - w\sin\psi = 0, \quad \dots \quad (1)$$

$$\mathrm{T}\frac{d\psi}{ds} + \frac{d\mathrm{U}}{ds} - w\cos\psi = 0, \quad \dots \quad (2)$$

$$\frac{d\mathrm{L}}{ds} + \mathrm{U} \qquad = 0. \quad \dots \quad (3)$$

Moreover, $\mathrm{L} = \mathrm{EI}/\rho = \mathrm{EI}\dfrac{d\psi}{ds}$, which with (3) gives

$$\mathrm{EI}\frac{d^2\psi}{ds^2} + \mathrm{U} = 0. \quad \dots \quad (4)$$

In these equations T is the tension, U the shear, L the bending moment, s the length of the curve measured along the arc from some fixed point, ψ the angle that the tangent makes with the horizontal, $1/\rho$ the curvature, w the weight of the chain per unit length, E Young's modulus, and I

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* Communicated by the Author.

the moment of inertia of the cross-section of the chain about a line through its centre of gravity perpendicular to the plane in which the chain hangs. The equations give

$$T = \frac{w \cos \psi - \dfrac{dU}{ds}}{\dfrac{d\psi}{ds}} = \frac{w \cos \psi + EI \dfrac{d^3\psi}{ds^3}}{\dfrac{d\psi}{ds}} = \frac{w \cos \psi + EI \dddot{\psi}}{\dot{\psi}}$$

where the . denotes differentiation with respect to s.

Hence

$$\frac{dT}{ds} = -w \sin \psi - \frac{w \cos \psi \ddot{\psi}}{\dot{\psi}^2} + EI \frac{\ddddot{\psi}\dot{\psi} - \dddot{\psi}\ddot{\psi}}{\dot{\psi}^2}$$

But from (1) and (4)

$$\frac{dT}{ds} = w \sin \psi + U\dot{\psi} = w \sin \psi - EI\dot{\psi}\ddot{\psi}$$

$$\therefore \cos \psi \frac{\ddot{\psi}}{\dot{\psi}^2} + 2 \sin \psi = \frac{EI}{w}\left[\dot{\psi}\ddot{\psi} + \frac{\ddddot{\psi}\dot{\psi} - \dddot{\psi}\ddot{\psi}}{\dot{\psi}^2} \right] \quad . \quad (5)$$

which is the differential equation of the curve in which the chain hangs. When the chain is perfectly flexible the right-hand side of (5) is zero, and we get

$$\cos \psi \frac{\ddot{\psi}}{\dot{\psi}^2} + 2 \sin \psi = 0.$$

Whence

$$\frac{\ddot{\psi}}{\dot{\psi}} = -2 \tan \psi . \dot{\psi} \; ; \; 1/\rho = \dot{\psi} = \frac{\cos^2 \psi}{c},$$

which, of course, represents the common catenary.

In the surveyor's tape or wire the flexibility is so nearly perfect that we may proceed to solve (5) by approximation, substituting in the right-hand side of that equation the value $\dot{\psi} = \dfrac{\cos^2 \psi}{c}$ obtained by neglecting the rigidity.

$$\dot{\psi} = \frac{\cos^2 \psi}{c},$$

$$\ddot{\psi} = -\frac{2 \sin \psi \cos^3 \psi}{c^2}.$$

$$\dddot{\psi} = -\frac{2}{c^3} \cos^4 \psi [\cos^2 \psi - 3 \sin^2 \psi],$$

$$\ddddot{\psi} = \frac{24 \sin \psi \cos^5 \psi}{c^4}[\cos^2 \psi - \sin^2 \psi].$$

$$\therefore \; \dot{\psi}\ddot{\psi} + \frac{\ddddot{\psi}\dot{\psi} - \dddot{\psi}\ddot{\psi}}{\dot{\psi}^2} = \frac{6}{c^3} \sin \psi \cos^3 \psi [5 \cos^2 \psi - 2].$$

Thus (5) becomes

$$\frac{\overset{\cdots}{\psi}}{\dot{\psi}} + 2 \tan \psi \cdot \dot{\psi} = \frac{6EI}{wc^3} \cos^3 \psi \left[5 \cos^2 \psi - 2\right] \sin \psi \cdot \dot{\psi}.$$

Integrating we get

$$\log \frac{c\dot{\psi}}{\cos^2 \psi} = \frac{6EI}{wc^3} \int \cos^3 \psi (5 \cos^2 \psi - 2) \sin \psi \, d\psi$$

$$= \frac{2EI}{wc^3} \cos^3 \psi (2 - 3 \cos^2 \psi)$$

$$\therefore \dot{\psi} = \frac{\cos^2 \psi}{c} \cdot e^{\frac{2EI}{wc^3} \cos^3 \psi \, (2 - 3 \cos^2 \psi)}.$$

It will be seen in the sequel that $\frac{EI}{wc^3}$ is an exceedingly minute quantity, and neglecting squares and higher powers of this we have, by the exponential theorem,

$$\frac{d\psi}{ds} = \dot{\psi} = \frac{\cos^2 \psi}{c} \left[1 - \frac{2EI}{wc^3} \ (3 \cos^2 \psi - 2)\right] \quad . \quad . \quad (6)$$

$$\therefore s = c \int \frac{d\psi}{\cos^2 \psi} + \frac{2EI}{wc^2} \int (3 \cos^2 \psi - 2) \cos \psi \, d\psi$$

$$= c \tan \psi + \frac{2EI}{wc^2} \sin \psi \cos^2 \psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is the intrinsic equation of the curve.

If x be the horizontal distance corresponding to s, then x is the quantity to be calculated from an observed value of s. We have :—

$$\frac{dx}{ds} = \cos \psi \ \therefore \ x = \int \cos \psi \, ds = c \int \frac{d\psi}{\cos \psi} + \frac{2EI}{wc^2} \int (3 \cos^4 \psi - 2 \cos^2 \psi) d\psi$$

Hence

$$x = c \log_e \frac{1 + \sin \psi}{\cos \psi} + \frac{EI}{4wc^2} \left[\psi + \tfrac{1}{2} \sin 2\psi (1 + 6 \cos^2 \psi)\right] \quad . \quad (8)$$

$$= c \sinh^{-1}(\tan \psi) + \frac{EI}{4wc^2} \left[\psi + \tfrac{1}{2} \sin 2\psi (1 + 6 \cos^2 \psi)\right]. (9)$$

The quantity $\frac{EI}{4wc^2} \left[\psi + \tfrac{1}{2} \sin 2\psi (1 + 6 \cos^2 \psi)\right]$ is not the correction to be applied when the rigidity is taken into account. The value of ψ is affected by the stiffness, so that $c \sinh^{-1}$ $(\tan \psi)$ will be different when the stiffness is regarded than in the case of perfect flexibility. We shall afterwards

develop a formula from which the correction can be calculated more readily than from (9), but before doing so some other results will be obtained.

It is no longer the case, as with the common catenary, that $T \cos \psi = wc$ and $T \sin \psi = ws$, for we have now to take account of shear and bending moment as well as of tension. All these quantities may be calculated with the aid of the equations already obtained.

Thus from (4) and (6) we get :—

$$L = \frac{EI\,d\psi}{ds} = \frac{EI \cos^2 \psi}{c}\left[1 - \frac{2EI \cos^3 \psi}{wc^3}(3 \cos^2 \psi - 2)\right]$$

$$= \frac{EI \cos^2 \psi}{c} \text{ approximately.}$$

$$U = -\frac{dL}{ds} = \frac{2EI}{c} \sin \psi \cos \psi \frac{d\psi}{ds}$$

$$= \frac{2EI \sin \psi \cos^3 \psi}{c^2}\left[1 - \frac{2EI \cos^3 \psi}{wc^3}(3\cos^2\psi - 2)\right]$$

$$= \frac{2EI \sin \psi \cos^3 \psi}{c^2} \text{ (to the same order of approximation).}$$

$$\frac{T}{w} = \frac{\cos \psi - \dfrac{1}{w}\dfrac{dU}{ds}}{\dfrac{d\psi}{ds}}$$

$$= \left[\cos \psi - \frac{2EI \cos^4 \psi}{wc^3}(\cos^2 \psi - 3 \sin^2 \psi)\right]$$

$$\times \left[\frac{c}{\cos^2 \psi} + \frac{2EI \cos \psi}{wc^3}(\cos^2 \psi - 2 \sin^2 \psi)\right]$$

$$= c \sec \psi + \frac{2EI}{wc^2} \sin^2 \psi \cos^2 \psi.$$

To this order of approximation we have :—

$$\left(\frac{T}{w}\right)^2 - s^2 = c^2 \sec^2 \psi + \frac{4EI}{wc}\sin^2 \psi \cos \psi - c^2 \tan^2 \psi - \frac{4EI}{wc}\sin^2 \psi \cos \psi$$

$$= c^2(\sec^2 \psi - \tan^2 \psi)$$

$$= c^2.$$

Hence to this order c is unaltered by the rigidity.

For some purposes it is convenient to introduce the auxiliary θ, such that

$$\frac{T}{w} = c \cosh \theta ; \quad s = c \sinh \theta.$$

These satisfy the relation just proved, viz., $\left(\dfrac{T}{w}\right)^2 - s^2 = c^2$, and they give $\tanh \theta = \dfrac{sw}{T}$ from which θ is determined when the weight of the chain and the tension at the end are known. In the unsymmetrical case when the tensions at the ends of the chain are not equal we have these equations to deal with :—

$$\frac{T_1}{w} = c \cosh \theta_1 \; ; \; s_1 = c \sinh \theta_1 \; ; \; \frac{s_1 w}{T_1} = \tanh \theta_1. \quad (\alpha)$$

$$\frac{T_2}{w} = c \cosh \theta_2 \; ; \; s_2 = c \sinh \theta_2 \; ; \; \frac{s_2 w}{T_2} = \tanh \theta_2, \quad (\beta)$$

$$\left(\frac{T_1}{w}\right)^2 - s_1^2 = c^2 = \left(\frac{T_2}{w}\right)^2 - s_2^2,$$

$$\therefore \frac{T_1^2 - T_2^2}{w^2} = (s_1 - s_2)(s_1 + s_2). \quad \ldots \quad (\gamma)$$

Either $s_1 - s_2$ or $s_1 + s_2$ is given, and when either is known the other can be calculated from (γ). Thus s_1 and s_2 can be found, and then θ_1 and θ_2 determined by means of (α) and (β). The horizontal distance x corresponding to any value of θ is very readily obtained if a table of hyperbolic functions is available. We have :—

$$\cosh \theta = \frac{T}{wc} = \sec \psi + \frac{2EI}{wc^3} \sin^2 \psi \cos^2 \psi$$

and in the terms of the first order we may put $\cos \psi = \operatorname{sech} \theta$; $\sin \psi = \tanh \theta$.

Thus we get

$$\cos \psi = \operatorname{sech} \theta + \frac{2EI}{wc^3} \frac{\sinh^2 \theta}{\cosh^5 \theta},$$

$$\frac{dx}{d\theta} = \frac{dx}{ds} \cdot \frac{ds}{d\theta} = \cos \psi \cdot c \cosh \theta$$

$$= c + \frac{2EI}{wc^3} \frac{\sinh^2 \theta}{\cosh^5 \theta}$$

$$\therefore \quad x = c\theta + \frac{2EI}{wc^3} \int \frac{\sinh^2 \theta}{\cosh^5 \theta} d\theta.$$

$$\int \frac{\sinh^2 \theta \, d\theta}{\cosh^5 \theta} = \int \frac{z^2 dz}{(1 + z^2)^3} \text{ where } z = \sinh \theta$$

$$= \frac{1}{8}\left[\tan^{-1} z + \frac{z}{1 + z^2} - \frac{2z}{(1 + z^2)^2} \right]$$

$$\therefore \quad x = c\theta + \frac{EI}{4wc^2}\left[\tan^{-1}(\sinh \theta) + \frac{\sinh \theta}{\cosh^2 \theta} - \frac{2 \sinh \theta}{\cosh^4 \theta} \right] \quad (10)$$

c and θ are unaltered by the rigidity, so that (to our order of approximation) the correction to be applied for stiffness is

$$\frac{EI}{4wc^2}\left[\tan^{-1}(\sinh\theta) + \frac{\sinh\theta}{\cosh^2\theta} - \frac{2\sinh\theta}{\cosh^4\theta}\right].$$

Putting $\sinh\theta = \tan\phi$, the correction is

$$\frac{EI}{4wc^2}\left[\phi - \frac{\tan\phi(2-\sec^2\phi)}{\sec^4\phi}\right].$$

It should be noticed that ϕ in this formula is not quite the same as ψ in (9).

We shall use this formula to find the correction for stiffness when a ten-chain steel tape or circular wire is suspended symmetrically with a tension of fourteen pounds at the ends.

Steel Tape.—Width $= \frac{1}{8}$ in. ; thickness $= \frac{1}{60}$ in.; $E = 3\times10^7$; 10·1 inches weighed 2·584 grammes $\therefore w = \dfrac{2\cdot584 \times 2\cdot2046}{10\cdot1\times10^3}$ pounds.

$$\mathrm{Sin}\,\phi = \tanh\theta = \frac{sw}{T} = \frac{3960\times2\cdot584\times2\cdot2046}{10\cdot1\times10^3\times14}$$

Whence

$$\phi = 9°\,11' = 0\cdot1603 \text{ radian}\,;\ \ \tan\phi = 0\cdot1617\,;\ \sec\phi = 1\cdot0130$$

$$c = \frac{s}{\tan\phi} = \frac{3960}{\cdot1617}$$

$$\frac{EI}{4wc^2} = \frac{3\times10^7\times10\cdot1\times10^3\times(\cdot1617)^2}{3\times8\times16\times(60)^3\times2\cdot584\times2\cdot2046\times(3960)^2}$$
$$= \cdot00000107$$

$$\phi - \frac{\tan\phi(2-\sec^2\phi)}{\sec^4\phi} = 0\cdot1603 - \frac{0\cdot1617(2-1\cdot0130^2)}{(1\cdot0130)^4}$$
$$= 0\cdot0107.$$

Hence the correction for stiffness in ten chains

$$= 2\times0\cdot00000107\times0\cdot0107 \text{ inch}$$
$$= 0\cdot000000023 \text{ in.}$$

This is only 0·184 inch in a million miles, so that the neglect of the stiffness of the chain need cause little anxiety to the surveyor.

Circular Wire.—We shall consider next one of the steel wires used in Jädern's experiments.

Diameter 2·66 mm. ; $w = 0\cdot04$ kilogramme per metre. Take $E = 3\times10^7$ and $T = 14$ pounds as before.

$$\sin\phi = \frac{sw}{T} = \frac{3960\times\cdot04\times2\cdot2046}{14\times39\cdot37}.$$

Whence

$$\phi = 39^\circ\, 19' = 0.6860 \text{ radian} ; \ \tan\phi = 0.8190 ; \ \sec\phi = 1.2926$$

$$\frac{\text{EI}}{4wc^2} = \frac{3 \times 10^7 \times 3.1416 \times (1.33 \times .03937)^4 \times 39.37 \times (.8190)^2}{16 \times .04 \times 2.2046 \times (3960)^2}$$

$$= .0008463$$

$$\phi - \frac{\tan\phi(2 - \sec^2\phi)}{\sec^4\phi} = 0.6860 - \frac{0.8190(2 - \overline{1.2926^2})}{(1.2926)^4}$$

$$= 0.5894.$$

Hence the correction for stiffness in ten chains

$$= 2 \times 0.0008463 \times 0.5894 \text{ inch}$$
$$= 0.0009976 \text{ in.}$$

This is slightly less than eight inches in a thousand miles.

If we wish to compare the corrections for two chains which differ only in the form of their cross-sections, we notice that if w, s, and T are the same, then ϕ and c are the same. The material being the same, E is also the same, so that the corrections are proportional to the moments of inertia of the cross-sections, *e. g.*, if a circular wire were made of the same area of cross-section as the tape considered above, the correction for the stiffness of the wire would be about seven times that for the tape.

It may be interesting to compare the correction for stiffness with that for stretching. If α be the area of cross-section, ds the unstretched length of an element of the chain, ds' the stretched length, we have by Hooke's law

$$\text{T} = \text{E}\alpha \frac{ds' - ds}{ds} \qquad \therefore \frac{ds'}{ds} = 1 + \frac{\text{T}}{\text{E}\alpha}. \qquad \text{Also } w\,ds = w'\,ds'.$$

The equations of equilibrium are $\text{T}\cos\psi = wc$, and $\text{T}\sin\psi = w's' = ws$;

$$\therefore \ s = c\tan\psi = c\sinh\theta ; \ \text{T} = wc\cosh\theta,$$

$$s' = \int\left(1 + \frac{\text{T}}{\text{E}\alpha}\right)ds = \int\left(1 + \frac{wc}{\text{E}\alpha}\cosh\theta\right)c\cosh\theta\,d\theta$$

$$= c\sinh\theta + \frac{wc^2}{2\text{E}\alpha}\left[\theta + \frac{\sinh 2\theta}{2}\right]$$

$$x = \int\cos\psi\,ds' = \int\frac{1}{\cosh\theta}\left[1 + \frac{wc}{\text{E}\alpha}\cosh\theta\right]c\cosh\theta\,d\theta$$

$$= c\theta + \frac{wc^2}{\text{E}\alpha}\sinh\theta.$$

The correction for stretching is

$$\frac{wc^2}{E\alpha}\sinh\theta = \frac{ws \cdot c}{E\alpha}$$

$$= \text{weight of chain} \times \frac{c}{E\alpha}.$$

For the ten-chain steel tape the correction is

$$\frac{\cdot0005653 \times 7920 \times 3960 \times 8 \times 60}{\cdot1617 \times 3 \times 10^7}$$

$$= 1\cdot757 \text{ inches.}$$

For the ten-chain circular wire (Jädern's) the correction is

$$\frac{\cdot04 \times 2 \times (3960)^2 \times 2\cdot2046}{39\cdot37 \times \cdot8190 \times 3 \times 10^7 \times 3\cdot1416 \times (1\cdot33 \times \cdot03937)^2}$$

$$= 0\cdot33182 \text{ inch.}$$

In any case the correction depends on the area of the cross-section of the wire, but not on its form.

XIX. *On a Simple Rheostat.* By G. F. C. Searle, *M.A., University Lecturer in Physics, Demonstrator in Experimental Physics, Cavendish Laboratory, Cambridge*[*].

1. IN many electrical experiments it is necessary to adjust the electric current flowing in a circuit to some definite value, and this is most easily done by varying the resistance of the circuit. But for accurate adjustment the resistance must be capable of *continuous* variation, and not merely of variation in steps, as is the case with a resistance-box. This continuous variation of the resistance is generally brought about by making a gradual alteration in the length of a piece of german-silver or platinoid wire included in the circuit, and the object of the present communication is to describe a rheostat in which the alteration is secured in a simple and effective manner.

The figure shows the general appearance of the rheostat. At the top of an upright there is a wooden wheel with a **V**-groove in its edge, the wheel turning about a horizontal axis, while at the foot of the upright there is a **U**-tube of glass containing a little mercury. The total length of the **U**-tube is about 30 cms. and the internal diameter of the tube is about ·8 cm. An endless band about 165 cms. in length, which passes over the wheel and through the **U**-tube,

is made up mainly of a piece of platinoid wire ·015 inch
(·038 cm.) in diameter. One end of the platinoid wire is
soldered to one end of a piece
of thin "flexible" stranded
wire L, while at the other end
of the platinoid wire a small
eye is formed. To complete
the band, one end of a piece
of silk cord is lashed to the
flexible wire near the soldered
joint, and the other end of the
cord is tied to the eye at the
end of the platinoid wire. By
means of the device described
below, the endless band is kept
tight enough to cause the
wheel to grip the band suf-
ficiently firmly to ensure that
the band shall not slip in the
groove when the wheel is
turned. Then, by turning the
wheel, the length of the plati-
noid wire included between the
surface of the mercury in the
right limb of the U-tube and
the end of the flexible wire
can be varied from practically
zero to about 165 cms., the
consequent change in resistance
being about 6 ohms. The other
end of the flexible wire L is
attached to a terminal screw
T, and L is long enough to
enable the joint to pass freely
over the wheel.

A second metallic connexion
with the platinoid wire is ob-
tained by a copper electrode,
which dips into the mercury
in the left limb of the U-tube.
This electrode is made in the
shape of a narrow strip of
sheet copper with a wider end,
like a spade. The wide part
is bent round so as to form a
nearly complete cylinder, which just fits into the glass tube,
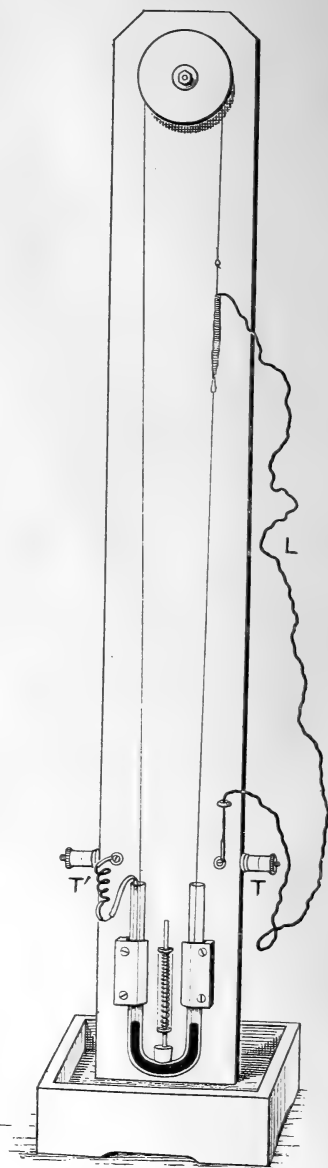
and the cylindrical part is amalgamated. This electrode is connected by a short piece of *flexible* wire with a terminal screw T′.

The object of making the effective part of the platinoid wire to enter the **U**-tube by the *right* limb and the copper electrode to dip into the *left* limb, is to avoid the accidental contacts between the platinoid wire and the part of the copper electrode above the surface of the mercury, which I found occurred when the electrode dipped into the right limb.

The radius of the wheel is somewhat greater than the radius of the bend of the **U**-tube, in order that the wire may not be in contact with the glass at the level of the surface of the mercury.

Since the joint in the endless band is thicker than the rest of the band, it is necessary, in order to allow the joint to pass easily over the wheel, to give the **U**-tube a little play in a vertical direction. The tube, therefore, slides quite freely through some simple wooden guides, and the requisite tension is given to the band by a plunger, which presses upon the bend of the **U**-tube and is actuated by a small coiled spring. The tube can thus rise slightly when the joint passes over the wheel, and the wire is always kept sufficiently tight and is never over-strained.

A rim $2\frac{1}{2}$ cms. high round the base board forms a trough, which catches the few small globules of mercury which are carried up by the wire and are sometimes shaken off it.

The resistance of the rheostat in a given position of the wire is very definite, and violent tapping or shaking has no effect upon it.

As mercury attacks solder, care must be taken not to wind the joint into the mercury. I find that the mercury also attacks german-silver so rapidly that a german-silver wire completely loses its strength after a few minutes' immersion in the mercury. On the other hand, so far as I can judge by three months' experience, platinoid wire is quite unaffected by the mercury.

Since the wire must be quite flexible if it is to draw easily round the bend of the **U**-tube, its thickness is limited, and hence this type of rheostat is not adapted for use with large currents. Yet a platinoid wire ·015 inch in diameter will carry 2 amperes without undue heating, and the heating leads to no inconvenience, for the wire, being exposed to the air, takes up almost instantly a steady temperature corresponding to the current passing through it.

The figure represents a specimen of the rheostat made for me by Messrs. W. G. Pye and Co., to whom I am indebted for some help in the details of the design.

XX. *On the Point-Discharge in Mixtures of Gases.*
By KARL PRZIBRAM, *Ph.D.* (*Graz*) *.

THE observations which led to the following investigation were made whilst experimenting on the point-discharge in different gases. The apparatus used in these experiments consisted of a thin platinum wire whose distance from a fixed circular metal plate could be altered at will, the whole being inclosed in a glass vessel, which could be filled with the gas to be investigated. A Wimshurst machine was used and the potential-difference measured with a Braun's electrometer. It was found, in accordance with the results of Warburg †, Precht ‡, and others, that the potential-difference of the point-discharge was greater when the point was positive than when it was negative. This holds for air, CO_2 and H_2. In CO_2 the potential-difference when the point is positive is considerably greater than in the same case for air, being for instance 4400 volts for air, 5200 for CO_2, the distance between the electrodes being in both cases 5 mm. If the point is negative, the potential-difference in CO_2 was found to be slightly less § than in air, 3700 volts for air, 3500 for CO_2.

When the apparatus had been filled with CO_2, and the gas afterwards replaced by air, then, the point being positive, the electrometer gave readings quite different to those previously obtained in air ; the values for the potential-difference were in these cases always too small. This was found to be due to traces of CO_2 that had not been removed. As one would expect an admixture of CO_2 rather to increase the potential-difference, the potential-difference in pure CO_2 being higher than that in pure air, the point-discharge in a mixture of air and CO_2 in varying proportions was investigated. The vessel used contained a platinum wire of 0·09 mm. diameter fused into a glass tube, so that only a few mm. were left free, inside a platinum cylinder of 54 mm. height and 15 mm. diameter, with a rectangular opening to allow the luminosity at the point to be observed. When the vessel contained air (dried over $CaCl_2$ and freed from dust by a plug of cotton-wool) and the point was positive, the luminosity was restricted to a small star just at the point. But if one now introduced a

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small amount of CO_2 into the vessel, the star immediately stretched out into a thin, steady brush without ramifications, which reached in a continuous curve to the wall of the surrounding cylinder. At the same time the electrometer showed an appreciable fall in the potential-difference.

The relation of the effect to the amount of CO_2 introduced was next investigated. A measured quantity of air was pumped out of the vessel and CO_2 let in until the original pressure was re-established. Thus a series of observations reaching from pure air to pure CO_2 was obtained, and also one in the reversed order, starting from CO_2 and introducing always more and more air. The approximate equality of the potential-differences in mixtures of equal composition obtained in these two ways showed that the gases had really mixed homogeneously ; to effect this one had to wait some time. Removing the CO_2 contained in atmospheric air by passing it through caustic potash had no appreciable effect on the discharge, this quantity being too small. But a quantity of CO_2 corresponding to something like 1 mm. of pressure produces the long brush and a decided fall in the potential-difference from 4500 to 4200 volt. If more CO_2 is introduced the brush contracts until, in pure CO_2, there is again only a star visible at the point ; the potential-difference rises rapidly until for about 20 mm. of CO_2 it reaches the same value as in pure air ; then it increases, if one allows for the experimental errors, nearly proportional to the amount of CO_2 present until it reaches the value of 5500 volts for pure CO_2.

Thus it seems that a gas in which the point-discharge requires a higher potential-difference than in air, lowers the potential-difference in air if introduced in small quantities, the electric strength—if this term may be applied to the point- as well as to the spark-discharge—being smaller in the mixture than in either of the constituents.

Lowering the pressure lessens the effect ; the long brush disappears, and the curve giving the connexion between the potential-difference and the quantity of CO_2 gets more smooth, until the minimum can no longer be observed.

Also when the point is negative the mixture of air and CO_2 shows a remarkable behaviour. In this case the introduction of a small amount of CO_2 causes the potential-difference to rise[*] from 3700 to 4400 volts. For the interval

[*] Warburg (*loc. cit.*) found a very large increase in the potential-difference of the negative point-discharge in nitrogen by the presence of very small traces of oxygen, but then the potential-difference for pure oxygen is higher than that for pure nitrogen, whilst in the case of CO_2 it is lower than for air, and yet addition of a little CO_2 increases the potential-difference in air for the negative point-discharge.

between 100 and 600 mm. of CO_2 the potential-difference seems nearly independent of the composition of the mixture and then drops down to 3600 volt in pure CO_2. There is no alteration in the luminosity, which, in this case, remains a mere star at the point, somewhat brighter than when the point is positive.

These facts having been established for CO_2, a number of other gases were tested and in most cases the same results were obtained. The lowering of the potential-difference for the positive point-discharge and the appearance of a long brush was observed in mixtures of

CO_2, Cl_2, CO, NH_3, H_2S with air, and of CO_2 with H_2.

All these gases gave about the same drop of potential of 200 to 300 volts. The effect could not be observed in mixtures of N_2 with H_2, and of Cl_2 with CO_2.

The figures show the curves giving the connexion between the potential-difference, in units of 100 volts, and the partial pressure of the mixture, in cms. of mercury. Fig. 1 is for a
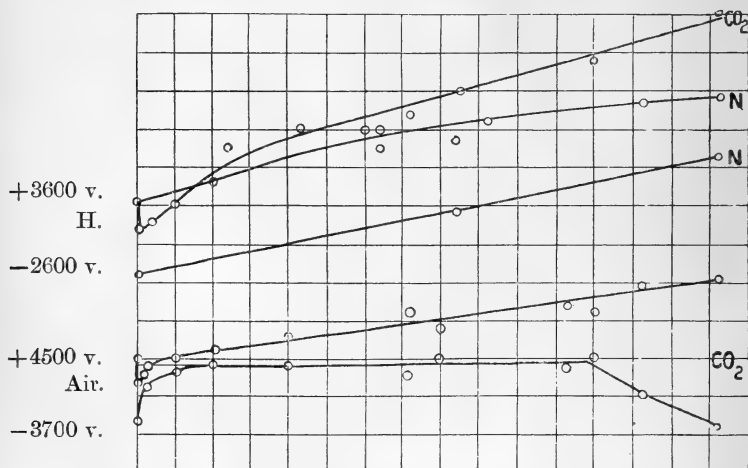
Fig. 2.

Fig. 1.

mixture of air and CO_2. Fig. 2 for hydrogen and nitrogen and CO_2 respectively.

In all cases the light from the brush was too weak to allow spectroscopical analysis. But if the gas admitted was CO, the colour appeared decidedly different to the naked eye, the brush having a greenish tint, whilst it was more violet in all other cases.

A number of organic vapours were tried next. Air which had been standing over the liquid concerned, so as to get saturated, was blown in varying quantities into the vessel.

The formation of a positive brush was observed on introducing vapours of CS_2, ether, ethyl alcohol, benzole ; acetone, no effect, or only a very slight one, was produced by chloroform and methyl iodide. Taking into account the vapour-pressure of the liquids, alcohol, benzole, and acetone are most effective, CS_2 and ether must be added in greater quantity.

The actions of two different admixtures to a third gas are not independent of one another. Thus, if a brush is produced in air by adding a little CO_2, this brush will disappear again if so much Cl_2 is added as would form the brush in absence of CO_2, so that the addition of Cl_2 has the same effect as adding more CO_2. So it is not possible to lower the potential-difference still more by adding a third gas.

If a small platinum disk as cathode was placed opposite to the point, which was made movable by passing the glass tube carrying it through a bit of rubber tubing drawn over the narrow end of the vessel, a well-developed brush could be obtained by just once quickly turning round the stopcock separating the vessel from the Kipp's apparatus in which the CO_2 was generated. The electrodes could be separated as much as 20 mm. without the brush ceasing to appear, whilst in pure air the electrodes had to be brought to a distance of 2 mm. from one another before a brush-discharge occurred, and then sometimes sparking would set in. Thus this small amount of CO_2 gives a brush about ten times as long as that in air. If air is drawn through the vessel, the long brush disappears immediately. When the brush is well-developed, one sees that it starts from a bright point and, widening a little and losing in intensity, goes straight across to the cathode, where there is an increase in luminosity. In the way the brush goes straight to the part of the cathode just opposite it differs from the brush in pure air, which can be produced to considerable length by slightly lowering the pressure and introducing a small spark-gap into the circuit. In this case the brush, which is very unsteady, forms a curve having its end on a prominent point on the sharp edge of the cathode. Adding a little CO_2 will immediately make the brush go straight to the middle of the cathode.

It would be rash at present to venture an explanation of the behaviour of the mixtures. I would only like to point out that the potential-difference of the discharge depending upon the amount of ionization and the mean free path in the gas, an easily ionized gas, which requires a higher potential-difference because of its smaller mean free path, might be expected to lower the potential-difference in another gas of longer mean free path by furnishing more ions, if it is only

added in such small quantities as not to alter the mean free path. This may possibly account for the effect produced on the positive point-discharge in air by a small admixture of CO_2 and other gases ; but it does not explain the different effect obtained when the point is negative.

It may also be noted that under the influence of the not uniform field a gas of higher specific inductive capacity, as for instance CO_2, will tend to concentrate round the point and to displace the air, so that the discharge may really pass through a mixture containing a larger proportion of CO_2 than observed in the experiments.

I am at present investigating the oscillatory discharge from a point, as observed by Himstedt, in gaseous mixtures, which may throw some more light on the subject.

Through the kindness of Professor J. J. Thomson I was enabled to carry on these experiments in the Cavendish Laboratory, Cambridge, and I take this opportunity of expressing my sincerest gratitude for the kind interest he has shown in my work.

XXI. *The Electric Intensity in the Uniform Positive Column in Air.* By HAROLD A. WILSON, *B.A., D.Sc., Fellow of Trinity College, Cambridge*[*].

IN the following paper some measurements of the electric intensity in uniform positive columns in air at low pressures are described, the object of which was to determine the variation of the electric intensity with the current density when the latter is very small.

In a previous paper [†] by the writer it was shown that the electric intensity in a uniform positive column in air falls off slowly with increasing current. When the current was less than 10^{-4} ampere, however, the intensity appeared to be nearly independent of the current. The value of this maximum intensity was found to be given by the equation $X = 35\sqrt{p}$ where X is the electric intensity in volts per cm. and p the pressure in millimetres of mercury.

When the current through an ordinary vacuum-tube containing air at about one mm. pressure is diminished much below 10^{-4} ampere, the discharge becomes intermittent, and on further diminishing the current goes out altogether.

In order to make measurements with very small currents it was therefore necessary to use a special device to prevent the discharge going out.

The apparatus used is shown in fig. 1.   A "vacuum-tube,"
AB, 15 cms. long and 2 cms. in diameter, having an aluminium

Fig. 1.
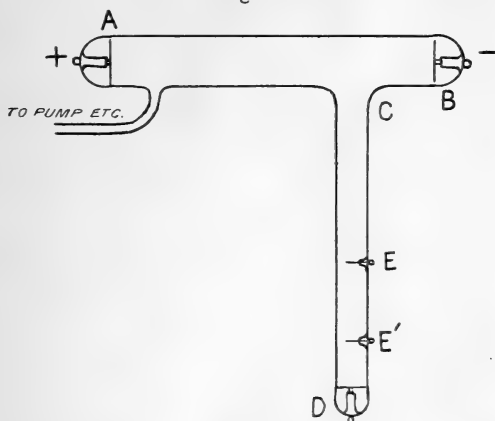
disk electrode at each end was connected to a Töpler pump,
McLeod gauge, and bulb containing phosphorus pentoxide.
A side tube, CD, was joined on to the tube AB.   This side
tube was about 15 cms. long and one sq. cm. in internal cross-
section.   Two small platinum wires were sealed into it at E
and E′ and an aluminium electrode at D.   A and B were
connected through a liquid resistance to a battery of small
secondary cells by means of which a discharge could be
passed between them.   The electrode D was also connected
to a point, on the same battery, through a liquid resistance
and galvanometer.   By adjusting the potential-difference be-
tween B and D a part of the current from B could be made
to pass down the side tube to D, and in this way a positive
column could be obtained in the side tube carrying any
desired current.

The discharge from A to B prevented the discharge in the
side tube going out even when the current in it was very
small.   In this way a positive column could be maintained
in the side tube with a current of less than 10^{-8} ampere.
With such small currents the light emitted by the positive
column was very faint, but it could be seen in a dark room
and it filled the tube with a faint uniform glow.   The elec-
trodes E and E′ were connected to a quadrant electrometer
by means of which the P.D. between them when the dis-
charge was passing was measured.   In this way the variation
of the electric intensity in the positive column in the side
tube with the current density could be obtained.   The follow-
ing tables give the results obtained in air :—

TABLE I.—Pressure 0·667 mm.

Current. (1=10⁻⁹ ampere).	Electric Intensity. (Volts per centim.)	Current. (1=10⁻⁹ ampere.)	Electric Intensity. (Volts per centim.)
2·5	52·8	78·4	64·9
4·9	55·3	150	65·5
7·4	58·6	225	66·8
9·8	59·6	245	66·3
13·2	61·3	181*	58·3
24·5	62·7	186*	49·0
39·2	64·1	220*	42·4
58·8	64·6	284*	38·8
68·6	64·9	289*	38·2

TABLE II.—Pressure 0·445 mm.

Current. (1=10⁻⁹ ampere.)	Electric Intensity. (Volts per centim.)	Current. (1=10⁻⁹ ampere.)	Electric Intensity. (Volts per centim.)
2·5	34·1	109	52·8
4·9	39·3	124·8	53·0
9·8	44·8	140	53·4
14·7	46·2	125*	44·3
19·6	47·6	125*	41·5
34·3	49·5	129*	36·9
46·8	51·1	152*	32·7
74·0	52·3	214*	24·2
93·6	52·5	410*	16·5

TABLE III.—Pressure 0·233 mm.

Current. (1=10⁻⁹ ampere.)	Electric Intensity. (Volts per centim.)	Current. (1=10⁻⁹ ampere.)	Electric Intensity. (Volts per centim.)
2·5	14·6	86	34·1
5·0	17·3	93	34·9
7·0	18·4	103	35·5
7·0	16·8	127	37·1
12	20·9	142	37·2
15	22·0	196	36·3
17	23·1	211	38·8
20	24·2	240	38·8
22	25·6	245	38·0
25	26·7	328	38·5
32	28·3	333	39·6
34	28·9	412	39·1
42	29·7	412	39·9
49	31·7	480	40·2
54	31·1	510	40·3
61	32·2	510	40·4
66	33·3	363*	29·9
74	33·0	377*	30·0
78	35·8	416*	28·9
83	35·2	455*	28·9

* Discharge intermittent.

These results are also shown in fig. 2. When the current

Fig. 2.
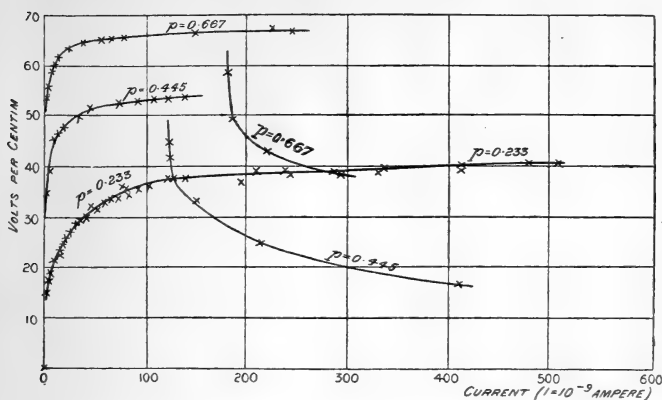
is very small the intensity rises very rapidly with the current, but the rate of rise falls off as the current increases. When the current is greater than a certain value the discharge becomes unstable and generally breaks down into an intermittent discharge in which the intensity is much smaller than in the steady discharge. A telephone was included in the circuit through the side tube by means of which the rate of intermittence could be estimated. The period of intermittence diminishes as the current is increased. At the upper end of the curves for the intermittent discharge the rate of intermittence is so slow that the successive discharges can be seen to follow each other at regular intervals of about one to two seconds, and the electrometer and galvanometer needles both oscillate. The electrometer shows that in between the discharges the intensity has the larger value corresponding to the steady discharge, and its deflexion drops at each discharge. The intermittent discharge consequently consists of a small steady current with momentary discharges superposed on it at regular intervals. When the current was increased sufficiently the electric intensity in the intermittent discharge became nearly independent of the current and was given by the formula $X = 35\sqrt{p}$.

As the current is increased the rate of intermittence increases, and the note emitted by the telephone rises until it becomes of very high pitch. With slightly more current the telephone becomes silent again, and the discharge seems to be perfectly steady. It was at this stage that the previous measurements (*loc. cit.*) were made.

It may be thought that the silence of the telephone at this stage merely shows that the rate of intermittence is so large that the note emitted by the telephone is of too high pitch to be audible. This may be the case, but it is more probable that when the successive discharges follow each other sufficiently closely they really blend into one continuous current. The high note in the telephone often suddenly disappears when the current is slightly increased, and the discharge at the same time acquires a much sharper outline as though it had suddenly really become continuous.

I shall now give an account of a theory of the positive column which leads to an explanation of the variation of X with the current shown in fig. 2.

Let n_1 and n_2 be the number of positive and negative ions respectively present in one c.c. of the uniform positive column. Then, since $\frac{dX}{dx} = 0$, where x is the distance measured along the axis of the discharge, we have $n_1 = n_2$. The current density ι is given by the formula

$$\iota = ne(v_1 + v_2),$$

where e is the charge carried by one ion and v_1 and v_2 the velocities of the positive and negative ions respectively. Also $v_1 = k_1 X$ and $v_2 = k_2 X$, where k_1 and k_2 are functions of the pressure only.

Let the number of positive or negative ions generated per c.c. per sec. be q : then we have

$$q = \beta n^2,$$

where β is the coefficient of recombination, and is a function of the pressure.

It has been shown by Townsend* that ionization is produced by negative ions when moving under the influence of an electric field. If α is the number of negative ions produced by one negative ion in moving one cm., then Townsend finds $\alpha = pf\left(\dfrac{X}{p}\right)$, and he has determined the values of $f\left(\dfrac{X}{p}\right)$ over a large range of values of $\dfrac{X}{p}$.

In the positive column another source of ionization besides the negative ions is also present. It was found by E. Wiedemann† that the positive column emits a kind of easily absorbed radiation which he termed " Entladungstrahlen," and

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* Phil. Mag. February 1901.
† Wied. *Ann.* lx. p. 269.

J. J. Thomson* has shown that this radiation produces ionization in gases such as air by which it is rapidly absorbed.

It seems probable that this radiation is emitted by the ions during the process of recombination. On this supposition the amount of radiation emitted per c.c. may be taken as proportional to the amount of recombination βn^2, and to the kinetic energy of the ions, which latter is proportional to X^2.

If, then, R_0 is the amount of energy emitted per c.c. as "Entladungstrahlen," we have $R_0 = A'\beta n^2 X^2$, where A' is a function of the pressure.

The radiation emitted by a small element of volume dv will be $R_0 dv$, and at a distance r from the element its intensity will be $\dfrac{R_0 dv \epsilon^{-\lambda r}}{4\pi r^2}$, where λ is the coefficient of absorption of the radiation by the gas.

At any point in the discharge-tube the intensity of radiation will be the integral throughout the positive column of $\dfrac{R\epsilon^{-\lambda r} dv}{4\pi r^2}$. It is easy to show that this gives for the intensity R at the axis of a discharge-tube of radius a

$$R = \frac{R_0}{\lambda}(1 - \epsilon^{-\lambda a}).$$

If a is not very small we may, therefore, take $\dfrac{R_0}{\lambda}$ as the intensity of radiation anywhere in the positive column except very near the walls of the discharge-tube. The ionization due to radiation, say q_r, will therefore be given by the equation

$$q_r = \frac{A\beta n^2 X^2}{\lambda},$$

where A is a function of the pressure only.

Denoting the ionization due to negative ions by q_c we have

$$q = q_c + q_r = \beta n^2.$$

Now q_c is evidently equal to $n\alpha v_2$, so that

$$n\alpha v_2 + \frac{A\beta n^2 X^2}{\lambda} = \beta n^2.$$

Eliminating n by means of the equation

$$\iota = ne(v_1 + v_2),$$

* Proc. Camb. Phil. Soc. vol. x. pt. ii. p. 74.

we get

$$\iota = \frac{eav_2(v_1 + v_2)}{\beta - \dfrac{A\beta X^2}{\lambda}}.$$

Now v_1 is known to be very small compared with v_2 at low pressures, and so may be neglected; also $v_2 = k_2 X$, and

$$a = pf\left(\frac{X}{p}\right),$$

hence

$$\iota = \frac{ek_2^2 X^2 p}{\beta\left(1 - \dfrac{AX^2}{\lambda}\right)} f\left(\frac{X}{p}\right).$$

This equation may be written

$$\iota = \frac{CX^2}{B - X^2} f\left(\frac{X}{p}\right),$$

where C and B are functions of p only.

In the following tables the observed values of X are compared with those calculated by means of this formula, taking appropriate values for C and B, and using Townsend's values of $f\left(\dfrac{X}{p}\right)$ taken from the curve on page 212, Phil. Mag. Feb. 1901.

The fourth column contains values of the current calculated by means of the formula from the values of X in the first column. The last column contains the values of X taken from the experimental curves for the calculated values of the current.

TABLE IV.

$p = 0.667.$ C $= 3.69.$ B $= 4530.$

X.	$\dfrac{X}{p}$.	$f\left(\dfrac{X}{p}\right)$.	ι (calculated) $1 = 10^{-9}$ ampere.	X (found for calculated value of i).
54	81	0·53	3·5	54
57	86	0·60	5·6	58
60	90	0·70	10	60
63	95	0·77	22	63
66	99	0·86	77	65
67	101	0·88	364	67

TABLE V.

$p = 0.445.$ $C = 5.5.$ $B = 3025.$

X.	$\frac{X}{p}$.	$f\left(\frac{X}{p}\right)$.	ι.	X.
34	76·5	0·47	1·6	33
40	90	0·69	4·3	38
45	101	0·90	10·0	45
50	112	1·12	29·4	49·5
52	117	1·23	56·1	52
54	121	1·33	203	54

TABLE VI.

$p = 0.233.$ $C = 26.$ $B = 1760.$

X.	$\frac{X}{p}$.	$f\left(\frac{X}{p}\right)$.	ι.	X.
20	86	0·60	5	16
25	107	1·00	14	23
30	129	1·50	41	30
33	142	1·80	76	34
35	150	1·96	117	36
38	163	2·25	269	39
39	167	2·35	387	39·5
40	172	2·46	640	40

Fig. 3.
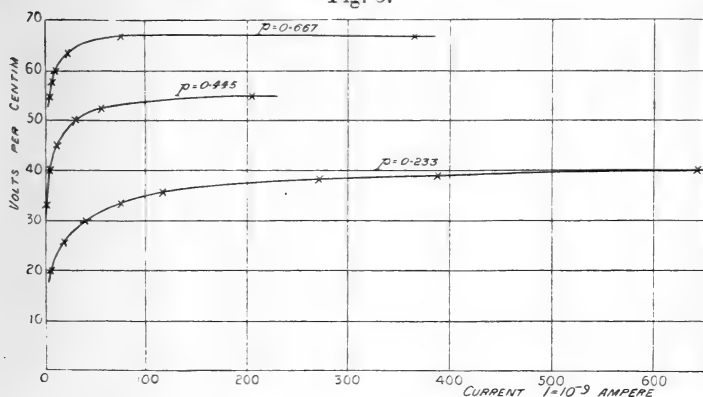
It will be seen that the values of X given in the last column agree fairly well with those in the first column. In fig. 3

the calculated values of X are represented, the scale being the same as in fig. 2.

The constant B varies nearly as the pressure, while C increases rapidly with diminishing pressure. If X_m is the maximum value of the intensity, then $X_m^2 = B$, so that X_m varies as $\sqrt{p}$.

C is equal to $\dfrac{\lambda e k_2 p}{A\beta}$ and $B = \dfrac{\lambda}{A}$, consequently, since C increases as the pressure diminishes, it follows that β the coefficient of recombination must rise with increasing pressure. This result is in agreement with Langevin's measurements [*] of the coefficient of recombination of the ions produced in air by Röntgen rays at low pressures.

The experiments described in this paper were done at the Cavendish Laboratory, and I wish to say that my best thanks are due to Prof. J. J. Thomson for his kindly interest and advice given during the course of the work.

XXII. *Note on the Construction and Attachment of Thin Galvanometer Mirrors. By* W. Watson, *D.Sc., F.R.S., Assistant Professor of Physics, Royal College of Science, London* [†].

AS has been pointed out by several persons (Rayleigh, Phil. Mag. xx. p. 360, 1885 ; Threlfall, Proc. Phys. Soc. xvi. p. 205, 1898) there are distinct advantages in increasing the sensitiveness of the optical arrangements used to measure the rotation of a galvanometer needle rather than increasing the electrical sensitiveness, *i. e.* the angular deflexion produced by a given small current. Excessive electrical sensitiveness implies a very weak controlling force. Such a weak controlling force involves a long period, and also disturbances produced by convection currents in the air and tremors in the supports of the instrument.

In order to increase the optical sensitiveness of a galvanometer it is necessary to increase the diameter of the mirror, while, to allow of its being carried by a fine suspension and to keep the period low, the weight of the mirror must be made as small as possible, yet at the same time its figure must remain good. With the ordinary form of mirror, consisting of a disk of glass silvered on the back, it is almost impossible to obtain a really thin mirror with a good figure unless the area of the mirror is excessively small. In the first place, the grinding and polishing of a very thin slip of glass is an

operation which taxes the skill of the optician to the utmost. More important, however, is the fact that the silver backing of the mirror has to be coated with some kind of varnish to protect the silver from the action of the impurities in the air. Although numberless kinds of varnish have been tried for this purpose none has yet been discovered which does not distort the mirror to some extent. A further difficulty in the case of the glass mirror is the attachment of the mirror to the stem which carries the magnets, the cement employed almost always producing some distortion. While designing a special form of mirror for use in a magnetograph I have been led to a method of constructing galvanometer mirrors which I believe entirely obviates the difficulties mentioned above.

Prof. Threlfall has already pointed out the advantages of quartz as a material from which to construct galvanometer mirrors, but he, I believe, employed silvered mirrors, and hence had to use some form of varnish, and also he cemented the mirrors to the rod which carried the magnets. He tried both crystalline quartz and fused silica, and states he found them equally suitable. In my case I was obliged to use fused silica, and when I attempted to use thin slices silvered on the back I found that there was a very considerable loss of light owing to the small bubbles which are always present in any but the smallest pieces of fused silica. It then occurred to me to try and use some material for the reflecting surface which would be unaffected by the air. Platinum naturally was the material first tried, and after testing practically all the methods of which I could hear for giving a bright film of platinum, I obtained from Messrs. Johnson and Matthey a platinizing solution which is entirely satisfactory, and gives without any polishing a perfectly bright surface of platinum. The film of platinum obtained is bright on the surface turned away from the silica, so that the reflexion takes place at the air-platinum surface, and hence the silica disk requires polishing on one surface only, a circumstance which, as will be seen, very much simplifies the construction of thin mirrors.

The method I adopt for making the mirrors is as follows:— A stick of fused silica is prepared having a diameter equal to the diameter of the mirror it is wished to prepare. This stick is cemented to a small piece of wood by means of pitch. A disk of tinned iron (No. 28, s.w.g.) about 12 centimetres in diameter is mounted on a mandrel and the edge turned true. A little diamond-dust mixed with thick oil is then spread round the edge of the disk, and while rotating the

disk at the rate of about five turns a minute a piece of flint
is pressed firmly against the edge. In this way the diamond
particles are driven into the iron and, at any rate after an
attempt or two, the disk will be satisfactorily armed. A
horizontal plate with a guide at one edge is attached to the
slide-rest of the lathe and the wooden base to which the
silica is attached is placed on this plate, and the silica pressed
firmly against the edge of the disk which is rapidly rotated.
Soap and water must be used as a lubricant, being supplied
by means of a brush. In this way a disk having a diameter
of one centimetre can be cut in a minute.

A small rod of fused silica is then fused to the edge of the
disk by means of a small oxyhydrogen flame, and all but the
last two or three millimetres cut off, so as to leave a small
tab of silica. The disk is then roughly ground flat on either
side by rubbing it on a flat plate of brass freely supplied with
powdered carborundum (of such a size as to pass through a
sieve of 250 threads to the inch) and water. The disk is
then annealed by being heated to a bright red in a small
scoop of platinum or thin sheet-iron over a Bunsen flame for
about 5 hours. That surface of the disk which, on inspection,
appears most free from bubbles is ground on a plate of thick
plate-glass with water and emery of gradually increasing
fineness. The disk is moved round and round in circular
sweeps, the end of the finger pressing it lightly down on the
surface of the glass. About ten minutes' grinding with each
of three grades of washed emery, finishing with the finest
which can be obtained, will generally be sufficient. The
grades of emery I use are such that the emery is deposited
from water in the following times:—No. 1 settles in 1
minute, No. 2 settles in 10 minutes, No. 3 settles in between
20 minutes and 60 minutes.

The surface is polished with rouge on a pitch form. It is
essential to obtain opticians' rouge, which has been well
washed, and not jewellers' rouge. The proper quality of
rouge I have obtained from Messrs. Cooke of York. To
form the pitch surface some pitch is melted, and when
thoroughly liquid is poured over the surface of a piece of
plate-glass and a second piece of plate-glass, the surface of
which has been thinly coated with dilute glycerine, is pressed
down on the top of the pitch. Weights are placed on the
upper glass till the pitch is cold, when the glass can be slid
off. The polishing is continued till on examining the surface
with a low-power microscope no pitting can be seen. Quite
a light pressure of the tip of the finger must be used during
the polishing, which will take from 10 to 30 minutes.

To give the polished surface its reflecting coating a very thin layer of the platinizing liquid is uniformly painted over the surface with a clean brush, and the disk is placed on a metal plate over a water-bath. When the coating is quite dry the disk is heated in a small muffle made by bending a sheet of thin sheet-iron and placing it over a large Bunsen flame. The heating must be continued till a fairly bright red is attained. It is an advantage to raise the temperature of the disk to a red heat as quickly as possible, for if the heating is slow the platinizing compound tends to volatilize before it decomposes. The surface thus obtained will be very bright and will not require polishing. If the film of platinum is too thin a second coating of the platinizing liquid can be given.

In this way a mirror is obtained which, however, is about a millimetre thick. To reduce it to a more suitable thickness the platinum surface is temporarily protected with a coating of pitch or some other varnish, and the other surface of the disk is ground away on the brass-surface plate with carborundum till the desired thickness is obtained. It will be found quite easy to prepare a mirror having a diameter of one centimetre and a thickness of two-tenths of a millimetre. Such a mirror will weigh about ·045 gram.

In order to support the mirror the tag is fused, by means of a small oxyhydrogen flame, to a thin rod of fused silica and the magnets are cemented to this rod.

The above method of constructing galvanometer mirrors avoids the necessity of grinding and polishing the surface of a very thin disk, an operation of great difficulty, and is only rendered possible by using a non-corrodible reflecting surface so that the light has not to traverse the quartz disk. Also the excessively small coefficient of thermal expansion of quartz enables us to obtain a disk of the material which is so free from internal strains that when we grind away one side the form of the other side is not appreciably altered. Lastly, the method of attachment of the mirror to the stem, since it avoids all cements, entirely does away with the risk of distortion due to the contraction of the cement.

Platinum surfaces prepared as above described have not quite such a large reflecting power as glass backed with silver. A comparison between a platinum surface which had been exposed quite unprotected to the air of the laboratory for a month and a silver on glass mirror made by Hilger showed that the reflecting power of the platinum was about seven-tenths of that of the silver. Any difficulty due to this

smaller reflecting power can be got over by using a transparent scale with opaque lines backed by a flame, in place of the ordinary opaque scale illuminated by diffuse light.

Platinum mirrors appear as if they might be of considerable use where a metallic mirror is required, and where silver would be likely to tarnish. When glass is used considerable care has to be used, in the first place not to increase the temperature so quickly as to crack the glass, and secondly not to raise the temperature to such a height as to alter the figure of the glass by softening. When crystalline quartz is used no softening is to be feared. The temperature has, however, to be altered *very slowly* or the quartz will crack. With care, however, very perfect mirrors have been obtained on quartz.

Platinum deposited in this way forms a very good coating for producing " half-silvered " mirrors for use in interference experiments and such instruments as those recently described by Sir Howard Grubb (Sci. Trans. R. Dublin Soc. vii. p. 385, 1902). The preparation of a surface having a given reflecting power is much easier in the case of platinum than in the case of silver, and once such a surface has been obtained it seems to be practically unalterable.

XXIII. *Notices respecting New Books.*

International Catalogue of Scientific Literature. First Annual Issue. C. Physics. Part I. London: Harrison and Sons. 1902. Pp. xiv + 239.

A S a ready means of ascertaining everything of importance that has been published on any particular subject of scientific inquiry, the new International Catalogue of Scientific Literature should prove of inestimable value to all engaged in original research. The present volume, which refers to Physics, contains schedules and indexes in four languages, an authors' catalogue, and a subject catalogue. The arrangement adopted should enable any one to find the particular section required without any trouble.

Subject List of Works on General Science, Physics, Sound, Music, Light, Microscopy, and Philosophical Instruments, in the Library of the Patent Office. London: Patent Office. 1903. Pp. 183.

This catalogue of books on physical subjects will be found useful by all who are in the habit of frequenting the Patent Office Library.
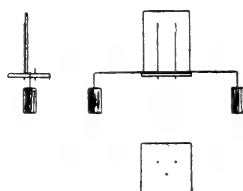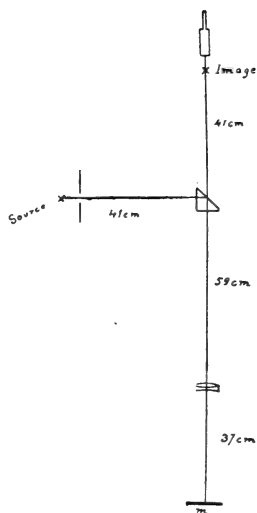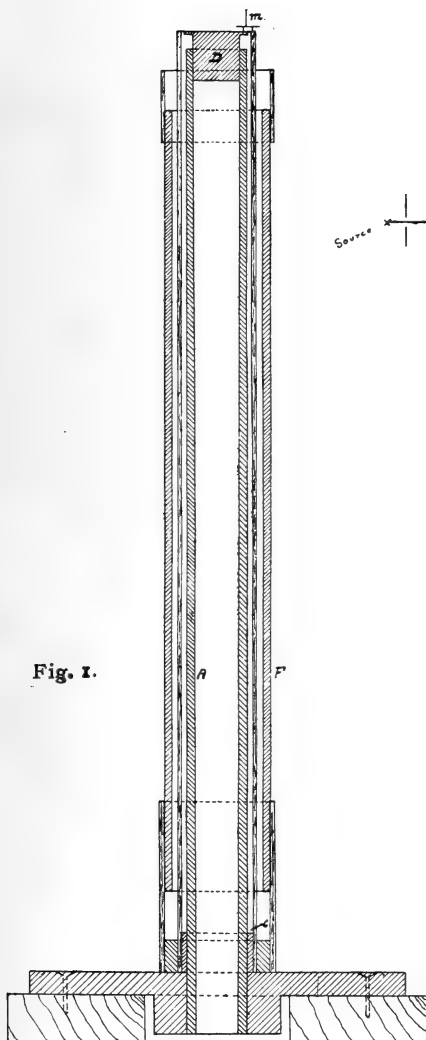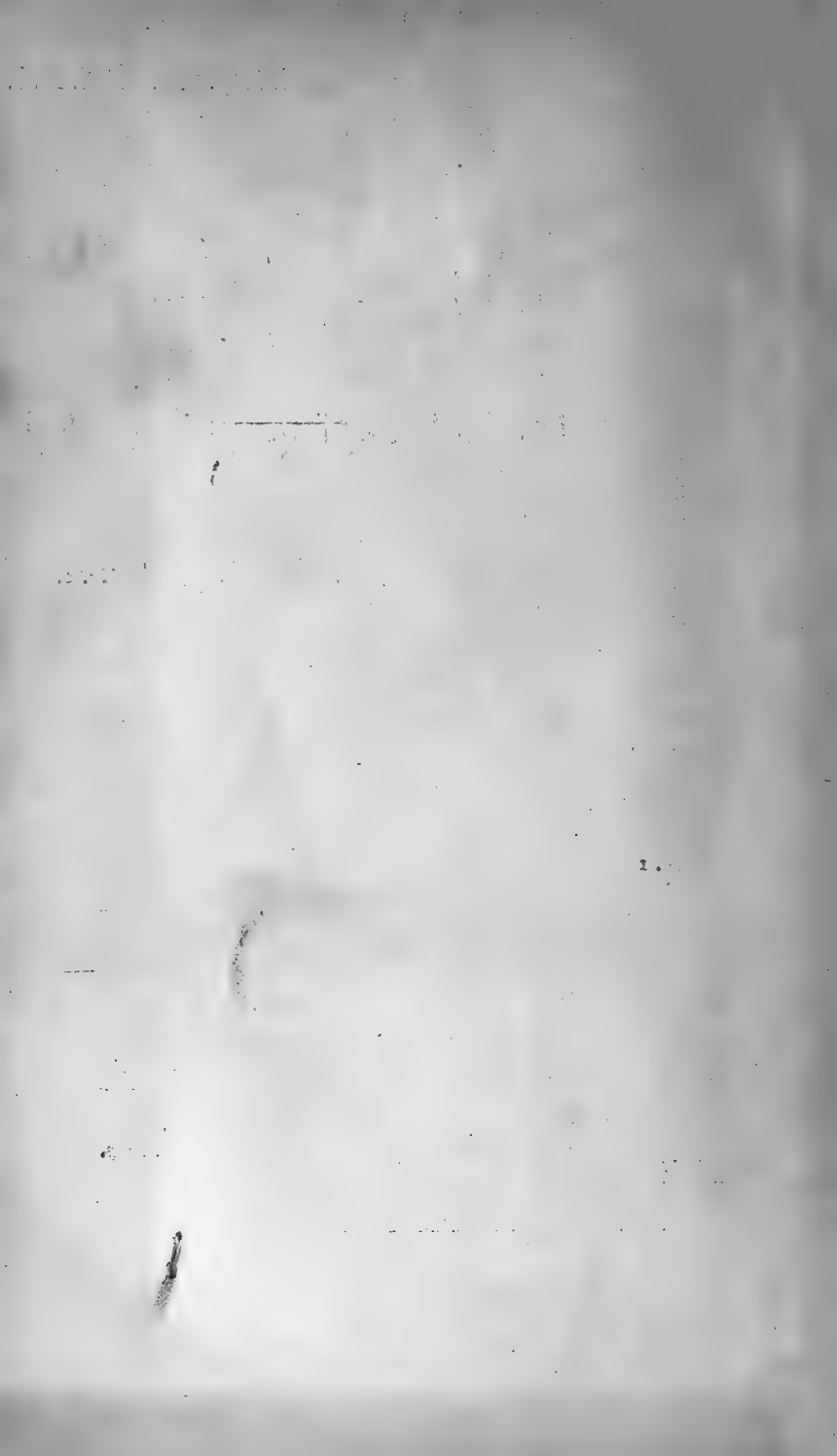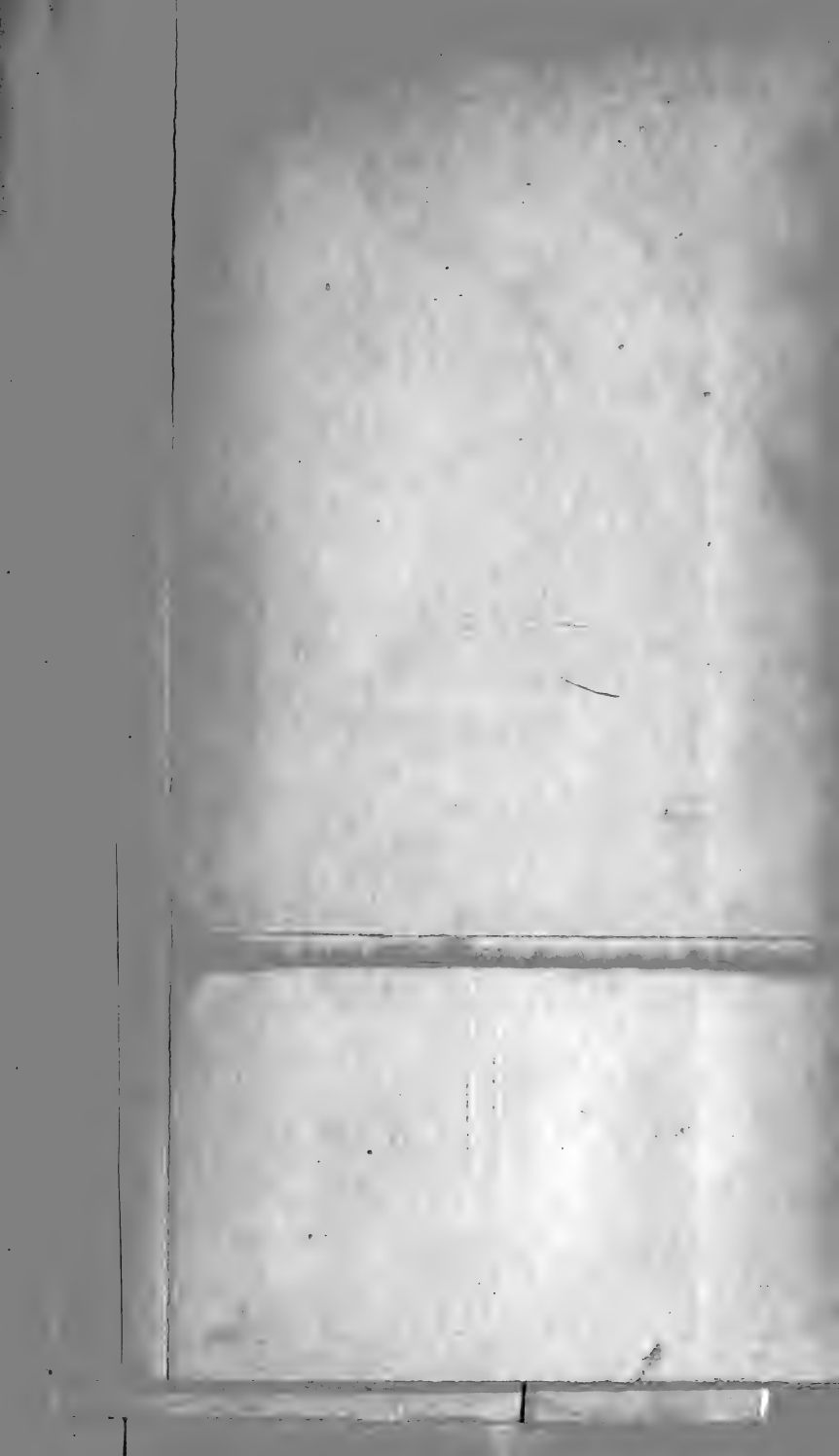
Fig. 1.
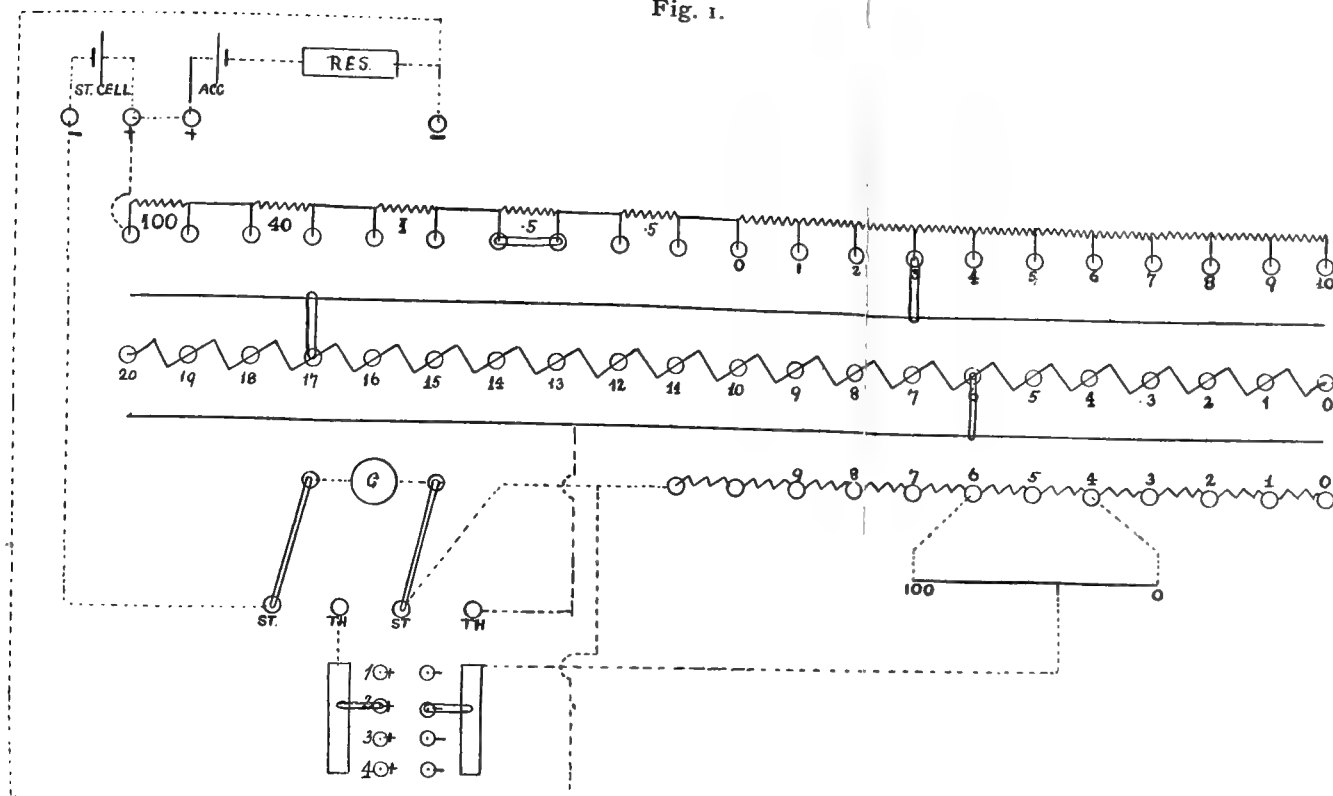
Fig. 2.
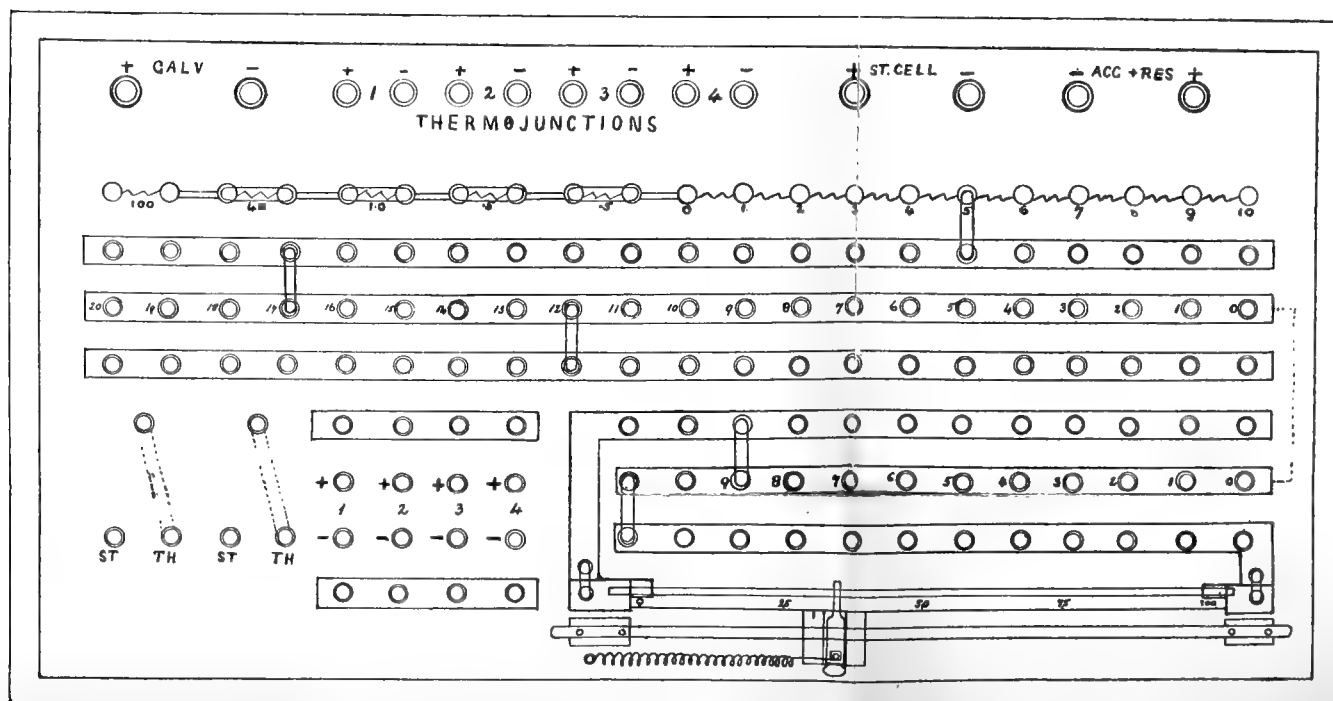
Fig 3.

Fig. 1.

Fig. 2.

$$d\ \frac{r\sqrt{a^2+h^2}}{\sqrt{\mu^2(a-r)^2+(\mu^2-1)h^2}-a}$$

$$y^2 = \frac{x^2}{a^2(\mu^2-1)}\left\{\frac{(ax+ra-a^2)^2-\mu(a-r)(x-a)^2}{(x-a)^2}\right\}$$

$$d = \frac{r}{\mu\sin\beta} = \frac{r\sqrt{a^2+h^2}}{\sqrt{\mu^2-1}\sqrt{a^2-r^2+h^2}-r}$$
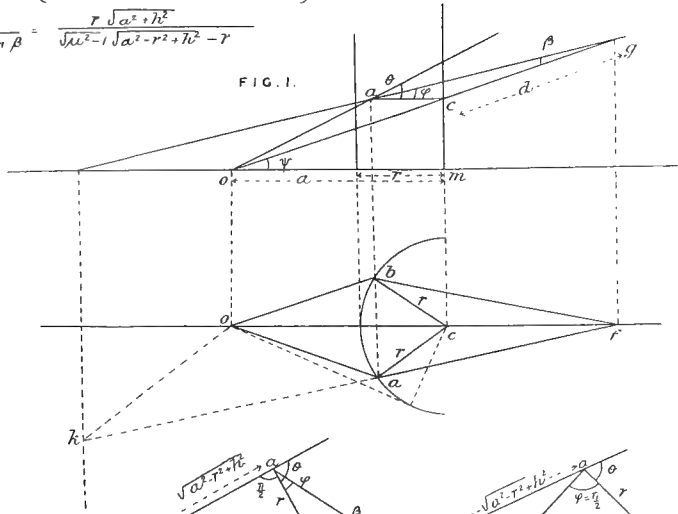
FIG. 1.
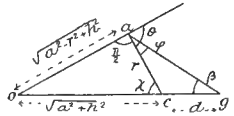
FIG. 2.
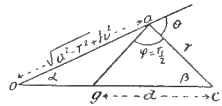
FIG. 3

FIG 4
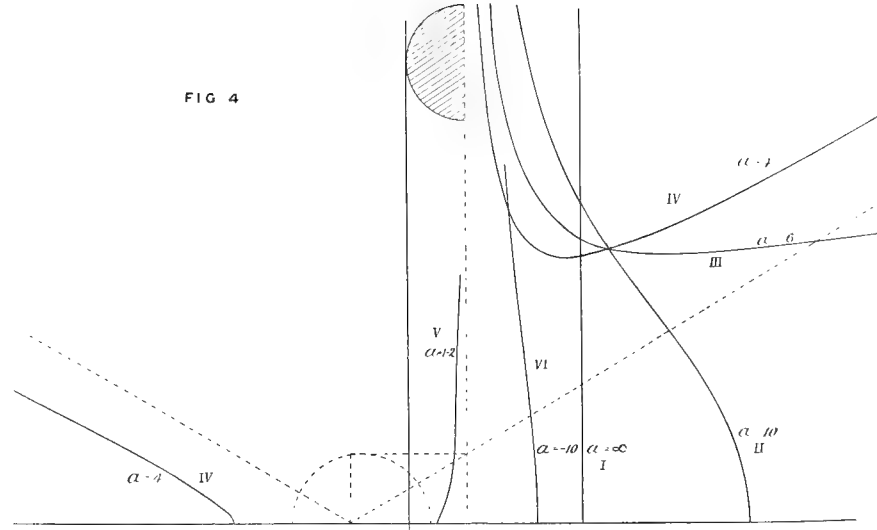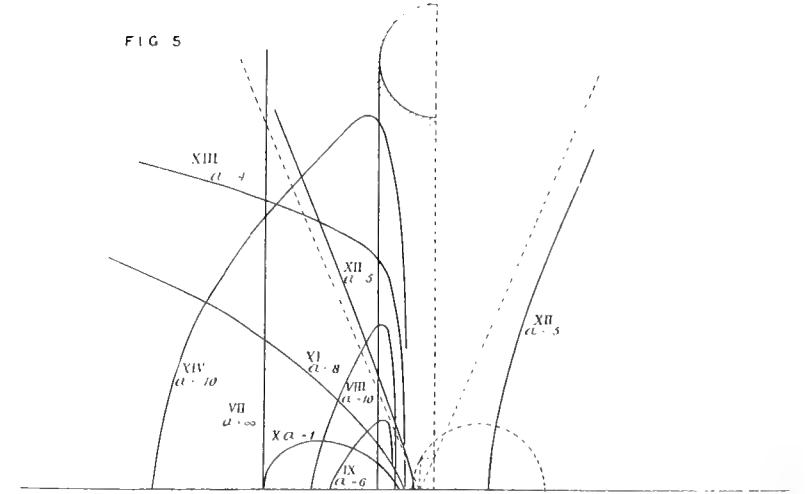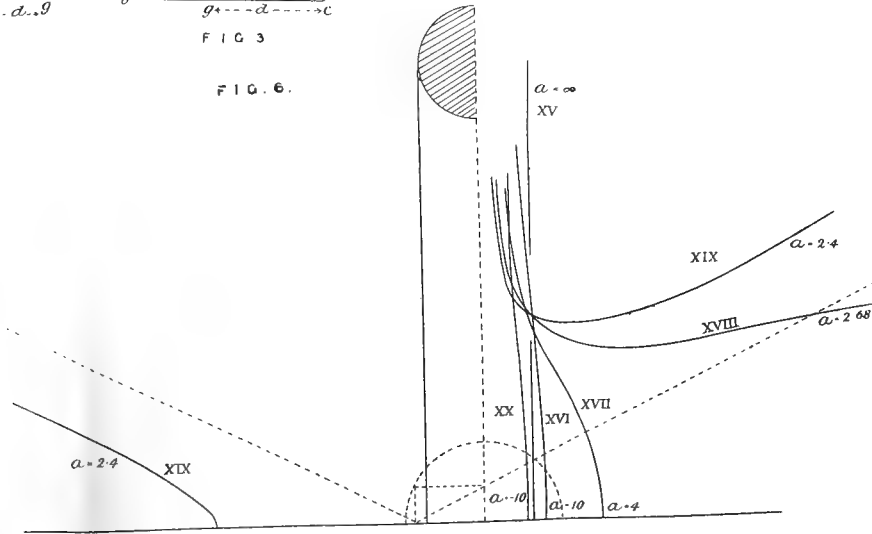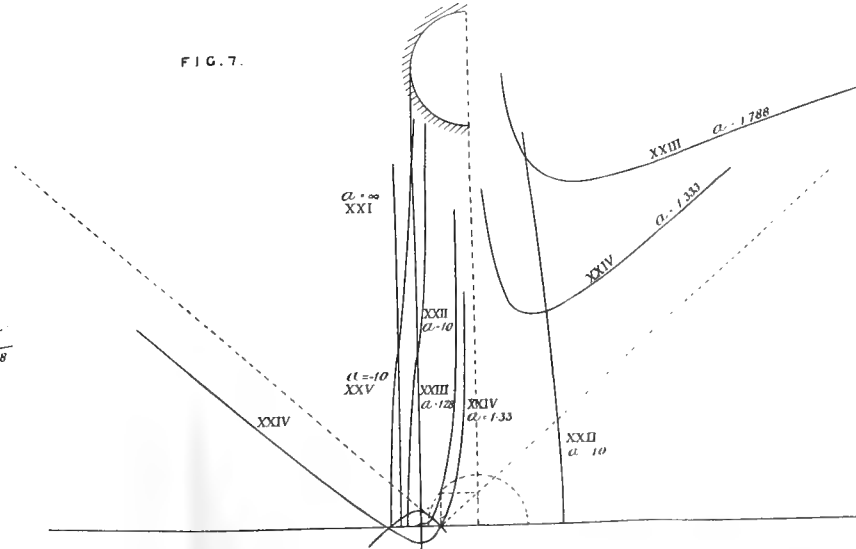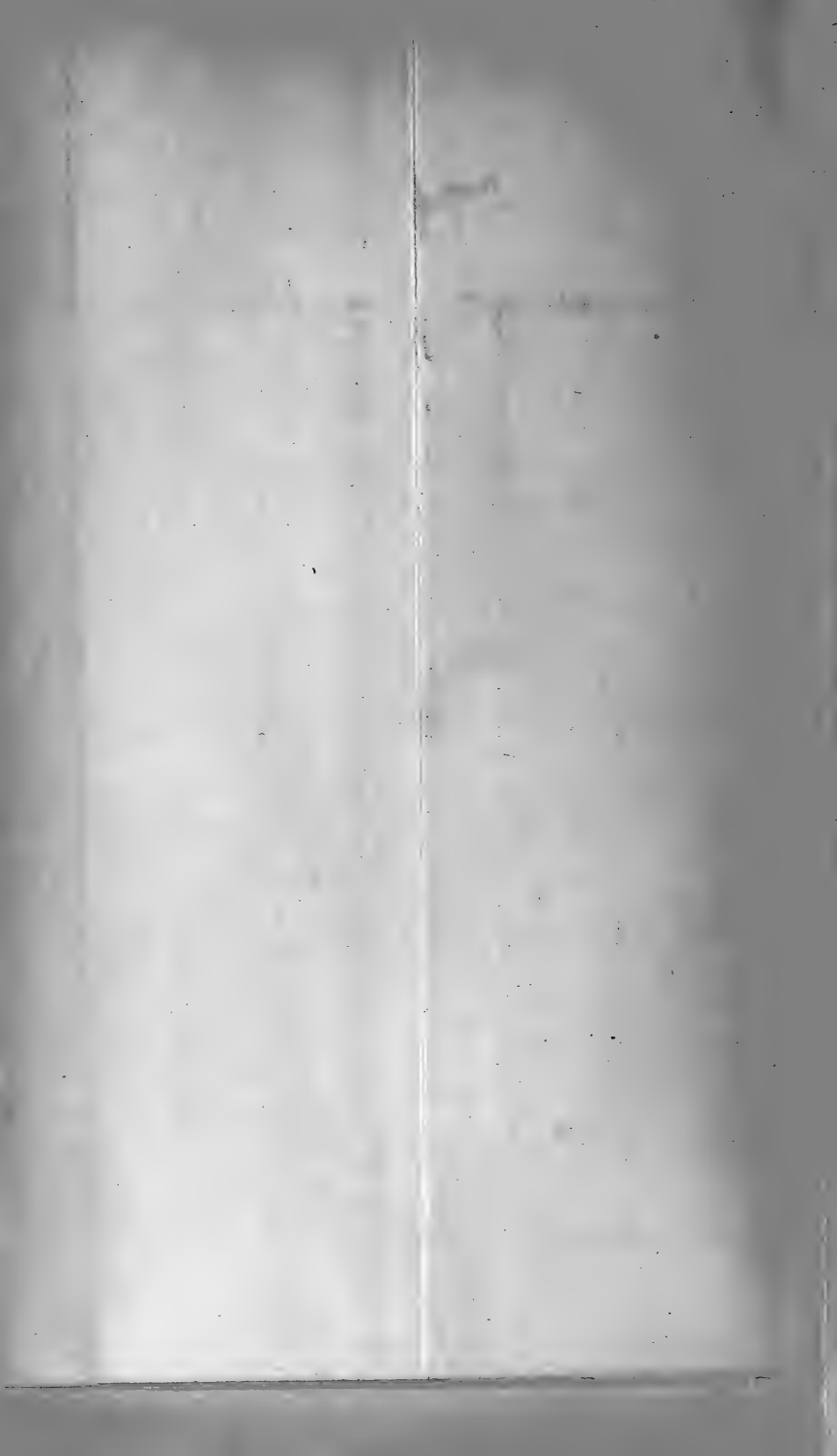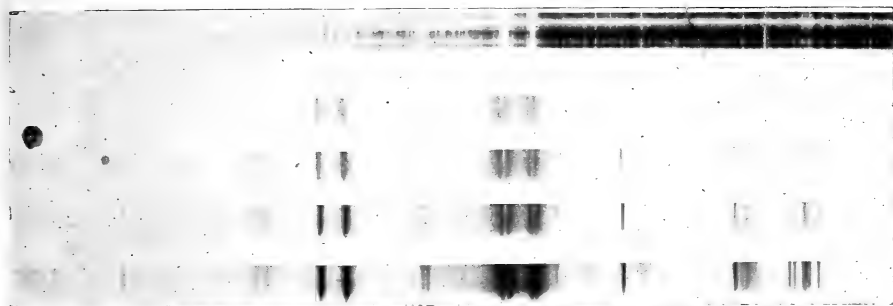
FIG 5
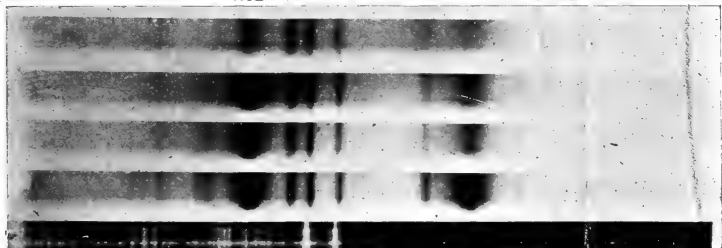
FIG. 6.

FIG. 7.

FIG. 3.

FIG. 4.

FIG. 5.
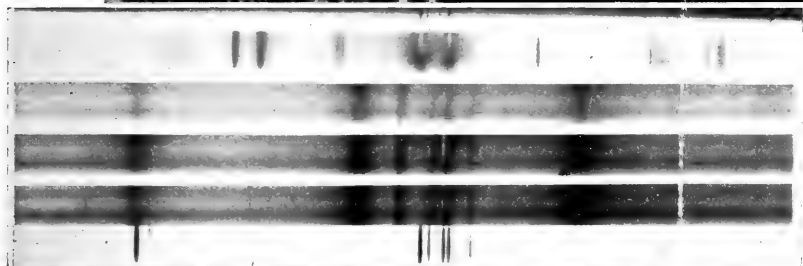
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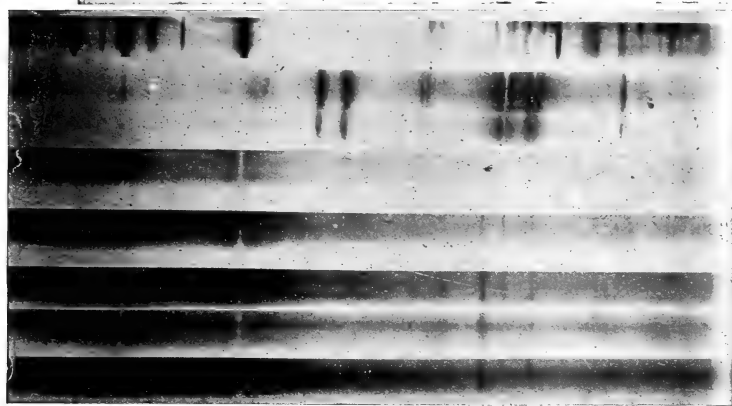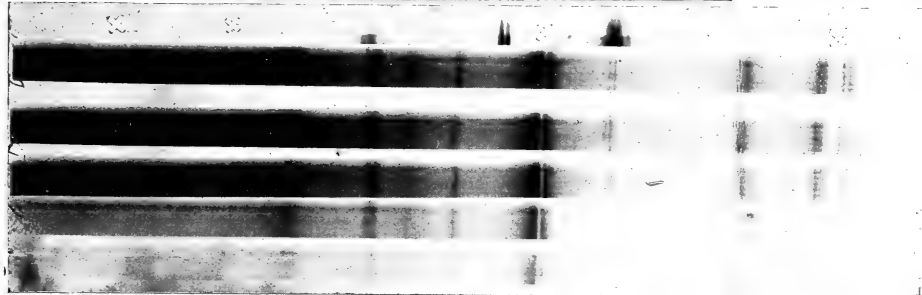
FIG. 7.

Note reversed lines on Figures 5, 6, and 7, marked with a cross.
Fig. 7 is in the ultra-violet portion of the spectrum.
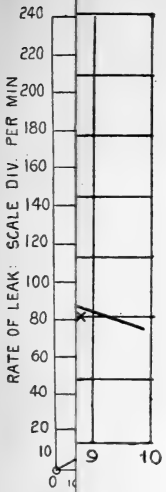
FIG. 5.
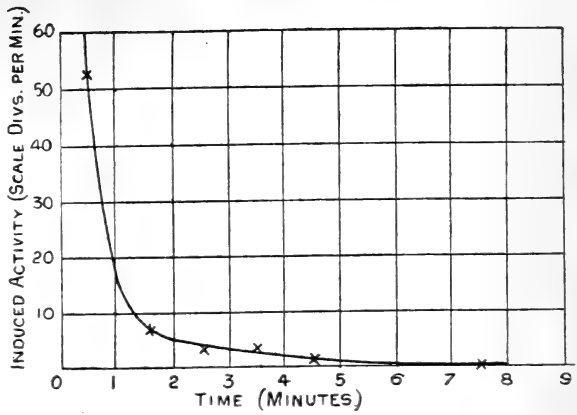
FIG. 6.
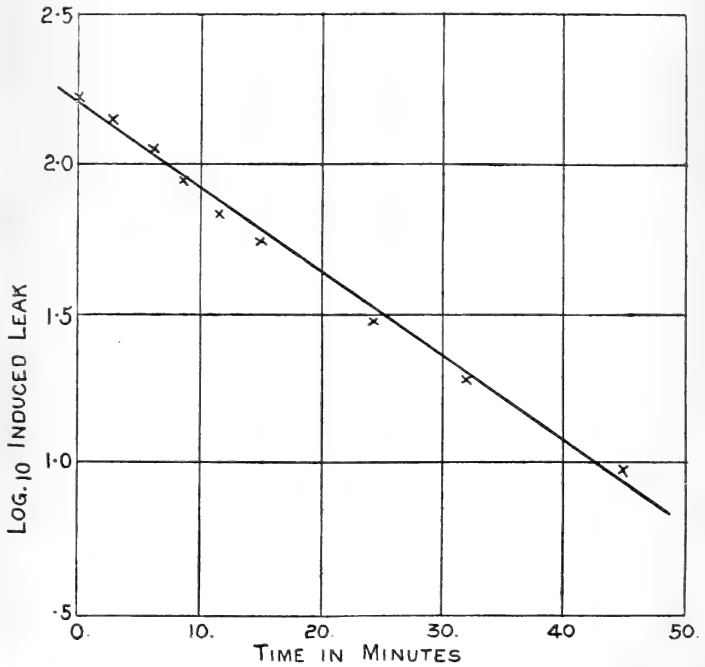
Fig. 1.

Fig. 3.

Fig. 5.

Fig. 2.

Fig. 4.

Fig. 6.

FIG. 7.

FIG. 6.

FIG. 8.

FIG. 3.

FIG. 5.

FIG. 9.

FIG. 10.
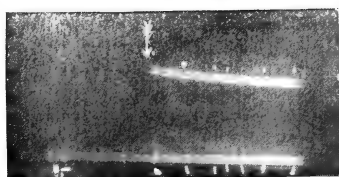
FIG. 11.

XXIV. *On a New Mode of Expressing Solutions of Laplace's Equation, in Terms of Operators involving Bessel Functions. By* L. N. G. FILON, *M.A., D.Sc., Lecturer in Mathematics and Fellow of University College, London* *.

1. IN a recent number of the R. A. S. 'Monthly Notices' (vol. lxii. no. 9, pp. 617–720), Mr. E. T. Whittaker has shown that the general solution of Laplace's equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0$$

could be written in the form

$$V = \int_0^{2\pi} f(x \cos v + y \sin v + iz, v)dv,$$

where f is an arbitrary function of the two arguments $x \cos v + y \sin v + iz$ and v and $i = \sqrt{-1}$.

It is further easy to show that the function f, which, by Mr. Whittaker's proof, must be an analytic function of the first argument, need not be an analytic function of the second argument. All that the proof assumes is that the function f, treated as a function of the second argument, shall be expansible in a Fourier Series, the coefficients of which are analytic functions of the first argument. This merely requires that f, as a function of the second argument, shall have only a finite number of turning-values or of finite discontinuities.

2. Let us assume that f is so expanded. Then V is expressed as the sum of typical integrals as follows :—

$$V = \int_0^{2\pi} f_0(x \cos v + y \sin v + iz) dv + \ldots \ldots$$

$$+ \int_0^{2\pi} f_m (x \cos v + y \sin v + iz) \cos mv \, dv + \ldots \ldots$$

$$+ \int_0^{2\pi} F_m(x \cos v + y \sin v + iz) \sin mv \, dv + \ldots \ldots$$

the f's and F's being all analytic functions over the range of x, y, z, v considered.

Transform to cylindrical coordinates

$$x = \rho \cos \phi, \quad y = \rho \sin \phi,$$

$$\int_0^{2\pi} f_m(x \cos v + y \sin v + iz) \cos mv \, dv$$

$$= \int_0^{2\pi} f_m(\rho \cos (v - \phi) + iz) \cos mv \, dv.$$

Let $v - \phi = u$.

The integral $= \int_{-\phi}^{2\pi - \phi} f_m (\rho \cos u + iz)(\cos m\phi \cos mu - \sin m\phi \sin mu) du.$

Consider

$$\cos m\phi \int_{-\phi}^{2\pi - \phi} f_m (\rho \cos u + iz) \cos mu \, du$$

$$= \cos m\phi \left[\int_0^{2\pi - \phi} + \int_{-\phi}^0 \right].$$

In the second integral write $u' = 2\pi + u$,

$$\int_{-\phi}^0 f_m (\rho \cos u + iz) \cos mu \, du$$

$$= \int_{2\pi - \phi}^{2\pi} f_m(\rho \cos u' + iz) \cos mu' du', \quad m \text{ being an integer.}$$

Hence the two integrals together

$$= \cos m\phi \int_0^{2\pi} f_m(\rho \cos u + iz) \cos mu \, du.$$

Similarly,

$$\int_{-\phi}^{2\pi - \phi} f_m (\rho \cos u + iz) \sin mu \, du$$

$$= \int_0^{2\pi} f_m (\rho \cos u + iz) \sin mu \, du$$

$$= \int_0^{\pi} f_m (\rho \cos u + iz) \sin mu \, du + \int_{\pi}^{2\pi} f_m(\rho \cos u + iz) \sin mu \, du.$$

In the second integral write $u = 2\pi - u'$,

$$\int_{-\phi}^{2\pi-\phi} f_m (\rho \cos u + iz) \sin mu \, du = \int_0^\pi f_m (\rho \cos u + iz) \sin mu \, du$$
$$+ \int_\pi^0 f_m (\rho \cos u' + iz) \sin mu' du'$$
$$= 0.$$

Hence

$$\int_0^{2\pi} f_m (x \cos v + y \sin v + iz) \cos mv \, dv$$
$$= \cos m\phi \int_0^{2\pi} f_m (\rho \cos u + iz) \cos mu \, du. \quad . \quad . \quad . \quad (1)$$

But the f's and F's being arbitrary functions, we may replace $f_m (\rho \cos u + iz)$ by $f_m (z - i\rho \cos u)$. The general expression for V then becomes

$$V = \int_0^{2\pi} f_0 (z - i\rho \cos u) du + \ldots\ldots$$
$$+ \cos m\phi \int_0^{2\pi} f_m (z - i\rho \cos u) \cos mu \, du + \ldots\ldots$$
$$+ \sin m\phi \int_0^{2\pi} F_m (z - i\rho \cos u) \cos mu \, du + \ldots\ldots \quad . \quad (2)$$

3. Use now the symbolic form of Taylor's Theorem,

$$f_m (z - i\rho \cos u) = e^{-i\rho \cos u \frac{d}{dz}} f_m(z). \quad . \quad . \quad . \quad (3)$$

V may be written symbolically

$$V = \left[\int_0^{2\pi} \left(e^{-i\rho \cos u \frac{d}{dz}} \right) du \right] f_0 (z) + \ldots\ldots$$
$$+ \cos m\phi \left[\int_0^{2\pi} \left(e^{-i\rho \cos u \frac{d}{dz}} \right) \cos mu \, du \right] f_m(z) + \ldots\ldots$$
$$+ \sin m\phi \left[\int_0^{2\pi} \left(e^{-i\rho \cos u \frac{d}{dz}} \right) \cos mu \, du \right] F_m(z) + \ldots\ldots \quad (4)$$

But it is easy to show that

$$\int_0^{2\pi} e^{-iq \cos u} \cos mu \, du = 2(-1)^m \int_0^\pi e^{iq \cos u} \cos mu \, du . \quad (5)$$

and

$$\int_0^\pi e^{iq \cos u} \cos mu \, du = \pi i^m J_m(q). \quad . \quad . \quad . \quad (6)$$

(See Gray & Matthews, 'Bessel Functions,' p. 89.)

So that

$$\left(\int_0^{2\pi} e^{-i\rho \cos u \frac{d}{dz}} \cos mu \, du \right) f_m(z) = 2\pi(-i)^m \mathrm{J}_m\left(\rho \frac{d}{dz} \right) f_m(z). \quad (7)$$

If we suppose the constant $2\pi(-i)^m$ absorbed inside the arbitrary function $f_m(z)$, the solution of Laplace's Equation may be written

$$V = \mathrm{J}_0\left(\rho \frac{d}{dz} \right) f_0(z) + \dots + \cos m\phi \mathrm{J}_m\left(\rho \frac{d}{dz} \right) f_m(z) + \dots$$
$$+ \sin m\phi \mathrm{J}_m\left(\rho \frac{d}{dz} \right) F_m(z) + \dots \quad . \quad . \quad . \quad (8)$$

the f's and F's being arbitrary functions.

4. By giving different forms to the functions f and F we obtain series of the various typical solutions of Laplace's equation.

Thus, if we write

$$f_m(z) = \mathrm{A} e^{kz} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

we get immediately the typical product form

$$\mathrm{A} \left. \begin{matrix} \cos \\ \sin \end{matrix} \right\} m\phi \, \mathrm{J}_m(k\rho) e^{kz}. \quad . \quad . \quad . \quad (10)$$

5. If, on the other hand, we assume

$$f_m(z) = z^n, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

then

$$\mathrm{J}_m\left(\rho \frac{d}{dz} \right) z^n$$
$$= \sum_{r=0}^{r=\infty} \left(\frac{\rho}{2} \frac{d}{dz} \right)^{m+2r} \frac{(-1)^r z^n}{(m+r)! \, r!}$$
$$= \sum_{r=0}^{r=\infty} \left(\frac{\rho}{2} \right)^{m+2r} z^{n-m-2r} \frac{(-1)^r n(n-1)\dots(n-m-2r+1)}{(m+r)! \, r!} \quad (12)$$

If n is fractional the series actually extends to infinity, and it is absolutely and uniformly convergent provided

$$|\rho| < |z|.$$

If n is integral the series terminates and the summation extends from $r=0$ to $r=\nu$, where

$$\nu = \frac{n-m}{2} \quad \text{if } n-m \text{ be even};$$

$$\nu = \frac{n-m-1}{2} \quad \text{if } n-m \text{ be odd}.$$

6. If we write

$$\rho = R\sin\theta, \qquad z = R\cos\theta,$$

$$J_m\left(\rho\frac{d}{dz}\right)z^n = R^n J_m\left(\sin\theta\,\frac{d}{d(\cos\theta)}\right)\cos^n\theta$$

$$= R^n J_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^n \quad . \quad . \quad (13)$$

if $\mu = \cos\theta$;

where the meaning of the operator is that the Bessel's function is expanded in powers as if $\dfrac{d}{d\mu}$, or $\dfrac{d}{d(\cos\theta)}$, were a symbol of quantity, then in each term $\left(\dfrac{d}{d\mu}\right)^n$ or $\left(\dfrac{d}{d(\cos\theta)}\right)^n$ is replaced by $\dfrac{d^n}{d\mu^n}$ or $\dfrac{d^n}{d(\cos\theta)^n}$ and made to operate on the quantity to the right of the original operator. Hence

$$R^n{\cos\atop\sin}\{m\phi\} \times \sum_{r=0}^{r=\infty}\left(\frac{\sin\theta}{2}\right)^{m+2r}(\cos\theta)^{n-m-2r}\frac{(-1)^r n(n-1)\ldots(n-m-2r+1)}{(m+r)\,!\,r\,!}$$

is a spherical harmonic of degree n and order m, the Σ stopping at $r=\nu$ if n is an integer,

Hence

$$\sum_{r=0}^{r=\infty}\left(\frac{\sin\theta}{2}\right)^{m+2r}(\cos\theta)^{n-m-2r}(-1)^r\frac{n(n-1)\ldots(n-m-2r+1)}{(m+r)\,!\,r\,!}$$

$$= J_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^n$$

must be a constant multiple of the Legendre's function of order m and degree n.

7. To identify this constant multiple

$$\pi i^m J_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^n = \int_0^\pi e^{(i\cos u\sqrt{1-\mu^2}\frac{d}{d\mu})}\mu^n\cos mu\,du.\ (14)$$

But

$$P_n^m(\mu) = \frac{(n+m)\,!}{n\,!}\frac{(-i)^m}{\pi}\int_0^\pi(\mu+\sqrt{\mu^2-1}\cos u)^n\cos mu\,du\ (15),$$

where $P_n^m(\mu)$ is the associated Legendre's coefficient of degree n and order m. (See Whittaker, 'Modern Analysis,' p. 235, or Todhunter, ' Functions of Laplace, Lamé, and Bessel.')

Comparing (15) and (14),

$$\mathrm{P}_n^m(\mu) = \frac{(n+m)\,!}{n\,!}\,\mathrm{J}_m\bigg(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\bigg)\mu^n \quad \cdot \ \cdot \quad (16)$$

when n is an integer, and it is

$$= \frac{\Pi(n+m)}{\Pi(n)}\,\mathrm{J}_m\bigg(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\bigg)\mu^n \quad \cdot \quad \cdot \quad \cdot \quad (17)$$

for other values of n.

Lord Rayleigh ('Theory of Sound,' vol. ii. p. 230 of the 1st edition) has given a curious parallel to this, namely,

$$\mathrm{J}_{n+\frac{1}{2}}(z) = \sqrt{\frac{2z}{\pi}}\,i^{-n}\mathrm{P}_n\bigg(\frac{d}{i\,dz}\bigg)\bigg(\frac{\sin z}{z}\bigg)$$

giving the Bessel's function of order half an odd integer as a Legendre coefficient operator acting on an elementary algebraic expression.

8. From this symbolic expression for the Legendre functions, it is easy to deduce that the Legendre coefficients $\mathrm{P}_n(\mu)$ are the coefficients in the expansion of $(1-2\mu t+t^2)^{-\frac{1}{2}}$ in powers of t. For since, putting $m=0$ in the last result, we have

$$\mathrm{P}_n(\mu) = \mathrm{J}_0\bigg(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\bigg)\mu^n$$

$$\mathrm{P}_0 + t\mathrm{P}_1 + t^2\mathrm{P}_2 + \ldots + t^n\mathrm{P}_n + \ldots$$
$$= \mathrm{J}_0\bigg(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\bigg)(1 + t\mu + \ldots + t^n\mu^n + \ldots)$$
$$= \mathrm{J}_0\bigg(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\bigg)\frac{1}{1-t\mu}$$
$$= \sum_{r=0}^{r=\infty} \frac{(\sqrt{1-\mu^2})^{2r}(-1)^r}{2^{2r}(r\,!)^2}\frac{d^{2r}}{d\mu^{2r}}\bigg(\frac{1}{1-t\mu}\bigg)$$
$$= \sum_{r=0}^{r=\infty} \frac{(\mu^2-1)^r t^{2r}}{(1-\mu t)^{2r+1}}\frac{(2r)\,!}{2^{2r}(r\,!)^2}$$
$$= \frac{1}{1-\mu t}\sum_{r=0}^{r=\infty}\bigg[\frac{(\mu^2-1)t^2}{(1-\mu t)^2}\bigg]^r\frac{(2r-1)\ldots 3\,.\,1}{2^r r\,!}$$
$$= \frac{1}{1-\mu t}\bigg(1 - \frac{(\mu^2-1)t^2}{(1-\mu t)^2}\bigg)^{-\frac{1}{2}}$$
$$= \frac{1}{\sqrt{1-2\mu t+t^2}}.$$

The expansion of $\dfrac{1}{\sqrt{1-2\mu t+t^2}}$ can be shown to be absolutely convergent if $|t| < 1$. And the expansion of

$\dfrac{1}{1-t\mu}$ is then also convergent, if $|\mu| \not< 1$, which is the case in physical applications, where $\mu = \cos\theta$. The convergence of the series for $\left(1 - \dfrac{(\mu^2-1)t^2}{(1-\mu t)^2}\right)^{-\frac{1}{2}}$ requires $\dfrac{t\sqrt{1-\mu^2}}{1-\mu t} < 1$, or $t < \dfrac{1}{\mu + \sqrt{1-\mu^2}}$. Since the greatest value of $\mu + \sqrt{1-\mu^2}$ is $\sqrt{2}$ when μ is the cosine of a real angle all the conditions for the validity of the expansions are satisfied if $t < \dfrac{1}{\sqrt{2}}$.

In like manner, when t is large, we can use an expansion in inverse powers of t, and we find

$$\frac{1}{t}\,\mathrm{P}_0 + \ldots + \frac{1}{t^{n+1}}\,\mathrm{P}_n + \ldots = \mathrm{J}_0\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\frac{1}{t-\mu}$$

$$= \frac{1}{\sqrt{1-2t\mu+t^2}} \text{ as before.} \quad . \quad (18)$$

9. We can, however, generalize this result and obtain all the associated Legendre coefficients as the coefficients in the expansion of a function of two variables.

Remembering the result

$$e^{\frac{z}{2}\left(u-\frac{1}{u}\right)} = \sum_{m=-\infty}^{m=+\infty} u^m \mathrm{J}_m(z) \quad . \quad . \quad . \quad (19)$$

(see Gray and Mathew's 'Bessel Functions,' p. 17),

let us write in this $z = \sqrt{1-\mu^2}\,\dfrac{d}{d\mu}$ and apply this operator to the function $\dfrac{1}{t-\mu}$,

$$e^{\left\{\frac{1}{2}\sqrt{1-\mu^2}\left(u-\frac{1}{u}\right)\frac{d}{d\mu}\right\}}\left(\frac{1}{t-\mu}\right) = \sum_{m=-\infty}^{m=+\infty} u^m \mathrm{J}_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\frac{1}{t-\mu}\ (20)$$

Now apply the symbolic form of Taylor's theorem to the left-hand side and expand $\dfrac{1}{t-\mu}$ on the right-hand side in powers of $1/t$, t being supposed large.

$$\frac{1}{t-\mu-\frac{1}{2}\sqrt{1-\mu^2}\left(u-\frac{1}{u}\right)} = \sum_{m=-\infty}^{m=+\infty}\sum_{n=0}^{n=\infty} u^m t^{-n-1}\mathrm{J}_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^n$$

$$= \sum_{m=-\infty}^{m=+\infty}\sum_{n=0}^{n=\infty} \frac{u^m t^{-n-1} n\,!}{(n+m)\,!}\,\mathrm{P}_n^m(\mu), \text{ by (16).} \quad . \quad (21)$$

Hence the associated Legendre's coefficient P_n^m is given as a

multiple of the coefficient of $u^m t^{-n-1}$ in the expansion of

$$\psi = \cfrac{1}{t-\mu-\frac{1}{2}\sqrt{1-\mu^2}\left(u-\dfrac{1}{u}\right)}, \quad \cdot \quad \cdot \quad \cdot \quad (22)$$

ψ being expanded in a Laurent series in u and in inverse powers of t.

10. We may now show by direct substitution that ψ satisfies the differential equation

$$\frac{d}{d\mu}(1-\mu^2)\frac{d\psi}{d\mu}+\frac{d}{dt}t^2\frac{d\psi}{dt}-\frac{u}{1-\mu^2}\cdot\frac{d}{du}\left(u\frac{d\psi}{du}\right)=0. \quad (23)$$

For

$$\frac{1}{\psi}=t-\mu-\tfrac{1}{2}\sqrt{1-\mu^2}\left(u-\frac{1}{u}\right),$$

$$\frac{1}{\psi^2}\frac{d\psi}{d\mu}=1-\frac{\mu}{2\sqrt{1-\mu^2}}\left(u-\frac{1}{u}\right),$$

$$\frac{1}{\psi^2}\frac{d\psi}{dt}=-1,$$

$$\frac{1}{\psi^2}\frac{d\psi}{du}=\tfrac{1}{2}\sqrt{1-\mu^2}\left(1+\frac{1}{u^2}\right);$$

whence

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\psi}{d\mu}\right\}=\frac{d}{d\mu}\left[\psi^2\left(1-\mu^2-\frac{\mu}{2}\sqrt{1-\mu^2}\left(u-\frac{1}{u}\right)\right)\right]$$

$$=2\psi^3\left\{1-\frac{\mu}{2\sqrt{1-\mu^2}}\left(u-\frac{1}{u}\right)\right\}\left\{1-\mu^2-\frac{\mu}{2}\sqrt{1-\mu^2}\left(u-\frac{1}{u}\right)\right\}$$

$$+\psi^2\left\{-2\mu-\left(\frac{\sqrt{1-\mu^2}}{2}-\frac{\mu^2}{2\sqrt{1-\mu^2}}\right)\left(u-\frac{1}{u}\right)\right\},$$

$$\frac{d}{dt}\left(t^2\frac{d\psi}{dt}\right)=-2t\psi^2+2t^2\psi^3,$$

$$u\frac{d}{du}\left(u\frac{d\psi}{du}\right)=\frac{\psi^2}{2}\sqrt{1-\mu^2}\left(u-\frac{1}{u}\right)+\frac{\psi^3}{2}(1-\mu^2)\left(u+\frac{1}{u}\right)^2.$$

From which

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\psi}{d\mu}\right\}+\frac{d}{dt}\left(t^2\frac{d\psi}{dt}\right)-\frac{u}{1-\mu^2}\frac{d}{du}u\frac{d\psi}{du}$$

$$=\psi^3\left[\begin{array}{l}\left\{-2\mu-\sqrt{1-\mu^2}\left(u-\dfrac{1}{u}\right)-2t\right\}\left\{t-\mu-\tfrac{1}{2}\sqrt{1-\mu^2}\left(u-\dfrac{1}{u}\right)\right\}\\[2mm] +2(1-\mu^2)\left(1-\dfrac{\mu}{2\sqrt{1-\mu^2}}\left(u-\dfrac{1}{u}\right)\right)\\[2mm] +2t^2-\tfrac{1}{2}\left(u+\dfrac{1}{u}\right)^2\end{array}\right]$$

$=0$ after some easy reductions.

Substitute now in the differential equation the expansion

$$\psi = \Sigma\Sigma \frac{u^m t^{-n-1} n!}{(n+m)!} P_n^m(\mu)$$

and equate to zero the coefficient of $u^m t^{-n-1}$ in the result; we obtain the differential equation satisfied by the associated Legendre's coefficient, namely,

$$\frac{d}{d\mu}(1-\mu^2)\frac{dP_n^m}{d\mu} + \left\{ n(n+1) - \frac{m^2}{1-\mu^2} \right\} P_n^m = 0. \quad . \quad (24)$$

11. The last result enables us to obtain various expressions for P_n^m.

For the expression

$$\frac{2(t-\cos\theta)}{\sin\theta} - \left(u - \frac{1}{u}\right)$$

may be broken up into factors

$$(\alpha - u)\left(\alpha + \frac{1}{u}\right)$$

where

$$\frac{\alpha^2 - 1}{\left\{\dfrac{2(t-\cos\theta)}{\sin\theta}\right\}} = \frac{\alpha}{1},$$

$$\alpha^2 - \frac{2\alpha(t-\cos\theta)}{\sin\theta} = 1,$$

$$\alpha = \frac{t-\cos\theta \pm \sqrt{t^2 - 2t\cos\theta + 1}}{\sin\theta}.$$

Let us take that sign of the square root which makes $|\alpha| = \infty$ when $|t| = \infty$. If then $\sqrt{t^2 - 2t\cos\theta + 1}$ denote that branch of the function which is real and positive when t is large, real, and positive we have to take the $+$ sign and we have

$$\alpha = \frac{t-\cos\theta + \sqrt{t^2 - 2t\cos\theta + 1}}{\sin\theta}. \quad . \quad . \quad (25)$$

Then

$$\frac{1}{t-\cos\theta - \dfrac{\sin\theta}{2}\left(u - \dfrac{1}{u}\right)} = \frac{2\alpha\,\mathrm{cosec}\,\theta}{(\alpha - u)\left(\alpha + \dfrac{1}{u}\right)}$$

$$= \left(\frac{u}{\alpha - u} + \frac{\alpha}{\alpha + \dfrac{1}{u}}\right)\frac{2\alpha\,\mathrm{cosec}\,\theta}{1 + \alpha^2}. \quad . \quad . \quad (26)$$

Suppose $|t|$ large, so that $|\alpha|$ is large.

Then we may, if $|u|$ be taken sufficiently small, expand $\dfrac{1}{\alpha-u}$ in powers of $\dfrac{u}{\alpha}$ and $\dfrac{1}{\alpha+\dfrac{1}{u}}$ in powers of $\dfrac{1}{u\alpha}$.

Hence

$$\frac{1}{t-\cos\theta-\dfrac{\sin\theta}{2}\left(u-\dfrac{1}{u}\right)}=\left(\sum_{m=1}^{m=\infty}\frac{u^m}{\alpha^m}+\sum_{m=0}^{m=\infty}\frac{(-1)^m}{u^m\alpha^m}\right)\frac{2\alpha\,\operatorname{cosec}\theta}{1+\alpha^2}. \quad (27)$$

Now

$$1+\alpha^2=2\alpha\,\operatorname{cosec}\theta\sqrt{t^2-2t\cos\theta+1}. \quad . \quad . \quad (28)$$

So that finally

$$\frac{1}{t-\cos\theta-\dfrac{\sin\theta}{2}\left(u-\dfrac{1}{u}\right)}=\frac{1}{\sqrt{t^2-2t\cos\theta+1}}\left[\sum_{m=1}^{m=\infty}\frac{u^m}{\alpha^m}+\sum_{m=0}^{m=\infty}\frac{(-1)^m}{u^m\alpha^m}\right].$$

The coefficient of u^m is

$$\frac{1}{\sqrt{t^2-2t\cos\theta+1}}\ \frac{\sin^m\theta}{(t-\cos\theta+\sqrt{t^2-2t\cos\theta+1})^m}$$
$$=(-1)^m\ \text{coeff. of } u^{-m}.$$

Hence $\dfrac{n!}{(n+m)!}\,P_n^m(\mu)$ is the coefficient of t^{-n-1} in the expansion of

$$\frac{1}{(t^2-2t\mu+1)^{\frac{1}{2}}}\ \frac{(1-\mu^2)^{\frac{m}{2}}}{(t-\mu+\sqrt{t^2-2\mu t+1})^m} \quad . \quad . \quad (29)$$

in inverse powers of t.

When $m=0$ this gives the Legendre's coefficients as the coefficients in the expansion of $(t^2-2t\mu+1)^{-\frac{1}{2}}$, as it should.

12. In the above t may be anything. Let us take t a complex variable. Then remembering the expansion is one about infinity, we have for the coefficient of t^{-n-1}

$$\frac{1}{2\pi i}\int\frac{(1-\mu^2)^{\frac{m}{2}}(t^2-2t\mu+1)^{-\frac{1}{2}}t^n}{(t-\mu+\sqrt{t^2-2\mu t+1})^m}dt, \quad . \quad . \quad (30)$$

where the integral is taken round a circle (or any contour) inclosing all the singularities of the integrand. Here the

singularities are the roots of

$$t^2 - 2t\mu + 1 = 0 ;$$

i. e.

$$t = \mu \pm \sqrt{\mu^2 - 1},$$

which are branch-points.

The function $t - \mu + \sqrt{t^2 - 2\mu t + 1}$ is not zero anywhere, unless $\mu = \pm 1$.

Since the contour is supposed carried round *both* branch-points $t = \mu \pm \sqrt{\mu^2 - 1}$, the function $\sqrt{t^2 - 2\mu t + 1}$ resumes its initial value after describing the contour.

Now $\dfrac{d}{dt} (t - \mu + \sqrt{t^2 - 2\mu t + 1}) = 1 + \dfrac{t - \mu}{\sqrt{t^2 - 2\mu t + 1}}.$

Hence

$$\frac{1}{2\pi i} \int \frac{t^n \, dt}{\sqrt{t^2 - 2\mu t + 1}\,(t - \mu + \sqrt{t^2 - 2\mu t + 1})^m} = (1 - \mu^2)^{-\frac{m}{2}} \frac{n\,!}{(n+m)\,!} \mathrm{P}_n^m$$

$$= \frac{1}{2\pi i} \int \frac{t^n}{(t - \mu + \sqrt{t^2 - 2\mu t + 1})^{m+1}} \frac{d}{dt} (t - \mu + \sqrt{t^2 - 2t\mu + 1}) dt. \quad (31)$$

Integrate by parts: then because the contour goes round both branch-points the integrated terms vanish, and the integral

$$= \frac{1}{2\pi i} \frac{n}{m} \int \frac{t^{n-1}}{(t - \mu + \sqrt{t^2 - 2t\mu + 1})^m} dt. \quad \cdot \quad \cdot \quad \cdot \quad (32)$$

Let us differentiate this with regard to μ in this new form:

$$\frac{n\,!}{(n+m)\,!} \frac{d}{d\mu} \left((1 - \mu^2)^{-\frac{m}{2}} \mathrm{P}_n^m(\mu) \right)$$

$$= + \frac{n}{2\pi i} \int \left\{ \frac{t^{n-1}}{(t - \mu + \sqrt{t^2 - 2t\mu + 1})^{m+1}} \right.$$

$$\left. + \frac{t^n}{\sqrt{t^2 - 2\mu t + 1}\,(t - \mu + \sqrt{t^2 - 2t\mu + 1})^{m+1}} \right\} dt. \quad (33)$$

But from (31) and (32)

$$\frac{1}{2\pi i} \int \frac{t^{n-1} dt}{(t - \mu + \sqrt{t^2 - 2t\mu + 1})^{m+1}} = \frac{m+1}{n} (1 - \mu^2)^{-\frac{m+1}{2}} \frac{n\,!}{(n+m+1)\,!} \mathrm{P}_n^{m+1}(\mu),$$

$$\frac{1}{2\pi i} \int \frac{t^n (t^2 - 2\mu t + 1)^{-\frac{1}{2}}}{(t - \mu + \sqrt{t^2 - 2t\mu + 1})^{m+1}} dt = (1 - \mu^2)^{-\frac{m+1}{2}} \frac{n\,!}{(n+m+1)\,!} \mathrm{P}_n^{m+1}(\mu) ;$$

$$\therefore \quad \frac{n\,!}{(n+m)\,!} \frac{d}{d\mu} \left((1 - \mu^2)^{-\frac{m}{2}} \mathrm{P}_n^m(\mu) \right) = \frac{(n+m+1)\,n\,!}{(n+m+1)\,!} (1 - \mu^2)^{-\frac{m+1}{2}} \mathrm{P}_n^{m+1}(\mu) ;$$

$$\therefore \quad (1 - \mu^2)^{-\frac{m+1}{2}} \mathrm{P}_n^{m+1}(\mu) = \frac{d}{d\mu} (1 - \mu^2)^{-\frac{m}{2}} \mathrm{P}_n^m(\mu). \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (34)$$

Repeating this m times, we find

$$(1-\mu^2)^{-\frac{m}{2}} P_n^m(\mu) = \frac{d^m}{d\mu^m} P_n(\mu), \quad . \quad . \quad . \quad (35)$$

a well-known relation.

13. If we integrate the right-hand side of the equation

$$P_n(\mu) = \frac{1}{2\pi i} \int \frac{t^n}{\sqrt{t^2 - 2\mu t + 1}} dt \quad . \quad . \quad (36)$$

n times with regard to μ, and equate $P_n(\mu)$ to the nth differential coefficient of this integrated right-hand side, we obtain:

$$P_n(\mu) = \frac{(-1)^n}{1 \cdot 3 \dots (2n-1)} \frac{1}{2\pi i} \frac{d^n}{d\mu^n} \int (1-2\mu t + t^2)^{\frac{2n-1}{2}} dt$$

$$= \frac{(-1)^n}{1 \cdot 3 \dots (2n-1)} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n \times \frac{1}{2\pi i} \int u^{2n-1}(1-u^{-2})^{\frac{2n-1}{2}} du, \quad (37)$$

where $u = (t-\mu)/\sqrt{\mu^2 - 1}$, and the integral is taken round a contour in the u-plane inclosing the points $u = \pm 1$.

Thus

$$P_n(\mu) = \frac{(-1)^n}{1 \cdot 3 \dots (2n-1)} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n \times \text{coeff. of } x^n \text{ in } (1-x)^{\frac{2n-1}{2}}$$

$$= \frac{1}{2^n n!} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Hence by the previous result

$$P_n^m(\mu) = \frac{(1-\mu^2)^{\frac{m}{2}}}{2^n n!} \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n. \quad . \quad . \quad (39)$$

So that all the well-known forms for the Legendre coefficients flow quite easily from their definition as multiples of

$$J_m\left(\sqrt{1-\mu^2} \frac{d}{d\mu}\right) \mu^n.$$

14. Further, it can be easily shown that the recurrence formulæ for the Legendre functions follow directly from the recurrence formulæ and the differential equation for Bessel functions. And in this it will not be necessary to suppose that n is an integer, but we may suppose it anything we please.

Using the extended symbolical form of Leibnitz's theorem, we have

$$f(D)uv = uf(D)v + Du \cdot \frac{d}{dD} f(D) \cdot v + \frac{D^2}{2!} u \cdot \frac{d^2}{dD^2} f(D) \cdot v + \dots \quad (40)$$

where $D = d/dx$.

Put in this $D = \frac{d}{d\mu}$,

$$f(D) = J_m\left(\sqrt{1-\mu^2} \frac{d}{d\mu}\right) = J_m(\sqrt{1-\mu^2}D) \; ;$$

and remember that, in differentiation with regard to D, $\sqrt{1-\mu^2}$ is to be treated as a constant.

We have then

$$J_m\left(\sqrt{1-\mu^2} \frac{d}{d\mu}\right) \mu \cdot \mu^{n-1}$$

$$= \mu J_m\left(\sqrt{1-\mu^2} \frac{d}{d\mu}\right)\mu^{n-1} + \sqrt{1-\mu^2} J_m{}'\left(\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\mu^{n-1}. \quad (41)$$

But we have

$$J_m{}' = \tfrac{1}{2}(J_{m-1} - J_{m+1}). \quad . \quad . \quad . \quad . \quad (42)$$

Let me denote $\dfrac{\Pi(n)}{\Pi(n+m)} P_n^m(\mu)$ by $P_n^m(\mu)$.

Then

$$P_n^m = J_m\left(\sqrt{1-\mu^2} \frac{d}{d\mu}\right) \mu^n; \quad . \quad . \quad . \quad (43)$$

and the equation above gives

$$P_n^m = \mu P_{n-1}^m + \frac{\sqrt{1-\mu^2}}{2}(P_{n-1}^{m-1} - P_{n-1}^{m+1}), \quad . \quad . \quad (44)$$

which is one of the recurrence formulæ.

But we have also

$$J_m{}'(z) = \frac{m}{z} J_m(z) - J_{m+1}(z). \quad . \quad . \quad . \quad (45)$$

Substituting this latter value of $\mathbf{J}_m'$, we have

$$\mathbf{J}_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^n = \mu\mathbf{J}_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^{n-1}$$

$$+ m\mathbf{J}_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\left(\frac{1}{\frac{d}{d\mu}}\right)\mu^{n-1} - \sqrt{1-\mu^2}\mathbf{J}_{m+1}\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^{n-1}$$

$$P_n^m = \mu P_{n-1}^m + \frac{m}{n}P_n^m - \sqrt{1-\mu^2}P_{n-1}^{m+1},$$

$$\frac{n-m}{n}P_n^m = \mu P_{n-1}^m - \sqrt{1-\mu^2}P_{n-1}^{m+1}, \quad . \quad . \quad . \quad (46)$$

another recurrence formula.

Eliminating P_n^m between (44) and (46), I find

$$\frac{2\mu m}{\sqrt{1-\mu^2}}P_{n-1}^m = (n-m)P_{n-1}^{m-1} + (n+m)P_{n-1}^{m+1};$$

replacing the P's by the $\mathbf{P}$'s and writing $n+1$ for n, this gives

$$(n-m+1)(n+m)\mathbf{P}_n^{m-1} + \mathbf{P}_n^{m+1} = \frac{2m\mu}{\sqrt{1-\mu^2}}\mathbf{P}_n^m, \quad . \quad (47)$$

a well-known recurrence form.

15. The recurrence formula in which m is constant and n varies by an integer may be similarly obtained.

$$\mathbf{J}_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^n = \mathbf{J}_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^2 \cdot \mu^{n-2}$$

$$= \mu^2\mathbf{J}_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^{n-2} + 2\mu\sqrt{1-\mu^2}\mathbf{J}_m'\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^{n-2}$$

$$+ (1-\mu^2)\mathbf{J}_m''\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)\mu^{n-2}.$$

But the Bessel's functions satisfy the relation

$$\mathbf{J}_m''(z) = -\frac{1}{z}\mathbf{J}_m'(z) + \left(\frac{m^2}{z^2} - 1\right)\mathbf{J}_m(z). \quad . \quad (48)$$

Hence

$$\mathbf{J}_m\left(\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\mu^n = \mu^2\mathbf{J}_m\left(\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\mu^{n-2}$$

$$+ 2\mu\sqrt{1-\mu^2}\,\mathbf{J}_m{}'\left(\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\mu^{n-2}$$

$$- \frac{\sqrt{1-\mu^2}}{n-1}\mathbf{J}_m{}'\left(\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\mu^{n-1} + \frac{m^2}{n(n-1)}\mathbf{J}_m\left(\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\mu^n$$

$$- (1-\mu^2)\mathbf{J}_m\left(\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\mu^{n-2}$$

Now substitute in this for $\mathbf{J}_m{}'$ from (45) and use (43),

$$P_n^m = \mu^2 P_{n-2}^m + \frac{2\mu m}{n-1}P_{n-1}^m - 2\mu\sqrt{1-\mu^2}P_{n-2}^{m+1}$$

$$- \frac{m}{n(n-1)}P_n^m + \frac{\sqrt{1-\mu^2}}{n-1}P_{n-1}^{m+1} + \frac{m^2}{n(n-1)}P_n^m - (1-\mu^2)P_{n-2}^m \quad (49)$$

But from formula (46)

$$\sqrt{1-\mu^2}\,P_{n-1}^{m+1} = \mu P_{n-1}^m - \frac{n-m}{n}P_n^m$$

$$\sqrt{1-\mu^2}\,P_{n-2}^{m+1} = \mu P_{n-2}^m - \frac{n-m-1}{n-1}P_{n-1}^m.$$

Substituting into (49) for P_{n-1}^{m+1}, P_{n-2}^{m+1}, it becomes, after reduction,

$$(n+m)(n-m)P_n^m + n(n-1)P_{n-2}^m - \mu n(2n-1)P_{n-1}^m = 0,$$

or in terms of P's, and writing $n+1$ for n,

$$(n-m+1)\mathrm{P}_{n+1}^m + (n+m)\mathrm{P}_{n-1}^m = \mu(2n+1)\mathrm{P}_n^m, \quad (50)$$

another well-known recurrence formula which includes, when $m=0$, the recurrence formula for Legendre functions.

16. The formulæ involving the derived function of the ordinary Legendre function $\mathrm{P}_n(\mu)$ can be obtained in a similar manner.

$$\frac{d\mathrm{P}_n}{d\mu} = \frac{d}{d\mu}\mathbf{J}_0\left(\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\mu^n.$$

Now

$$\mathrm{D}(f(\mu)\mathrm{D})^r = \mathrm{D}\{f(\mu)\}^r\mathrm{D}^r$$

$$= r\{f(\mu)\}^{r-1}f'(\mu)\,.\,\mathrm{D}^r + \{f(\mu)\}^r\mathrm{D}^{r+1}$$

$$= \frac{\partial}{\partial\mu}[\{f(\mu)\}^r\mathrm{D}^r] + [\{f(\mu)\}^r\mathrm{D}^r]\mathrm{D},$$

$\dfrac{\partial}{\partial \mu}$ denoting partial differentiation when μ and D are independent variables, and D having the same meaning as before. Hence, for any function F which can be expanded in powers

$$DF(f(\mu)D) = \frac{\partial}{\partial \mu} F(f(\mu)D) + F(f(\mu)D) \cdot D.$$

$$\therefore \quad \frac{dP_n}{d\mu} = \frac{-\mu}{\sqrt{1-\mu^2}} J_0''\left(\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\frac{d}{d\mu}\mu^n$$
$$+ J_0\left(\sqrt{1-\mu^2}\frac{d}{d\mu}\right)\frac{d}{d\mu}\mu^n.$$

But $J_0' = -J_1$;

$$\therefore \quad \frac{dP_n}{d\mu} = nP_{n-1} + \frac{\mu n}{\sqrt{1-\mu^2}}P^1_{n-1}.$$

Now putting $m=0$ into (46) and remembering that $P_n^0 = P_n$, we have

$$\sqrt{1-\mu^2}P^1_{n-1} = \mu P_{n-1} - P_n.$$

Hence, substituting for P^1_{n-1},

$$\frac{dP_n}{d\mu} = n\left[\frac{P_{n-1}}{1-\mu^2} - \frac{\mu P_n}{1-\mu^2}\right]. \quad . \quad . \quad . \quad (51)$$

From this

$$(1-\mu^2)\left(\frac{dP_{n+1}}{d\mu} - \frac{dP_{n-1}}{d\mu}\right) = (n+1)[P_n - \mu P_{n+1}]$$
$$- (n-1)[P_{n-2} - \mu P_{n-1}]. \quad (52)$$

But from (50), putting $m=0$,

$$(n+1)P_{n+1} + nP_{n-1} = \mu(2n+1)P_n$$
$$nP_n + (n-1)P_{n-2} = \mu(2n-1)P_{n-1}.$$

Substituting for P_{n+1} and P_{n-2} into (52), we obtain, after reduction,

$$\frac{dP_{n+1}}{d\mu} - \frac{dP_{n-1}}{d\mu} = (2n+1)P_n. \quad . \quad . \quad (53)$$

17. An interesting identity of two series of Legendre associated coefficients, one proceeding in a series of varying m, the other in a series of varying n, follows at once from the expression of P_n^m as a Bessel operator on μ^n.

For consider

$$\mathbf{J}_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)(1-\mu^2)^n$$

$$=\mathbf{J}_m\left(\sqrt{1-\mu^2}\,\frac{d}{d\mu}\right)(1-n\mu^2+\ldots+(-1)_n^r\mathbf{C}_r\mu^{2r}+\ldots+(-1)^n\mu^{2n}),$$

$$\therefore\quad(1-\mu^2)^{\frac{n}{2}}\sum_{r=0}^{r=\rho}\frac{\mathbf{P}_n^{m+2r-n}(-1)^{r+n}n!}{2^{m-n+2r}r!\,(m+r)!}$$

$$=\sum_{s=\sigma}^{s=n}\mathbf{P}_{2s}^m(-1)^s\frac{n!}{s!\,(n-s)!},$$

where $\rho=\left(n-\dfrac{m}{2}\right)$ if m be even,

$\qquad\rho=\left(n-\dfrac{m+1}{2}\right)$ if m be odd,

$\qquad\sigma=\dfrac{m}{2}\qquad$ if m be even,

$\qquad\sigma=\dfrac{m+1}{2}\qquad$ if m be odd.

Hence

$$\sum_{r=0}^{r=\rho}\frac{(-1)^{r+n}n!\,(1-\mu^2)^{\frac{n}{2}}\mathbf{P}_n^{m+2r-n}}{(m+r)!\,r!\,2^{m+2r-n}}=\sum_{s=\sigma}^{s=n}\frac{(-1)^s n!\,(2s)!\,\mathbf{P}_{2s}^m}{s!\,(m+2s)!\,(n-s)!}.\quad(54)$$

18. It might be thought that analogous symbolic expressions could be obtained for the Q-functions.

Thus we know that if

$$\mathbf{P}_n=\int_c\frac{t^n}{\sqrt{1-2\mu t+t^2}}\,dt$$

taken round a contour surrounding the two branch-points $t=\mu\pm\sqrt{\mu^2-1}$, then

$$\mathbf{Q}_n=\int_c\frac{t^n}{\sqrt{1-2\mu t+t^2}}\,dt$$

taken round a figure of eight, each of whose loops incloses one of these branch-points.

Now in the case of the first contour

$$(1-2\mu t+t^2)^{-\frac{1}{2}}=\frac{1}{t-\mu}\left(1-\frac{\mu^2-1}{(t-\mu)^2}\right)^{-\frac{1}{2}}$$

can be expanded in a convergent series of powers of

$$(\mu^2 - 1)/(t - \mu)^2,$$

for $\left| \dfrac{\mu^2 - 1}{(t - \mu)^2} \right|$ can be kept < 1 at all points of the contour. We then find

$$(1 - 2\mu t + t^2)^{-\frac{1}{2}} = J_0\left(\sqrt{1 - \mu^2}\,\frac{d}{d\mu}\right)\frac{1}{t - \mu};$$

$$\therefore \quad P_n = J_0\left(\sqrt{1 - \mu^2}\,\frac{d}{d\mu}\right)\int \frac{t^n}{t - \mu}\,dt$$

$$= J_0\left(\sqrt{1 - \mu^2}\,\frac{d}{d\mu}\right)\mu^n.$$

But in the case of the figure of eight there must be points on the contour nearer to $t = \mu$ than the two branch-points. This is obvious, since the contour must cut the line joining the points $\pm\sqrt{\mu^2 - 1}$ *between* these points. At these intersections $\left| \sqrt{\mu^2 - 1} \right| > \left| t - \mu \right|$.

But further, the loop must also cut the same line *outside* the points, and at these intersections $\left| \sqrt{\mu^2 - 1} \right| < \left| t - \mu \right|$.

Hence the same expansion will not serve for all parts of the contour, and the method fails.

19. Some of the associated Legendre's functions of the second kind, however, can be obtained by expanding the function

$$\psi = \frac{1}{t - \mu - \frac{1}{2}\sqrt{1 - \mu^2}\left(u - \dfrac{1}{u}\right)}.$$

For if we write this

$$\psi = \frac{u}{\frac{1}{2}\sqrt{1 - \mu^2}(1 - u^2) + u(t - \mu)}.$$

then, if u and t be small enough, ψ can be expanded in an absolutely convergent series of positive powers of u and t.

And in virtue of the differential equation (23) satisfied by ψ, the coefficient of $u^m t^n$ will be in all cases a solution of the differential equation of the associated Legendre functions.

The denominator of ψ may be broken up, as before, into factors, and we find

$$\psi = \frac{-2u\sqrt{1 - \mu^2}}{(u\sqrt{1 - \mu^2} - t + \mu - \sqrt{t^2 - 2t\mu + 1})(u\sqrt{1 - \mu^2} - t + \mu + \sqrt{t^2 - 2t\mu + 1})}. \quad (55)$$

But

$$\frac{-2x}{(x-\alpha)(x-\beta)} = \frac{2}{\alpha-\beta}\left(\frac{1}{1-\frac{x}{\alpha}} - \frac{1}{1-\frac{x}{\beta}}\right)$$

$$= \sum_{m=1}^{m=\infty} \frac{2x^m}{(\alpha-\beta)}\left(\frac{1}{\alpha^m} - \frac{1}{\beta^m}\right).$$

Hence coefficient of u^m in (52)

$$= \frac{1}{\sqrt{t^2-2t\mu+1}}\left(\frac{(\sqrt{1-\mu^2})^m}{(t-\mu+\sqrt{t^2-2t\mu+1})^m} - \frac{(\sqrt{1-\mu^2})^m}{(t-\mu-\sqrt{t^2-2t\mu+1})^m}\right)$$

$$= \frac{(-1)^{m+1}}{\sqrt{t^2-2t\mu+1}}\left[\frac{(t-\mu+\sqrt{t^2-2t\mu+1})^m - (t-\mu-\sqrt{t^2-2t\mu+1})^m}{(1-\mu^2)^{\frac{m}{2}}}\right]$$

$$= \frac{(-1)^{m+1}}{(1-\mu^2)^{\frac{m}{2}}}\left\{m(t-\mu)^{m-1} + \frac{m(m-1)(m-2)}{3!}(t-\mu)^{m-3}(t^2-2\mu+1) + \ldots\right\}.(56)$$

The coefficients of the various powers of t in this are of the form

$$\frac{1}{(1-\mu^2)^{\frac{m}{2}}} \times \text{rational integral function of } \mu,$$

the coefficient of t^{m-1} being simply a constant multiple of

$$\frac{1}{(1-u^2)^{\frac{m}{2}}}.$$

It follows that these functions become infinite when $\mu = \pm 1$.

Hence the coefficients of t^{m-1}, t^{m-2}, ... are multiples of

$$Q^m_{m-1}, \quad Q^m_{m-2}, \quad \ldots \quad Q^m_1, \quad Q^m_0.$$

If we write down the known expressions

$$Q_n(\mu) = \tfrac{1}{2}P_n(\mu)\log\frac{\mu+1}{\mu-1} - W_{n-1}(\mu) \quad . \quad . \quad (57)$$

where W_{n-1} is an algebraic polynomial of degree $n-1$, we see, on differentiating m times, where $m > n$, that the logarithm must disappear and the points $\mu = \pm 1$ are simple poles.

20. Let us now return to our general solution for V,

$$V = \Sigma \frac{\cos}{\sin}(m\phi)J_m\left(\rho\frac{d}{dz}\right)f_m(z).$$

P 2

If in this we put $f_m(z) = \log z$ we obtain harmonics of zero degree.

These harmonics are given by the series

$$
\begin{aligned}
\mathrm{J}_m\left(\rho\,\frac{d}{dz}\right)\log z &= \sum_{r=0}^{r=\infty} \left(\frac{\rho}{2z}\right)^{m+2r} \frac{(-1)^{m+r-1}(m+2r-1)!}{r!\,(m+r)!} \\
&= \sum_{r=0}^{r=\infty} \left(\frac{\tan\theta}{2}\right)^{m+2r} \frac{(-1)^{m+r-1}(m+2r-1)!}{r!\,(m+r)!}. \quad . \quad (58)
\end{aligned}
$$

except when $m=0$, when the first term $=\log z$.

Another interesting sequence of harmonics is that given by putting

$$
f_m(z) = z^n \log z.
$$

If n be integral, these may be expanded in a series homogeneous in ρ and z, the coefficients being harmonics of zero degree, except the last $(n-m)$, which involve $\log z$. Thus :

$$
\begin{aligned}
&\mathrm{J}_m\left(\rho\,\frac{d}{dz}\right)z^n \log z \\
&= z^n \mathrm{J}_m\left(\rho\,\frac{d}{dz}\right)\log z \\
&\quad + n\rho z^{n-1}\mathrm{J}_m{}'\left(\rho\,\frac{d}{dz}\right)\log z \\
&\quad + \frac{n(n-1)}{1\,.\,2}\rho^2 z^{n-2}\mathrm{J}_m{}''\left(\rho\,\frac{d}{dz}\right)\log z \\
&\quad + .\ \ .\ \ .\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . \\
&\quad + \frac{n(n-1)\ldots(m+1)}{(n-m)!}\rho^{n-m}z^m \mathrm{J}_m{}^{(m)}\left(\rho\,\frac{d}{dz}\right)\log z \\
&\quad + \ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ .\ \ . \\
&\quad + \rho^n \mathrm{J}_m{}^{(n)}\left(\rho\,\frac{d}{dz}\right)\log z.\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . \ (59)
\end{aligned}
$$

and using the reduction formulæ for Bessel coefficients, the quantities multiplying the various powers of ρ may be exhibited as the sum of terms of the form

$$
\mathrm{J}_r\left(\rho\,\frac{d}{dz}\right)\log z.
$$

Such functions will be suitable in problems where the plane $z=0$ is a locus of discontinuity.

21. Although the greater number of the results obtained

here are not substantially new, it appeared of some interest to show how they could be deduced, in a simple manner, from the general solution of Laplace's equation. The method seems to me to throw some new light upon the relation of the Legendre to the Bessel functions. The latter, according to the view here set forth, should be looked upon as in some ways the more fundamental of the two.

There are two papers which deal with the expression of Legendre functions in terms of Bessel's functions. One of these is by Mr. Schafheitlin (Math. *Annalen*, vol. xxx.) on the representation of a hypergeometric function by definite integrals. Mr. Schafheitlin obtains the hypergeometric function as a definite integral involving a Bessel's function under the integral sign. The Legendre function is then obtained as a special case of the hypergeometric function.

The other paper is by Mr. Steinthal "On the Solution of the Equation $(1-x^2)\dfrac{d^2u}{dx^2} - 2x\dfrac{du}{dx} + n(n+1)u = 0$" (Quarterly Journal of Pure and Applied Mathematics, vol. xviii.). The Legendre coefficients P_n and Q_n are obtained as definite integrals involving the Bessel's function of order $n+\frac{1}{2}$.

Neither Mr. Schafheitlin nor Mr. Steinthal makes any use of what I have called "Bessel operators," and neither has considered the associated Legendre's coefficients.

I understand from Mr. G. F. C. Searle, M.A., of St. Peter's College, Cambridge, that the expression $J_0\left(\sqrt{1-\mu^2}\,\dfrac{d}{d\mu}\right)\mu^n$ for $P_n(\mu)$ has been previously arrived at by Mr. Oliver Heaviside, by considering the series for $P_n(\mu)$ directly. This result, however, is now seen to be merely a particular case of a more general theorem.

XXV. *The Anomalous Dispersion and Selective Absorption of Fuchsin. By* W. B. CARTMEL, *M.A., former Fellow in Physics at the University of Nebraska**.

[Plates IX. & X.]

ALTHOUGH fuchsin is the substance in which anomalous dispersion was first observed, and although it shows this phenomenon much more decidedly than any other substance upon which it has been found possible thus far to make anything like reliable measurements, its optical constants

have not yet been fully determined except by indirect methods. Very complete absorption-curves have indeed been given for solutions of different concentration, but not for solid fuchsin.

It was therefore thought that it might be of interest to determine the absorption and dispersion directly, both of these upon the same identical fuchsin, and as a very good determination of the dispersion-curve has already been given by Pflüger *, who measured the deviation produced by a thin wedge of solid fuchsin, it was decided to redetermine this, using interferential means. This would present the advantage of a redetermination by a different method, and, furthermore, the measurements of absorption and dispersion could both be made upon the same film.

Films were therefore prepared in the usual way by dipping glass plates into an alcoholic solution of fuchsin and allowing the alcohol to evaporate. The fuchsin upon which the first experiments were made was some that had been purchased for general laboratory purposes, and it was found that the dispersion and absorption had values very much lower than those given by Pflüger. Some fuchsin of the same kind as that which Pflüger had used was therefore imported from Kahlbaum in Berlin, and upon this were carried out the experiments which form the basis of the following paper.

As already pointed out by Wood†, the great difficulty in determining the dispersion of strongly absorbing substances by interferential methods is that the ray which passes through the substance is so reduced in intensity that it is not capable of causing interference when it meets the undiminished light of the other ray. This may be easily conceived to be the case with fuchsin, when we consider that a film of fuchsin a wave-length thick transmits only five parts in one hundred million of the incident light within the absorption-band. Experiments were therefore made with the object of reducing if possible the intensity of the light in one of the paths of the interferometer without producing any change in its optical length, in order that the fuchsin might be placed in the more intense beam. However, none of the various plans tried were adopted. It was then decided to use a form of interferometer in which the light does not return upon itself, and for two reasons. First, because in traversing the film twice the diminution in intensity is squared while the retardation is only doubled ; and, second, because the enormous reflexion from the surface of the film obscures the fringes in the ordinary form of interferometer, but in the type used the reflected light does not reach the observer's eye at all.

* Wied. *Ann.* xlv. p. 203 (1898).　　† Phil. Mag. i. p. 43 (1901).

After a number of trials to determine the best adjustment it was found possible, by using this type of interferometer and making the films sufficiently thin, to obtain distinct fringes throughout the spectrum. An unsymmetrical arrangement was first tried, in which the partly silvered plates were very lightly silvered or not silvered at all, and the fringes observed in a direction at right angles to that at which the light entered the instrument. In this way fringes were obtained from beams of unequal intensity, though the method was finally abandoned for the following, which is more satisfactory.

The partly silvered mirrors of the interferometer are silvered so as to reflect and transmit equally, in order that the light in the two paths may be of equal intensity, and then the fuchsin-film is introduced in one of the paths and an absorbing screen in the other. Good fringes may now be seen because the intensity of the light in the two paths is reduced. The retardation may be measured by the shift of the fringes on the removal of the fuchsin-film. But the presence of the absorbing screen causes the fringes to be indistinct when the fuchsin-film is removed, so the absorbing screen should have only half the absorption of the fuchsin-film, in order that the fringes may be seen with equal distinctness whether the fuchsin-film is in or out of the interferometer. In practice the fuchsin-film was placed in the upper or lower half of one path of the interferometer, and the absorbing screen in both upper and lower halves of the other path, so that with the interferometer adjusted for vertical fringes two sets were seen one above the other, but one set displaced with respect to the other.

Sunlight from a slit S (Pl. IX. fig. 1) was brought to a focus by means of the lens L, so that an image of the slit fell upon the glass plate upon which the fuchsin had been deposited, and thus light of very great intensity was concentrated upon a strip of the film only a millimetre wide, a portion narrow enough for the thickness to be determined definitely. With a wider film only an average thickness could have been obtained.

The light after leaving the interferometer was brought to a focus upon the slit of a small spectroscope by means of the lens L'. By observing through the eyepiece of the spectroscope spectral bands could be seen.

At this point it may be as well to mention that it takes very careful adjustment of this type of interferometer to be able to observe the bands, either with a spectroscope or a telescope. Even with the naked eye it was found that the

bands might, under certain circumstances, be seen with the eye in one position, but with the eye nearer or farther from the interferometer they were invisible. Or, again, by moving the eye nearer to or farther away from the interferometer a position might be found in which two sets of bands could be seen crossing one another at right angles, though this only occurred when the instrument was very carelessly adjusted. The following method of adjustment enabled very satisfactory fringes to be produced. A telescope having a fairly well-corrected objective was focussed as carefully as possible for parallel rays, and then an object at least two or three hundred metres away was observed by means of this telescope in such a way that light from the distant object reached the telescope directly and at the same time by reflexion from two of the mirrors of the interferometer. If the mirrors were not quite parallel two images of the object could be seen, so that all that was necessary to make the mirrors parallel was to adjust them till the two images seen in the telescope coincided. In this way, by comparing one mirror with another, all the mirrors of the instrument could be brought into almost perfect parallelism. The final adjustment was made by observing with the naked eye the reflexion of a pointed object held near the instrument in such a way that reflexions of the object reached the eye by way of the two paths of the instrument, and the images thus seen brought into coincidence by moving one of the mirrors parallel to itself by a screw motion. When the images are thus brought into coincidence the two paths of the instrument are equal, and the coloured fringes of white light can be seen. A simple adjustment of one of the mirrors by trial will widen or narrow down the bands at will, or rotate them through any azimuth, though they cannot be so widened or narrowed or rotated through any azimuth if the mirrors are not nearly parallel.

After having arranged the interferometer so that the fringes were all that could be desired, no difficulty whatever was experienced in seeing them with a telescope, but there was still trouble in seeing spectral bands when the apparatus was set up as shown in Pl. IX. fig. 1. The difficulty was found to be in the lenses L and L', which, though achromatic and well corrected, were of very short focus ; but on changing these for lenses of 25 cms. focal length, the spectral bands were so bright and clear that they could evidently be very much dimmed by the introduction of the fuchsin, and still be visible.

In order to determine what kind of spectroscope would give the best results when viewing very faint bands, a trial

was made using various spectroscopes. The conclusion reached was that a spectroscope of very low dispersive power having a low-power eyepiece was the most satisfactory, and therefore that kind was used in this work.

The bands as viewed with the system just described presented the appearance represented diagrammatically in Pl. IX. fig. 3, and showed the anomalous dispersion of fuchsin in a general way at a glance.

In discussing these bands we will, for the sake of clearness, call those bands affected by the fuchsin the fuchsin-bands and the others the air-bands. The bands as shown in fig. 3 correspond to a film too thin to produce a whole band retardation, so at the two points where the two sets of bands coincide the index of refraction is unity. To determine the index of refraction in any other part of the spectrum one of the mirrors is moved parallel to itself by means of a screw till a fuchsin-band appears in the required portion of the spectrum, unless indeed one is already there. Now by means of a compensator an air-band is brought into coincidence with the fuchsin-band, and the amount of retardation introduced by the compensator is equivalent to the retardation produced by the fuchsin. In the case of a film thick enough to produce more than one band displacement, the fraction of a band is measured by means of the compensator, and the whole number of bands added to this.

It was found that when an attempt was made to produce an arbitrary shift of a definite fraction of a band, either by moving one of the mirrors or rotating a compensator, the bands were disturbed by the mere touching of the mirror or compensator. A very thin mica compensator, so thin that it had to be rotated through about 20 degrees to produce a shift of one band, was therefore used instead of the usual thick glass ones, and with this no such disturbing effect was observed since an infinitesimal movement of this would not visibly affect the bands. To overcome the difficulty of mica having three different indices of refraction, the piece of mica was cut so that one of its axes of elasticity was the axis of rotation. By using plane-polarized light, and making the plane of polarization parallel to the axis of rotation of the mica, the retardation introduced by the mica when at different angles was always proportional to just the one index of refraction. It was found incidentally that the introduction of the nicol prism N, fig. 1, made the bands much more distinct. On rotating the nicol prism through a right angle, the bands became more confused than with the nicol out.

The compensator (C, fig. 1) was made by cutting a thin

piece of mica in two and attaching one of the pieces to a
device by means of which it could be rotated through an
angle and the angle read off.   These two pieces of mica were
placed in the same path as the fuchsin, the fixed piece and
the fuchsin in the lower half of the path, and the movable
piece in the upper half, the line of separation of the mica
precisely on a level with the edge of the fuchsin; in both
upper and lower halves of the other path was a very thin film
of fuchsin that served as an absorbing screen.   When both
pieces of mica were at right angles to the beam of light their
effect was null, because they affected both sets of bands alike,
being of equal thickness.   It could be ensured that they
were equally thick by removing the fuchsin and observing
whether the upper and lower spectral bands coincided when
both pieces of mica were in the same plane.

It is usual to calibrate a compensator by turning it through
different angles and noticing the angles corresponding to
different numbers of bands displacement.   However, for this
work, as the displacement that had to be measured was
always less than a single band, it did not help to know the
angle corresponding to one band displacement or two bands
displacement, because the greatest change of curvature of the
calibration-curve comes between zero and one.

A formula was therefore used which may be deduced as
follows:—If a piece of mica of thickness t is traversed by a
ray which meets it at an angle of incidence i, as shown in
fig. 2, Pl. IX., the length of the path of the ray will be increased
by $\mu l - d$, if the index of refraction of the mica is represented
by μ.

Also
$$l = t \sec r = \frac{t}{\sqrt{1 - \sin^2 r}}$$
$$d = l \cos (i - r)$$
$$\mu l - d = l[\mu - \cos (i - r)],$$

which reduces to
$$t(\sqrt{\mu^2 - \sin^2 i} - \cos i).$$

The introduction of the mica at right angles to the ray will
shorten its path by $(\mu - 1)t$.   The number of wave-lengths
retardation produced by a piece of mica in the path of the
ray, when the mica is moved from a position at right angles
to the ray through an angle i, is the difference of these two
quantities, and we have at once

$$n\lambda = t(\sqrt{\mu^2 - \sin^2 i} - \cos i) - (\mu - 1)t.$$

In using a compensator with this type of interferometer

there is a shift of the bands due to the lateral displacement of the beam of light, so it is usual to use a double compensator, one piece in each path, thus overcoming the difficulty by moving both pieces together and giving an equal lateral displacement to each beam. The compensator that was used could be turned through 40 degrees, producing a retardation of three or four bands before any effect due to this lateral displacement could be observed, and as the greatest shift that had to be measured was less than a band, it was decided to use a single compensator for the sake of convenience.

Using the above formula the values of i that were observed for one, two, and three bands displacement were found to be consistent. Assuming μ to be constant and equal to 1·58, the value of t was found from the formula, and then by substituting back this value the various values of n corresponding to the different values of i were found, and a curve of $n\lambda$ and i plotted. From this the fraction of a band displacement corresponding to any value of i for any part of the spectrum could be determined. There would be a slight error due to the variation of the index of refraction of the mica for light of different wave-lengths, but as this was less than one per cent. from one end of the spectrum to the other, the compensator could be calibrated for one part of the spectrum, and a linear correction made in any other part. As a matter of fact, it was calibrated in the red end of the spectrum, and as the index of refraction of fuchsin is nearly unity at the violet end of the spectrum, and as $(\mu - 1)$ is the quantity that is measured it does not matter very much whether the correction is made or not, considering that there are much graver errors which are unavoidable.

Knowing $n\lambda$ the retardation produced by the fuchsin in light of any wave-length, we may determine μ the index of refraction of fuchsin for light of that wave-length by the formula

$$n\lambda = (\mu - 1)t,$$

provided we know t the thickness of the fuchsin-film. This was determined by a method which will be described later. The measurements were made upon a film whose thickness was 580 $\mu\mu$, with the exception of those values falling between $\lambda = 460 \mu\mu$ and $\lambda = 590 \mu\mu$, which were made upon a film 192 $\mu\mu$ thick.

The results are plotted in fig. 1, Pl. X., and in the following table they are compared with the values obtained by Pflüger[*]

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[*] Phil. Mag. i. p. 43 (1901).

by the prism method, and by Walter* by the method of total reflexion.

λ.	Pflüger.	Walter.	Interference method.
A ...	...	2·019	2·055
a......	...	2·086	2·12
706...	2·31	...	2·15
B ...	...	2·161	2·22
Li ...	2·34	...	2·27
C ...	...	2·310	2·35
634...	...	2·412	2·48
D ...	2·64	2·684	2·70
535...	1·95	...	1·98
E ...	...	1·912	1·85
F ...	1·05	1·074	1·05
Sr ...	0·83	...	0·82
455...	...	0·847	0·83
G ...	1·04	0·95	1·00
425...	...	1·00	1·11
H ...	...	1·32	

The curve is uncertain within the absorption-band and perhaps also to some extent in the violet, but in the red end of the spectrum it is established beyond a doubt that the values given by Pflüger do not apply to the fuchsin upon which my measurements were made. The fact that Pflüger obtained the same values using prisms of different angles seems to show either that the two methods lead to different results, or that the composition of the two samples of fuchsin was different. There are a number of red triphenyl methane dyes of distinctly different compositions, all of which go under the name of fuchsin. Wernicke† obtained the dispersion-curve of a fuchsin by the prism method, and found the indices to range from 1·31 to 1·90. The one on which the writer's first work was done had indices ranging from less than unity to about two. Kundt experimented on a fuchsin having two absorption-bands in the visible spectrum. However, there is no doubt but what the fuchsin in question was made by the same process as that which Pflüger used, though having been purchased seven years later it was probably made in a different batch, and hence there is a possibility of the composition being slightly different.

The agreement with Walter in the red is extremely satisfactory. The two curves run uniformly parallel to one another, the author's being a little higher than Walter's, which may be accounted for by an error in the thickness or

* Wied. *Ann.* lvii. p. 396 (1896).
† Pogg. *Ann.* clv. (1875).

a systematic error of some kind. A slight change in the thickness would make the two curves almost identical.

The thickness of the films was determined as follows:—A portion of the film was washed away with alcohol leaving a clean sharp edge. In general, when a film is thus washed away, there is a concentration of fuchsin at the edge, due to a flowing of the solvent. This is of course undesirable, and was eliminated by using a piece of blotting-paper with which to clean away the fuchsin. This piece of blotting-paper was itself cut to a clean sharp edge and slightly damped with alcohol. The glass near the film was wiped with it, a new edge of the blotting-paper was prepared, dampened, and the operation was repeated until a satisfactory edge of fuchsin was obtained.

Now a plate of glass was placed upon the glass upon which the fuchsin had been deposited. There were thus two layers of air of different thickness, one between the fuchsin and the glass and the other between the two pieces of glass. The difference in the thickness of the two films of air gives the thickness of the fuchsin. The thickness of the air-films may be obtained very accurately by the interference of thin films by a method first used by Wernicke* and modified by Wiener. White light was allowed to fall on the two air-films, and the reflected light was examined with a spectroscope, by means of which interference-bands could be seen in the spectrum.

The arrangement of the apparatus is shown by Pl. IX. fig. 4. The glass plates inclosing the air-film are shown at G. P is a totally reflecting prism, one edge of which forms an edge of the slit of the collimator. A Rowland grating, R, sent a spectrum into the observing telescope, and the distance from band to band was measured by means of the micrometer eye-piece.

The two glass plates, G, were so arranged before the slit that the film of air between the fuchsin and the glass was before the lower part of the slit, and the film of air between the two bare plates before the upper part. By a suitable mechanical contrivance the plates were adjusted so that the bands seen at the eyepiece were vertical.

On examining the two sets of bands it was noticed that while those corresponding to the film between the two glass surfaces were regular and became uniformly narrower towards the violet, those corresponding to the air-film between the glass and the fuchsin showed an irregularity.

* Wied. *Ann.*, Pogg. *Ann.* Ergbd. viii. p. 65 (1878).

For instance, when there was a film of fuchsin whose thickness was about a fourth of a wave-length of red light, the bands of one set were displaced with respect to the other set by half a band in the red, and this displacement regularly increased, till a point in the blue-green was reached where it was a whole band. At this point there was a sudden change of half a band and then a regular change took place going from short to shorter wave-lengths, till in the violet there was again a whole band displacement.

This brings up the question of phase-change. There was evidently about half a band phase-change in the blue-green, but in the violet end the red fuchsin reflects like a transparent body, and it is likely that the difference of phase-change between it and the glass is zero. Pflüger has gone into this point very thoroughly, and says that from the experiments of Wernicke it is safe to assume that the phase-change by reflexion from fuchsin is the same as that from glass for wave-lengths longer than $640 \mu\mu$. The measurements were therefore made with the light comprised between the B and C lines. Sunlight was used and a number of gelatine screens interposed at S, fig. 4, so as to keep out light of those wavelengths not needed, because the films bleach easily.

The thickness of the fuchsin may be determined from the two sets of interference-bands in several ways, but the method involved in the following formulæ, due to Wiener, was the one used.

We have $\qquad 2t = m\lambda_m = (m+1)\lambda_{m+1},$

where t is the thickness of the film between the two pieces of glass, λ_m the wave-length corresponding to the centre of one band, and λ_{m+1} that of the next band towards the violet. Taking one of the bands due to the film of air between the glass and the fuchsin, which falls between the mth and $(m+1)$th of the other bands, and calling it λ'_m we have

$$nt' = m\lambda'_m,$$

where t' is the thickness of the film of air corresponding to this case. From these two equations follow at once

$$t - t' = \frac{\lambda_m - \lambda'_m}{\lambda_m - \lambda_{m+1}} = \frac{\lambda_{m+1}}{2},$$

which gives the thickness of the fuchsin $(t - t')$ directly.

The thickness of the air-films was regulated so that about five or six interference-bands fell between the B and C lines. Thus there were several sets of bands upon which independent measurements could be made.

Some idea of the accuracy of the measurement can be obtained from the following set of measurements made upon one of the films, in which is included every measurement that was made upon that film.

Wave-length.	Half the measured shift.	Thickness.	Wave-length.	Half the measured shift.	Thickness.
661	36	625	660	37	623
661	46	615	660	48	612
667	57	610	665	67	608
667	60	606	665	60	605
672	70	602	671	70	601
672	67	604	671	72	604
677	92	585	676	89	587
677	89	588	676	82	594

This makes the probable error of a single observation 8 $\mu\mu$ and of the mean 2 $\mu\mu$, which is about the order of accuracy that interference methods usually give.

There is some little uncertainty sometimes as to whether the shift refers to a fraction of a band, or to one or two whole bands plus a fraction of a band. In the above instance the shift is about eight-tenths of a band, and the total displacement one and eight-tenths, giving a thickness of about nine-tenths. If the displacement had been assumed to be 2·8 bands the resulting thickness would have been 1·4 wavelengths. To decide this point definitely a number of films were made of different thickness. The first of this series was made from a saturated solution and the rest of the films were made, each succeeding one thinner than those which preceded it, by continually diluting the solution from which they were deposited. Starting from the thinnest (which was 320 $\mu\mu$ thick), the thickness of each succeeding one was measured till the thickest was reached (617 $\mu\mu$). The continuity of the series furnished data from which the thickness was determined definitely, and observation upon the films with a spectrophotometer confirmed the conclusions.

The same trouble was experienced in the measurement of the retardations with the interferometer, but when the thickness was definitely known a comparison with a film sufficiently thin to show bands all through the spectrum cleared up this point, for with the very thin film the retardation was never so much as a whole wave-length, for reasons already mentioned. The matter might have been cleared up by viewing the fringes of white light in the usual way and observing the shift of the central fringe produced by the fuchsin, but this was found impracticable on account of the selective absorption of the

fuchsin-film, which made the central fringe look just like any other fringe. This plan, however, would have furnished a very decisive solution of the difficulty with regard to the thickness, for by making the plate upon which the fuchsin was deposited the back mirror of an interferometer, and comparing the fringes due to the reflexion from the bare plate with those due to the reflexion from the top surface of the fuchsin, the shift of the central fringe could have been easily determined.

The photometric measurements were made by means of a Brace spectrophotometer. The methods of adjusting the instrument have been already given by Tuckerman*.

The extremely great absorption gave rise to some difficulties which do not occur in ordinary photometric work. For instance, a little of the red light (for which fuchsin is transparent) reached the eye as stray light, by reflexion in the telescope-tubes and the comparison prism. This, too, in spite of the fact that the telescope-tubes were well blackened and diaphragmed. This small amount of red light was more intense than the green light that was being measured, since the green light had been cut down a thousand times or more by the absorption of the film.

This was remedied by using a screen having absorption-bands in the red and the blue, but which transmitted green quite freely. In this way green light alone entered the instrument when measurements in the green were being made.

Fig. 5 shows the general arrangement of the apparatus. A beam of sunlight is brought to a focus on each of the totally reflecting prisms p and p' by means of a split lens S. The prisms reflect the light on to the mirrors m and m', and these direct the light on to the collimator slits. Part of the light from the right-hand collimator reaches the observing telescope by reflexion from a silvered strip cemented in the comparison prism P. The rest passes above or below the strip, and is lost. On the other hand, of the light which comes from the left-hand telescope, only that which passes above or below the silvered strip reaches the observing telescope, with the result that in the eyepiece may be seen a field illuminated from the left-hand collimator, with the exception of a central strip illuminated from the right-hand collimator, and when the intensity of the illumination from both these sources is alike the field appears uniform. This uniformity may be brought about by varying the width of either of the slits S or S'. If the fuchsin-film is placed before the slit S it will cause the centre of the field to seem

* Wied. *Ann.* xxxi. p. 629 (1888).

dark, and a match may again be produced by narrowing the slit S′, and if we know the original width of the slit the ratio of the change in width to the original width gives very closely the proportion in which the introduction of the fuchsin has diminished the intensity of the light. If instead of diminishing the intensity at S′ by narrowing the slit a revolving sector is used which cuts down the light by the proper amount, the error due to the lack of proportionality between intensity and slit-width does not enter in. However, it is found more convenient to use a sector cut into an arbitrary number of parts, as shown in fig. 6, and as with this we can in general only make an approximate match, the varying of the slit-width is used as a fine adjustment. This gives practically as good results as can be obtained with a variable sector. A sector of eighty notches was used, except for the measurements within the absorption-band, in which case a tin disk, as shown in fig. 7, was used having a slit cut in the edge 0·7 mm. wide and another about a centim. from the edge 1·5 mm. wide. This disk was rotated very rapidly before the collimator slit by means of a small motor, and thus the intensity of the light incident upon the collimator slit was diminished by 99·856 or 99·682 per cent., according to which of the slits of the disk was used. Since the edges of the slits of the disk are parallel, care had to be taken to have the collimator-slit at a definite distance from the centre of the disk. This was done by using the disk close up against the collimator-slit, and then by holding a sharp edge on a line B or C scratched on the disk, the shadow of this edge would fall on the centre of the collimator slit when the centre of the disk was at a distance OC or OB from the ray entering the collimator.

The use of this tin disk, which cut down the intensity of the light so greatly, made it possible to match very accurately, although the light had been reduced by the fuchsin to such an enormous degree. Since the slope of the absorption-curve was very steep, it was necessary to have the collimator-slit S quite narrow, not more than half a millimetre wide at most, in order that the readings should be made with sufficiently homogeneous light, otherwise the absorption as observed would have too low a value. With the slit S, half a millimetre wide, S′ would have been less than a five thousandth part of a millimetre wide when the adjustment was made using no sector, and accurate settings of the slit at this width would have been impossible. This feature of the photometer, together with the fact that a bare flame or direct sunlight can be used instead of a flame or sunlight behind ground or

opal glass which would cause a loss of 95 to 98 per cent., and the absence of Nicol prisms which would cause a further loss, makes it singularly well adapted for work on strongly absorbing substances.

The diminution in intensity caused by transmission through the fuchsin-film is due to two factors, reflexion from its surfaces and absorption within the film. To determine the absorption it is necessary either to eliminate the reflexion by measuring the differential absorption of two films of different thickness or by making a separate determination of the reflexion and subtracting it. Preparations were made to measure the reflexion, but this part of the work was not finished, though I expect to continue this work and measure the reflexion. Not having the reflexion measurements I must be content with computing them. The effect of an error in the reflexion upon the final values of the absorption-coefficients will be greatest within the absorption-band, and even there a ten per cent. change in the reflexion only changes the absorption coefficient one per cent.

For computing the reflexion from an absorbing substance into air Cauchy's theory leads to the following formula :—

$$R = \frac{(\mu - 1)^2 + K^2}{(\mu + 1)^2 + K^2},$$

while the electromagnetic theory leads to

$$R = \frac{\mu^2(1 + K^2) + 1 - 2\,\mu}{\mu^2(1 + K^2) + 1 + 2\,\mu},$$

which may be seen to be identical with the previous formula when we remember that in Cauchy's theory the quantity K is identical with μK of the electromagnetic theory. For the reflexion at the interface between an absorbing substance and a transparent substance whose indices of refraction are μ and μ' respectively, the electromagnetic formula becomes

$$\frac{(\mu - \mu')^2 + \mu^2 K^2}{(\mu + \mu')^2 + \mu^2 K^2}.$$

According to this formula the reflexion is the same on either side of the interface between the media, while with Cauchy's formula the reflexion on the two sides will in some instances be different, which would lead to the loss by reflexion being different according to whether the light went through the glass plate and then through the fuchsin or through the fuchsin first, an effect which the writer has been unable to observe. The values of the reflexion as computed in this way are plotted in Pl. X. fig. 3.

If r_1, r_2, and r_3 represent the reflexion at the fuchsin-air, fuchsin-glass, and air-glass surfaces respectively, and I_0 the

original intensity of the incident light, the effect of the reflexion is to make the transmitted light have the value $(1 - r_1)(1 - r_2)(1 - r_3) I_0$, neglecting the effect of multiple reflexions which does not enter in within the absorption-bands. For those colours for which the film is transparent the effect of multiple reflexion is easily allowed for to the degree of accuracy possible when working with strongly absorbing substances.

Knowing the thickness of the film we may compute the absorption-coefficient μK from the following formula

$$\frac{I}{I_0(1 - r_1)(1 - r_2)(1 - r_3)} = e^{-\frac{4\pi\mu K t}{\lambda}},$$

in which $\dfrac{I}{I_0}$ is the ratio of the intensities as determined by the spectrophotometer, t is the thickness of the film, and λ the wave-length in vacuo. The photometric measurements were all made upon a film $192\,\mu\mu$ thick. The transmission of this film is plotted in Pl. X. fig. 2. Using these values the curve given in Pl. X. fig. 1 was computed. In the following table these values are compared with Pflüger's and Walter's results :—

Values of μK.

λ.	Pflüger.	Walter.	Cartmel.
589	0·79	0·792	·86*
527	1·22	1·419	1·354*
486	0·98	1·168	1·07*
455	0·43	0·533	0·56*

* By interpolation.

Walter used Cauchy's formulæ for the elliptic polarization which is associated with metallic reflexion, and as this could only be applied to those radiations which are strongly absorbed by the fuchsin, he only obtained the four values given above. In order to make a comparison with Walter's work Pflüger measured the same four quantities directly, using the difference in transmission of two films of different thickness.

Q 2

XXVI. *On the Mathematics of Bees' Cells.*
    By Prof. J. D. EVERETT, *F.R.S.*[*]

FOR the literature of this subject reference may be made
    to a critical summary by Dr. Glaisher in the Philo-
sophical Magazine for August 1873, which contains an
exposure of several popular errors. Later results of obser-
vation will be found in Darwin's 'Origin of Species,'
pp. 221–227.

A bees' comb consists of cells combined in double layers.
Each cell has the form of a regular hexagonal prism with 3
of the 6 corners at one end sliced off so as to give a pointed
apex. The 3 faces which meet at the apex are rhombuses
symmetrically arranged, and 3 corners of each rhombus lie
upon edges of the hexagonal prism. Three alternate edges
of the prism are accordingly longer than the other three, and
we shall denote this difference of length by h. Then h will
also denote the distance of the apex from the plane of the
ends of the 3 longer edges. The length of a side of the
regular hexagon which is the cross-section of the prism will
be denoted by s.

Any number of equal and similar sharpened prisms, con-
structed according to this specification, can be fitted accu-
rately together, in two layers, in such a way that the apex of
a member of one layer is inserted between three apexes
belonging to the opposite layer. The axis of each prism will
then be in line, not with an axis, but with the common edge,
of 3 prisms belonging to the other layer.

The annexed diagram is specially designed to set forth the
relations of the parts in the clearest manner by projecting
them on a plane perpendicular to the axes of the prisms.
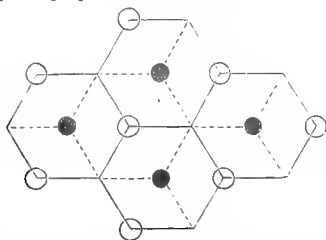
The ends of the longer edges of both sets of prisms lie in
the same plane, which is the plane of the paper. The two

sets of apexes are represented by the two symbols ● and ○, and lie at equal distances h from the plane of the paper on opposite sides of it. All points of meeting not so marked lie in the plane of the paper. Two sets of hexagons are indicated, one by continuous and the other by dotted lines. They are the projections of the two sets of prisms. The rhombuses into which the figure is divided are projections of the rhombuses which meet at the several apexes; each rhombus-edge being in the plane of one of the prism-faces. The plane of one set of apexes is at a distance $2h$ from the plane of the other set.

The projected length of a rhombus-edge is s, and its actual length $\sqrt{(s^2 + h^2)}$.

The long diagonal of a rhombus lies in the plane of the paper, and is $s\sqrt{3}$: the shorter diagonal is $\sqrt{(s^2 + 4h^2)}$. If θ denote an acute angle of a rhombus we have

$$\cos \tfrac{1}{2}\theta = \frac{\sqrt{3}}{2} \frac{s}{\sqrt{(s^2 + h^2)}},$$

whence

$$\cos \theta = \frac{s^2 - 2h^2}{2(s^2 + h^2)}.$$

If ϕ denote the acute angle which a prism-edge makes with the rhombus-edge that it meets, we have

$$\cos \phi = \frac{h}{\sqrt{(s^2 + h^2)}}.$$

Changes in the value of h will not affect the volume of a cell, but they will affect the quantity of wax required for building the cell. If h were zero the prism-edges would all terminate in the plane of the paper, which would also contain the rhombuses. As compared with this standard of reference, the area of a prism-face is less by $\tfrac{1}{2}sh$, and the area of a rhombus is greater by $\tfrac{1}{2}s\sqrt{3}\sqrt{(s^2 + 4h^2)} - \tfrac{1}{2}s^2\sqrt{3}$. To give proper weights to these two items on opposite sides of the account, we must know the ratio of the number of prism-walls to the number of rhombuses. Each cell has twice as many prism-sides as rhombuses; and in the comb, except at the outside, each rhombus, as well as each prism-wall, is common to two cells. At the outside, the walls exposed also comprise twice as many prism-walls as rhombuses. We must therefore reckon two prism-faces to one rhombus, and the net saving in area is

$$sh - \tfrac{1}{2}s\sqrt{3}\{\sqrt{(s^2 + 4h^2)} - s\},$$

Omitting a constant term and a constant factor, the quantity to be made a maximum is

$$h - \frac{\sqrt{3}}{2} \sqrt{(s^2 + 4h^2)},$$

provided that the rhombuses have the same thickness as the prism-walls. If they are n times as thick as the walls, the quantity to be made a maximum is

$$u = h - n \frac{\sqrt{3}}{2} \sqrt{(s^2 + 4h^2)} \; ;$$

whence, putting $\dfrac{du}{dh} = 0$, we find $s^2 = (12n^2 - 4)h^2$,

or

$$\frac{s^2}{12n^2 - 4} = \frac{h^2}{1} = \frac{s^2 + h^2}{12n^2 - 3} = \frac{s^2 - 2h^2}{12n^2 - 6},$$

giving

$$\cos \theta = \frac{2n^2 - 1}{4n^2 - 1}, \qquad \cos \phi = \sqrt{\frac{1}{3(4n^2 - 1)}}.$$

The usual calculation assumes $n = 1$, giving

$$\cos \theta = \tfrac{1}{3}, \qquad \cos \phi = \tfrac{1}{3}.$$

The trihedral angles, of which there is one at each apex, and one at the end of each of the shorter prism-edges, are each composed of 3 plane angles whose cosine is $-\tfrac{1}{3}$, the inclinations of their planes being 120°. The form thus deduced is regarded as the normal form of bees' cells.

It is closely related to the most compact system of piling of equal spheres. In this system each sphere touches 3 spheres in the layer below, 3 in the layer above, and 6 in its own layer. If we omit the 3 upper or the 3 lower, the tangent planes at the 9 remaining points of contact represent the 9 walls which bound a cell.

Again, the 9 walls (including 3 pairs of parallel walls) are perpendicular to the 6 edges of a regular tetrahedron; and the 12 lines of junction of these walls (including 3 pairs of parallels and 1 set of 6 parallels) are perpendicular to the 4 faces of the same tetrahedron.

According to observations cited by Darwin, the actual value of n averages about $\tfrac{3}{2}$. This gives

$$\cos \theta = \tfrac{7}{16}, \qquad \cos \phi = \sqrt{\tfrac{1}{24}},$$

for minimum consumption of material.

XXVII. *On the Ionization in Air at different Temperatures and Pressures. By* J. Patterson, *B.A., Professor of Physics, Muir Central College, Allahabad, India;* 1851 *Exhibition Science Scholar, Emmanuel College, Cambridge* *.

ELSTER and Geitel[†] and C. T. R. Wilson[‡] have shown that ions are being continuously produced in air and other gases confined in a closed vessel without being exposed to any known ionizing agent.

Hitherto it has not been known whether this ionization, which has been called spontaneous ionization, was due to some property in the air itself, to the vessel being slightly radioactive, or to very penetrating rays that easily pass through the walls of the vessel.

Wilson[§], in his experiments, used very small volumes of gas, and he found that, except for high pressures, the ionization in air and CO_2 was proportional to the pressure, while for hydrogen it varied as the pressure throughout the range of the experiments. These results led him to suspect that the ionization was due to the walls of the vessel being slightly radioactive, the radiation having only moderate penetrating power; consequently, if this is the cause one should get the ionization independent of the pressure over a considerable range by using a large volume of gas.

Rutherford[||] has shown that certain rays of radium can penetrate great thicknesses of metal, and suggests that the cause of the ionization in closed vessels may be due, in part at least, to some very penetrating rays.

If the ionization were due to some property of the air itself one would expect that it would be some function of the temperature and pressure, consequently, by measuring the ionization at different temperatures and pressures and using a large volume of gas one should discover the cause of the ionization. It was with this object that the present investigation was undertaken.

The results of the experiments showed that the ionization was due to a slight radioactivity in the walls of the vessel. This conclusion was communicated to the Cambridge Philosophical Society in a preliminary note " On the Ionization in Air at different Temperatures and Pressures "[¶].

Wilson and Elster and Geitel in their experiments measured the amount of the ionization with specially constructed electroscopes. This method of determination requires considerable time for a single observation, and, moreover, could not be used to measure the ionization at different temperatures. Rutherford [*], by using a large volume of gas and a fairly sensitive electrometer, succeeded in measuring the ionization. This latter method was employed in the present investigation.

In the experiments at different temperatures it was hoped that by using quartz one would be able to get good insulation at temperatures up to 400° or 500° C. A quartz tube with walls about 1 mm. thick was made, but it would not insulate above 250° C., consequently all the insulation had to be in the cold.

The air was contained in a wrought-iron cylinder 30 cms. in diameter, 20 cms. long, and containing 12,800 c. c. The general arrangement of the apparatus is shown in fig. 1. A is the cylinder supported on a tripod (not shown in fig.) resting on ebonite blocks. It was heated with a gas furnace.

Fig. 1.
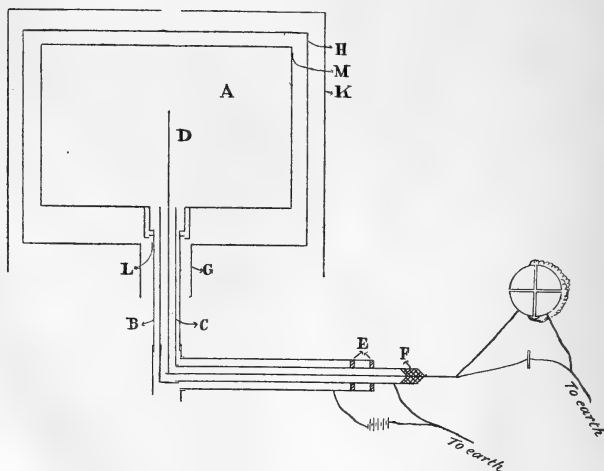
The tube B was screwed into the cylinder: inside this was the tube C insulated and supported by two ebonite rings E: inside this again was the aluminium electrode D, insulated and supported by the ebonite plug F, which was kept cool by means of a water jacket. The horizontal tubes were screwed into the uprights at the elbow and could thus be

* Bulletin of the American Physical Society, vol. ii. No. 4, p. 59.

placed in position. This arrangement prevented the hot air being driven against the insulation. In order that the cylinder might be heated as uniformly as possible and protected from the flames a second cylinder H surrounded it. The tube C was brought through the bottom of this cylinder in the pipe G, which extended below the burners and so protected C from the flames. The legs of the tripod were brought through in a similar manner. The cylinder and tripod were thus screened from the flames so that they held a charge for some time. Around the cylinder H was another K, so that the hot gas from the flames passed up between H and K and out at the top.

The temperature of the cylinder was measured by two Cu-Ni couples, one at M the other at L, both being on the outside of the cylinder and held firmly against it. The cold ends of the wires dipped into mercury cups placed in a water-bath at a known temperature. The two cups for each couple were connected through a galvanometer. 100 ohms resistance was added to the galvanometer circuit, so that the increase of resistance due to heating could be neglected. The couples were calibrated by observing the galvanometer deflexions for known temperatures.

The cylinder A was connected to one pole of a battery of small storage-cells, the other pole being earthed : C was earthed and D was permanently connected to one pair of quadrants of a quadrant electrometer, the other pair being to earth. The current between the walls of the cylinder and the electrode D was measured. By this arrangement there could be no leakage from the charged cylinder to the insulated electrode. Any leakage due to imperfect insulation would make the current through the gas greater than that which was measured.

As very small potential-differences had to be measured it was necessary to guard very carefully against external electrostatic disturbances. All connecting wires, keys, and electrometer were inclosed in conductors connected to earth. The quadrants were opened by a mercury key which could be worked from a distance by means of a cord. At first mercury cups in paraffin wax were used, but it was found impossible to keep the paraffin from charging up, and consequently it was discarded and steel thimbles mounted on sealing-wax were used instead. These were very satisfactory.

A Dolezalek electrometer was used. The needle of silver-paper was suspended by a very fine quartz fibre and charged to about 80 volts by touching with a fine wire connected to the battery. The needle held its charge very well, and only

required recharging every other day. It was found that about 80 volts gave the maximum sensitiveness. The deflexions were measured by a telescope and scale about two metres distant. A deflexion of one scale-division was equivalent to 0·000006 volt P. D. between the quadrants, or a P. D. of one volt between quadrants would give a deflexion of 15,000 scale-divisions about.

The effective capacity of the electrometer and its connexions depends almost entirely on the charge on the needle, and in the experiments varied from about 400 to 200 cms. two days after charging. Before comparative readings could be taken it was thus necessary to determine the capacity for each set of observations.

The electrometer needle had a very long period, and the air-damping was sufficient to make it dead-beat. The zero-position was not very constant but drifted in one direction. It was impossible to get insulation absolutely free of charge, so if the quadrants were opened they gained a slight charge but always in one direction. It was thus necessary to observe the rate at which the quadrants charged up when the cylinder was maintained at a constant potential and the quadrants opened. In order that the movement might become steady the time was taken from the instant a certain scale-division crossed the spider line in the telescope, this division was usually about 200 div. from the zero-position. By taking equal intervals of time and observing the deflexions with the cylinder charged alternately positive and negative any extraneous effects would be eliminated.

To measure the capacity of the electrometer and its connexions the number of ions produced per second in the cylinder was taken as constant during the observation, and hence the rate at which the system charged up would be proportional to the capacity. The rate at which the system charged up was determined and then a condenser of known capacity was inserted in parallel and the rate again determined. If α and β represent the deflexion of the needle in scale-divisions per second without and with the condenser in circuit, C the capacity of the electrometer and its connexions, and C_1 the capacity added, then

$$\alpha C = \beta (C_1 + C),$$

from which C can be determined.

The saturation current between the walls of the cylinder and the electrode was first obtained, and is shown in fig. 2 for air that had remained for some time in the cylinder. It will be seen that the air is practically saturated when the

cylinder is charged to 280 volts. The cylinder was maintained at this potential in all the temperature experiments,

Fig. 2.
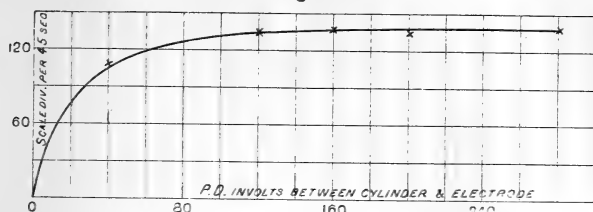
but 1000 volts were used in some of the experiments on pressure.

In taking the observations the current through the gas was measured and the capacity of the electrometer and connexions determined. The cylinder was then heated until the temperature became steady and the current again measured. The results of a set of observations are given in the following table. A number of readings were taken at each temperature and the average is given. Many more observations were taken but they were all similar to the set given.

Ionization at different temperatures.

Temp.	Current between Cylinder and Electrode.
20° C.	$2 \cdot 6 \times 10^{-4}$
195 ,,	2·7 ,,
230 ,,	2·8 ,,
370 ,,	2·83 ,,
410 ,,	2·7 ,,
460 ,,	3·0 ,,
520 ,,	$\begin{cases} +3 \cdot 2 \ ,, \\ -4 \cdot 8 \ ,, \end{cases}$

These results are plotted in fig. 3, where the abscissæ

Fig. 3.
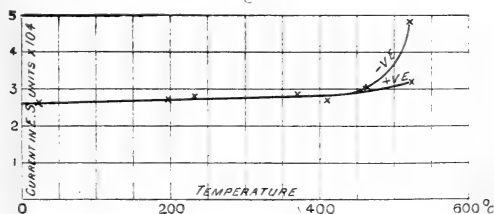
represent the temperatures and the ordinates the current in E.S. units.

It will be seen from the above that the ionization varied

but slightly with the temperature (pressure constant) up to 450° C., but above this it increased very rapidly, due to ions being given off by the walls of the vessel, and the current being much greater when the cylinder was charged negative than when positive. The observations were not continued beyond this.

The air in the cylinder was at atmospheric pressure in all the experiments, consequently it was necessary to determine the variation of the ionization with pressure before deducing its variation with temperature. This was done as follows :—

The same cylinder and aluminium electrode were used, but the arrangement was slightly modified as shown in fig. 4. It was impossible to get the vessel air-tight, but the pressure

Fig. 4.
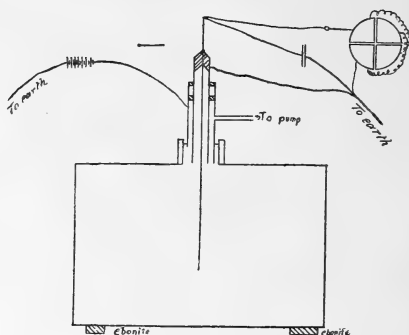
did not vary more than one or two millimetres while taking the readings.

The cylinder was connected to a water-pump and a mamometer for measuring the pressure. Between the cylinder and the water-pump a tube filled with calcium chloride and having a plug of glass wool at each end was inserted so that the air in the cylinder was fairly dry and dust free.

Experiments were also made with the cylinder filled with hydrogen. The hydrogen was made from pure zinc. The cylinder was first exhausted and then filled with hydrogen ; this was repeated two or three times so that the hydrogen would be fairly free from air.

The method of observation was the same as before. For pressures below 300 mm. of mercury the cylinder was charged to a potential of 340 volts. and for pressures above this 1000 volts were used.

The results for a set of observations on air and hydrogen are given in the following table. The results are plotted in

fig. 5, where the abscissæ represent the pressure in mm. of mercury and the ordinates the current in E.S. units.

Ionization in Air and Hydrogen at different pressures.

Air.		Hydrogen.	
Pressure in mm. of Hg.	Current between Electrode and Cylinder in E.S. units.	Pressure in mm. of Hg.	Current between Cylinder and Electrode in E.S. units.
765	$3 \cdot 05 \times 10^{-4}$	790	$2 \cdot 88 \times 10^{-4}$
435	3·22 ,,	422	2·33 ,,
299	2·99 ,,	242	1·84 ,,
291	2·85 ,,	116	1·13 ,,
179	2·66 ,,		
136	2·54 ,,		
105	2·17 ,,		
80	1·95 ,,		
59	1·42 ,,		
27	0·69 ,,		

It would seem as if the air in the vessel was hardly saturated even when the cylinder was charged to a potential of

Fig. 5.
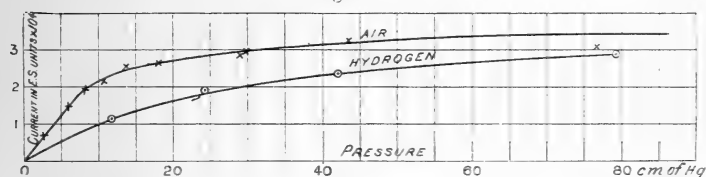
1000 volts, but from the shape of the curve it will be seen that the current varies very slowly with the pressure down to a pressure of 130 mm. of mercury; and below 80 mm. the current is proportional to the pressure, while above 300 mm. the current is almost independent of the pressure. In hydrogen the current varies rapidly with the pressure but not proportionally. This is partly due to the hydrogen not being pure, and as the vessel was not quite air-tight there would be a greater proportion of air at the lower pressures than at the higher.

These results would show that the walls of the vessel were slightly radioactive as Wilson suggested, and that in the cylinder used the radiation was all absorbed before it penetrated the air in the vessel. In the case of hydrogen the radiation was not all absorbed by the gas at the pressures

used. This is what one would expect from our knowledge of other types of radiation.

The experiments with temperature show that the ionization is independent of the temperature, or that the radiation from the walls of the vessel is not affected by temperature up to 450° C. at least.

Since these experiments were performed Strutt[*] and McLennan and Burton[†] have confirmed the conclusion that the ionization is due to the walls of the vessel being radioactive by showing that the radiation varies with the material of the vessel, and the latter have further shown that there are very penetrating rays that can pass through the walls of the vessel. The radiation is thus evidently complex in character, and the greater part of the ionization is produced by the easily absorbed rays. It thus resembles the radiations from thorium and radium.

The average of a large number of observations gave 61 as the number of ions produced per c. c. per sec. in the iron cylinder at atmospheric pressure, $3 \cdot 4 \times 10^{-10}$ E.S. units[‡] as the charge on an ion, and the volume of the cylinder as 12,800 c. c. Using this new value for e, and reducing the values of other experimenters, we find that Wilson (*loc. cit.*) by using a glass flask with a slight coat of silver obtained 36 for n, Rutherford (*loc. cit.*) with zinc cylinders 27, while Harm[§] with a large glass flask obtained values between 53 and 63. The difference in these results is most probably due to the different materials used in the several experiments.

In conclusion I desire to express my sincere thanks to Prof. Thomson for his kindly interest and suggestions during the course of the investigation.

Cavendish Laboratory, Cambridge.

XXVIII. *On the Theory of the Quadrant Electrometer.* **By** GEORGE W. WALKER, *M.A., A.R.C.Sc., Fellow of Trinity College, Cambridge* ‖.

FOR the purpose of some experiments which I am taking up, I have found it necessary to examine the theory of a symmetrical quadrant electrometer more carefully than I have formerly had occasion to do. The results seem to me to be of considerable importance.

* 'Nature,' vol. lxvii. p. 369 (1903).
† Ibid. p. 391 (1903).
‡ Phil. Mag. vol. v. p. 346 (1903).
§ *Physikalische Zeitschrift*, iv. Jahrgang, No. 1, 1902.
‖ Communicated by the Physical Society : read Jan. 23, 1903.

The late Dr. John Hopkinson [*] pointed out the imperfection of the usual formula given in Maxwell [†], and also gave an empirical formula which closely represented his experiments. The general result is well known, namely, that the sensibility of the electrometer rises to a maximum as the potential of the needle is raised, and that any further increase in the potential of the needle reduces the sensibility.

The same effect occurs in the extremely sensitive electrometers made by Bartels, of Göttingen, several of which have been in use in the Cavendish Laboratory for some time. In these instruments the needle is made of silvered paper, and hung by a single quartz fibre. The quadrants are about 5 cms. radius by 1 cm. deep, and the air-space between the quadrants is about 1 mm. The quadrants are mounted on ebonite, and are not adjustable.

There is no guard-tube for the fibre, and no leyden-jar attached, the insulation being extremely good. With the needle charged to about 100 volts, a deflexion of 1000 mms. per volt, on a scale about 1 metre from the mirror, can readily be obtained, the needle being nearly dead-beat at this sensibility, and quite steady. The shift of the needle during charging is generally but a small fraction of the deflexion for 1 volt. A maximum sensibility occurs at about 100 volts, but this of course depends on the fineness of the fibre. The sensibility seems to go on diminishing after this, at least until very high voltages are used.

In examining the theory I found that Hopkinson's formula could be readily explained.

The referee to whom my paper was sent pointed out that my conclusions conflicted with the experiments of Ayrton and Sumpner [‡] on a White pattern electrometer. I have therefore added to my paper a discussion of their results.

It will be convenient to give my modified theory of a symmetrical instrument first, and then compare my conclusions with Ayrton and Sumpner's results.

Let us first indicate the usual theory. Suppose V_1, V_2, and V_3 are the potentials of the two pairs of quadrants and the needle respectively.

The energy of the system is given by

$$E = \tfrac{1}{2}c_{11}V_1^2 + \tfrac{1}{2}c_{22}V_2^2 + \tfrac{1}{2}c_{33}V_3^2 + c_{12}V_1V_2 + c_{13}V_1V_3 + c_{23}V_2V_3, \quad (1)$$

and the force in the direction θ is given by

$$\frac{1}{2}\frac{\partial c_{11}}{\partial \theta}V_1^2 + \frac{1}{2}\frac{\partial c_{22}}{\partial \theta}V_2^2 + \frac{1}{2}\frac{\partial c_{33}}{\partial \theta}V_3^2 + \frac{\partial c_{12}}{\partial \theta}V_1V_2 + \frac{\partial c_{13}}{\partial \theta}V_1V_3 + \frac{\partial c_{23}}{\partial \theta}V_2V_3. \quad (2)$$

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[*] Phil. Mag. [5] vol. xix. 1885, p. 291.
[†] 'Electricity and Magnetism,' vol. i.
[‡] Phil. Trans. 1891, vol. clxxxii. p. 519.

It is next proved, somewhat doubtfully, that

$$\frac{\partial c_{11}}{\partial \theta} = -\frac{\partial c_{22}}{\partial \theta} = -\frac{\partial c_{13}}{\partial \theta} = +\frac{\partial c_{23}}{\partial \theta}$$

$$\frac{\partial c_{12}}{\partial \theta} = \frac{\partial c_{33}}{\partial \theta} = 0,$$

and hence, if the couple due to the fibre is $F\theta$, we get

$$F\theta = \frac{\partial c_{23}}{\partial \theta}(V_2 - V_1)\{V_3 - \tfrac{1}{2}(V_1 + V_2)\}. \quad . \quad . \quad (3)$$

In the above process no exception can be taken to (1) and (2); but in the remaining part of the proof the values of the differential coefficients are calculated for $\theta = 0$.

This is not valid. Formula (2) is only true provided the quantities are reckoned for the displaced position.

In what follows I shall assume perfect symmetry of the arrangements.

Let $\qquad c_{11} = a_0 + \Sigma_1^\infty a_n \theta^n,$

where a_0 etc. are constants independent of θ.

Then by symmetry we obtain

$$c_{22} = a_0 + \Sigma_1^\infty (-)^n a_n \theta^n.$$

Again, let

$$c_{13} = b_0 + \Sigma_1^\infty b_n \theta^n,$$

the condition of symmetry gives us

$$c_{23} = b_0 + \Sigma_1^\infty (-)^n b_n \theta^n.$$

Further, it is clear that c_{12} and c_{33} must be even functions of θ. Hence let us take

$$c_{12} = c_0 + \Sigma_1^\infty c_{2m} \theta^{2m}$$

$$c_{33} = d_0 + \Sigma_1^\infty d_{2m} \theta^{2m}.$$

It is clear, since the zero of potential is arbitrary, that an equal increase in each of the potentials must leave the force unchanged.

We thus get from formula (2)

$$\left. \begin{array}{c} \dfrac{\partial c_{11}}{\partial \theta} + \dfrac{\partial c_{12}}{\partial \theta} + \dfrac{\partial c_{13}}{\partial \theta} = 0 \\[2mm] \dfrac{\partial c_{22}}{\partial \theta} + \dfrac{\partial c_{23}}{\partial \theta} + \dfrac{\partial c_{12}}{\partial \theta} = 0 \\[2mm] \dfrac{\partial c_{33}}{\partial \theta} + \dfrac{\partial c_{13}}{\partial \theta} + \dfrac{\partial c_{23}}{\partial \theta} = 0 \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence, if we substitute the assumed forms for the co-efficients in the system (4), we obtain

$$\Sigma_1^\infty n(a_n+b_n)\theta^{n-1}+\Sigma_1^\infty 2mc_{2m}\theta^{2m-1}=0, \quad \cdots \quad (5)$$

$$\Sigma_1^\infty (-)^n n(a_n+b_n)\theta^{n-1}+\Sigma_1^\infty 2mc_m\theta^{2m-1}=0, \quad \cdots \quad (6)$$

$$\Sigma\, 2m\,d_{2m}\theta^{2m-1}+\Sigma_1^\infty nb_n\theta^{n-1}+\Sigma_1^\infty (-)^n nb_n\theta^{n-1}=0. \quad (7)$$

We may therefore equate the coefficients of the various powers of θ to zero.

Equations (5) and (6) lead to the same results, namely,

For n odd $\qquad\qquad a_n+b_n=0.$

For n even $=2m \qquad a_{2m}+b_{2m}+c_{2m}=0.$

While equation (7) gives

For n even $=2m \qquad d_{2m}+2b_{2m}=0.$

Hence we get for the coefficients of capacity

$$c_{11}=a_0+\Sigma_1^\infty a_n\theta^n,$$

$$c_{22}=a_0+\Sigma_1^\infty (-)^n a_n\theta^n,$$

$$c_{13}=b_0-\Sigma_1^\infty a_{2n-1}\theta^{2n-1}+\Sigma_1^\infty b_{2n}\theta^{2n},$$

$$c_{23}=b_0+\Sigma_1^\infty a_{2n-1}\theta^{2n-1}+\Sigma_1^\infty b_{2n}\theta^{2n},$$

$$c_{12}=c_0-\Sigma_1^\infty (a_{2n}+b_{2n})\theta^{2n},$$

$$c_{33}=d_0-2\Sigma_1^\infty b_{2n}\theta^{2n}.$$

We are now in a position to calculate the values of the differential coefficients, and since for the present purpose we are concerned with small displacements only, we shall proceed to first powers of θ. We thus get

$$\frac{\partial c_{11}}{\partial\theta}=a_1+2a_2\theta, \quad \frac{\partial c_{13}}{\partial\theta}=-a_1+2b_2\theta, \quad \frac{\partial c_{12}}{\partial\theta}=-2(a_2+b_2)\theta,$$

$$\frac{\partial c_{22}}{\partial\theta}=-a_1+2a_2\theta, \quad \frac{\partial c_{23}}{\partial\theta}=+a_1+2b_2\theta, \quad \frac{\partial c_{33}}{\partial\theta}=-4b_2\theta.$$

Substituting in (2) we obtain for the force on the needle

$$a_1(V_2-V_1)\{V_3-\tfrac{1}{2}(V_1+V_2)\}+\theta\{a_2(V_2-V_1)^2-2b_2(V_3-V_1)(V_3-V_2)\}$$

Hence, instead of the ordinary equation

$$F\theta=a_1(V_2-V_1)\{V_3-\tfrac{1}{2}(V_1+V_2)\},$$

we obtain

$$\{F+2b_2(V_3-V_1)(V_3-V_2)-a_2(V_2-V_1)^2\}\theta$$
$$=a_1(V_2-V_1)\{V_3-\tfrac{1}{2}(V_1+V_2)\}.$$

In the ordinary method of working V_1 is actually zero, and V_2 only slightly greater than zero, while V_3 is comparatively high. We thus have approximately the formula

$$\{ F + 2b_2 V_3^2 \} \theta = a_1 (V_2 - V_1) V_3. \quad . \quad . \quad . \quad (8)$$

This equation exactly represents the results in Bartel's electrometer, and shows that if b_2 is positive there will be a maximum sensibility depending on the potential of the needle. The higher the attempted sensibility, by reducing F and increasing V_3, the more important does the term $b_2 V_3^2$ become.

Can we then account for a positive value of b_2?

Let us consider how c_{33} varies with the displacement. If there were no air-gap between the quadrants, it is clear that c_{33} would remain constant in all positions of the needle, but in practice the air-gap is not negligible. For example, suppose the quadrants are 5 cms. radius, and the air-gap 1 mm. wide, then the ratio of the air-gap to the area of the quadrants is about 2 sq. cms. to 75 sq. cms.
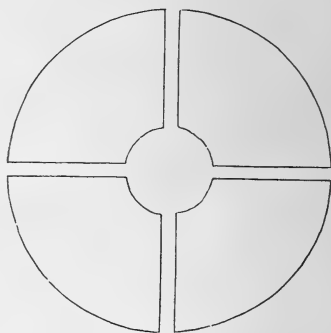
The capacity of the needle c_{33} will thus vary, depending on the number of lines of force which escape the quadrants. We clearly have stationary values at $\theta = 0$ and $\theta = \frac{\pi}{4}$ or

$$\frac{\partial c_{33}}{\partial \theta} = 0 \quad \text{for } \theta = 0 \quad \text{and} \quad \theta = \frac{\pi}{4}.$$

First suppose the angle of the needle 90°. Then as we move from the position $\theta = 0$ both edges of the needle approach an air-space, and therefore c_{33} diminishes. It will continue to diminish to $\theta = \frac{\pi}{4}$, where it is least. The curve will therefore be as shown in fig. (1), and the quantity b_2 is positive.

Second, suppose the needle to be a mere wire, then c_{33} continually increases from $\theta = 0$ to $\theta = \frac{\pi}{4}$, and the curve is as shown in fig. 2.

In this case b_2 is negative. For angles of the needle near 90° we may therefore suppose that b_2 is $+$, while it is probable

that there is an angle between $0°$ and $90°$ for which b_2 might be made actually zero.
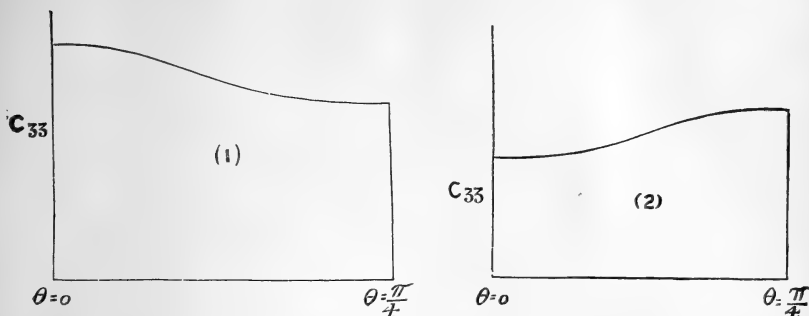
Further it is clear that b_2 must diminish as the air-gap is reduced, and hence the potential for maximum sensibility increased, other things being equal.

So far the argument is qualitative, and we have now to get a quantitative estimate of the effect of the air-gap.

Let us take the case of a quartz fibre 5 cms. long, ·009 mm. diameter. For this $F = 8 \times 10^{-3}$. Now if $2b_2 = 10^{-22}$, and $V_3 = 100$ volts, we get $2b_2 V_3{}^2 = 10^{-2}$. This is a quantity of the same order as F. Hence if $2b_2$ is only $\frac{1}{10}$ of an electrostatic unit of capacity, there would in this case be a maximum sensibility at about 100 volts.

The solution of the electrical distribution for a system like the quadrant electrometer is a well-nigh hopeless problem. I now propose to discuss a two-dimensional problem, which in some respects corresponds to the actual case considered in the preceding pages. I have succeeded in solving the problem completely, and the result confirms my view that the air-gap is sufficient to account for a maximum sensibility.

We shall take four semi-infinite plates to correspond to the upper and lower plates of the quadrants, and a plate of finite breadth to correspond to the needle.
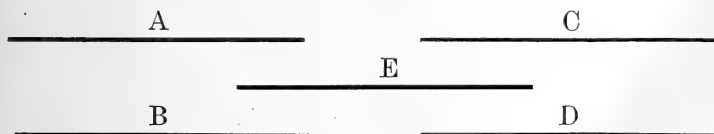
The cross-section of the arrangement is shown in the diagram : and it is to be understood that the plates A and B extend indefinitely to the left, while C and D extend indefinitely to the right. The plate E is situated midway

between the two pairs of plates, and displaced from the symmetrical position.

This problem is soluble by the Schwarzian method, so that I think it would be superfluous to give the details of the solution. I shall therefore indicate the transformations, and give the final solution :—
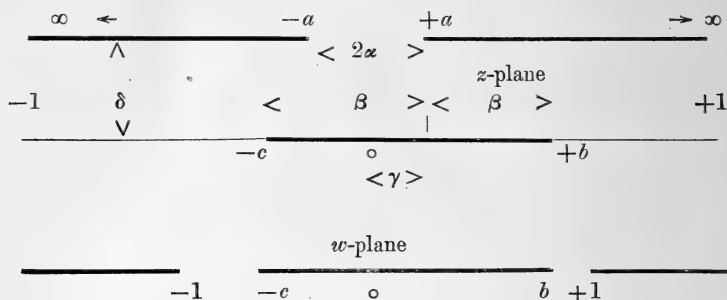
The plane of the needle is a plane of symmetry, and we may transform the given problem in the z-plane into the problem indicated in the w-plane, the points of correspondence being marked as I have shown.

The transformation is

$$z = -\frac{2\delta}{\pi}\left\{ \frac{w}{a^2-1} - \frac{1}{2}\log\frac{w+1}{w-1} - i\frac{\pi}{2} \right\}.$$

Let 2α be the breadth of the air-gap,
  2β ,,   ,,   ,,   needle,
  γ ,,   displacement from the zero position,
  2δ ,,   distance between the upper and lower pair of plates.

Then we have the following equations to determine a, b, and c :—

$$\frac{a}{a^2-1} - \frac{1}{2}\log\frac{a+1}{a-1} = \frac{\pi}{2}\cdot\frac{a}{\delta} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9)$$

$$\frac{1}{2}\log\frac{1+b}{1-b} - \frac{b}{a^2-1} = \frac{\pi}{2}\cdot\frac{\beta+\gamma}{\delta} \quad \cdot \quad \cdot \quad \cdot \quad (10)$$

$$\frac{1}{2}\log\frac{1+c}{1-c} - \frac{c}{a^2-1} = \frac{\pi}{2}\frac{\beta-\gamma}{\delta}. \quad \cdot \quad \cdot \quad \cdot \quad (11)$$

We may note that these give a real value for a greater than unity, and real values for b and c less than unity when γ is $<\beta$. When γ is zero, the values of b and c are equal.

The solution of the w-problem can be effected by the transformation

$$\frac{\partial \phi + \iota \psi}{\partial w} = \frac{i\mathrm{B}}{\sqrt{(w^2-1)(w-b)(w+c)}},$$

where ϕ and ψ are the potential and stream-function respectively.

The latter transformation is reducible by means of elliptic functions. When we write the condition that the finite plate is at unit potential, while the semi-infinite plates are at zero potential, we readily find that the charge on the finite plate is $\dfrac{4\mathrm{K}'}{\mathrm{K}}$, where K' and K are the usual quarter periods of the elliptic functions, and the modulus is given by

$$k'^2 = \left\{ \frac{\lambda' - b\lambda}{\lambda - b\lambda'} \right\}^2$$

and

$$(\lambda^2 + \lambda'^2)(b-c) - 2\lambda\lambda'(1-bc) = 0.$$

Let us put

$$b = b_0 + \delta_0$$
$$c = b_0 - \delta_0$$

and take the root

$$\frac{\lambda}{\lambda_1} = \frac{1 - b_0^2 + \delta_0^2 + \sqrt{(1-b_0^2+\delta_0^2)^2 - 4\delta_0^2}}{2\delta_0} = \frac{\xi}{2\delta_0} \text{ say,}$$

we get

$$k'^2 = b_0^2 \left\{ \frac{\xi}{\xi - 2\delta_0^2} \right\}^2.$$

If δ_0 be small we get

$$k'^2 = b_0^2 \left\{ 1 + \frac{2\delta_0^2}{1 - b_0^2} \right\}.$$

Let b_0' be the root of

$$\frac{1}{2} \log \frac{1 + b_0'}{1 - b_0'} - \frac{b_0'}{a^2 - 1} = \frac{\pi}{2} \cdot \frac{\beta}{\delta}.$$

Then we find from equations (10) and (11)

$$b_0 = b_0' \left\{ 1 - \frac{a^2-1}{(a^2 + b_0'^2 - 2)(1 - b_0'^2)} \delta_0^2 \right\}$$

and

$$\delta_0 = \frac{\pi\gamma}{2\delta} \frac{(1 - b_0'^2)(a^2 - 1)}{(a^2 + b_0'^2 - 2)},$$

and hence

$$k'^2 = b_0'^2 \left\{ 1 - \frac{1}{2} \frac{\pi^2\gamma^2}{\delta^2} \frac{(1 - b_0'^2)(a^2 - 1)^2}{(a^2 + b_0'^2 - 2)^3} \right\}.$$

The variation in k'^2 will thus be negative or positive according as $a^2 - 1 >$ or $< 1 - b_0'^2$. The formula would be invalid if $(a^2 - 1) = (1 - b_0'^2)$. A brief examination shows that with a finite breadth of needle $a^2 - 1$ is $> 1 - b_0'^2$.

Consider the function

$$f = \frac{1}{2} \log \frac{1+b}{1-b} - \frac{b}{a^2 - 1}.$$

Then $f = 0$ when $b = 0$,

and $f = +\infty$ when $b = 1$,

$$\frac{\partial f}{\partial b} = \frac{a^2 - 1 + b^2 - 1}{(1 - b^2)(a^2 - 1)}.$$

Now $\dfrac{\partial f}{\partial b}$ will vanish when $a^2 - 1 = 1 - b^2$ or $b^2 = 2 - a^2$. Hence, if a^2 is < 2, there is a real value for which $\dfrac{\partial f}{\partial b}$ vanishes ; but if a^2 is > 2, there is no real value. There are thus two cases, A and B, as shown in the figure.
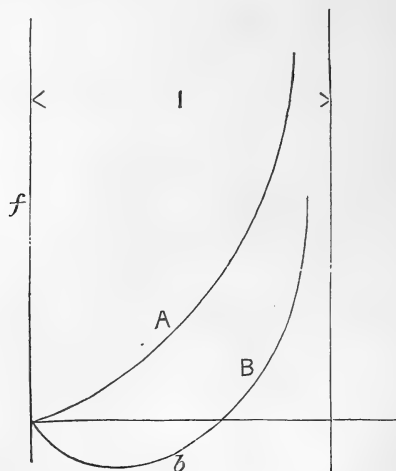
Now the root b_0' is given by the intersection of the curve

$$f = \frac{1}{2} \log \frac{1+b}{1-b} - \frac{b}{a^2 - 1}$$

with the straight line

$$f = \frac{\pi}{2} \frac{\beta}{\delta}.$$

Hence, at the intersection $\dfrac{\partial f}{\partial b}$ must be positive, that is, $a^2 - 1$ is $> 1 - b_0'^2$.

We have seen that the capacity of the plate per unit of length perpendicular to the plane of the paper is

$$\frac{4\mathrm{K}'}{\mathrm{K}},$$

and K' is the same function of k'^2 that K is of k^2.

Now let $\Delta k'^2$ be the variation of k'^2 from its value in the symmetrical position, then

$$\Delta k^2 + \Delta k'^2 = 0.$$

Hence the capacity

$$= \frac{4\mathrm{K}_0'}{\mathrm{K}_0}\left\{ 1 + \left(\frac{1}{\mathrm{K}'} \frac{\partial \mathrm{K}'}{\partial k'^2} + \frac{1}{\mathrm{K}} \frac{\partial \mathrm{K}}{\partial k^2} \right)_0 \Delta k'^2 \right\}.$$

By a curious formula in elliptic functions we have

$$\frac{1}{\mathrm{K}'} \frac{\partial \mathrm{K}'}{\partial k'^2} + \frac{1}{\mathrm{K}} \frac{\partial \mathrm{K}}{\partial k^2} = \frac{\pi}{4 k^2 k'^2 \mathrm{K} \mathrm{K}'}.$$

Hence the capacity

$$= \frac{4\mathrm{K}_0'}{\mathrm{K}_0}\left\{ 1 + \frac{\pi}{4 k^2 k'^2 \mathrm{K}_0 \mathrm{K}_0'} \Delta k'^2 \right\},$$

and we have

$$k_0'^2 = b_0'^2$$

$$k_0^2 = 1 - b_0'^2$$

and

$$\Delta k'^2 = -\frac{1}{2} \frac{\pi^2 \gamma^2}{\delta^2} \frac{k_0^2 k_0'^2 (a^2 - 1)^2}{(a^2 + b_0'^2 - 2)^3},$$

where the suffix 0 refers to the values when $\gamma = 0$.
Hence the capacity is

$$= \frac{4\mathrm{K}_0'}{\mathrm{K}_0}\left\{ 1 - \frac{\pi^3}{8} \frac{(a^2 - 1)^2}{\mathrm{K}_0 \mathrm{K}_0'(a^2 + b_0'^2 - 2)^3} \frac{\gamma^2}{\delta^2} \right\}. \quad \cdot \quad (12)$$

This formula corresponds to the formula which I gave in the early part of this paper for the capacity of the needle, viz.

$$c_{33} = d_0 - 2b_2\theta^2,$$

θ being the angular deflexion.

I showed there that if b_2 was a positive quantity of order $\frac{1}{10}$ of an electrostatic unit, the experimental peculiarity of the electrometer for increasing potential of the needle could be accounted for both qualitatively and quantitatively.

I have therefore prepared the following numerical table, making use of Legendre's Tables for K and Hutton's Tables of Naperian Logarithms. Throughout I have taken δ as 1 cm.

No.	a_0	b_0'	K_0'	K_0	α in cms.	β in cms.	$\dfrac{4K_0'}{K_0}$	$\dfrac{\pi^3}{8}\dfrac{(a^2-1)^2}{K_0 K_0'(a^2+b_0'^2-2)^3}$	$\dfrac{\pi^3}{2}\dfrac{(a^2-1)^2}{K_0'^2(a^2+b_0'^2-2)^3}$
(1)	1·8	·98	3·0233	1·5866	·112	1·18	7·60	·333	2·53
(2)	1·8	·99	3·3577	1·5806	·112	1·40	8·48	·291	2·46
(3)	1·8	·999	4·5006	1·5715	·112	2·13	11·2	·213	2·38
(4)	2	·9999	5·658	1·570	·074	2·94	14·4	·143	2·06
(5)	3	·9999	5·658	1·570	·0108	3·07	14·4	·054	·777
(6)	3	·98	3·0233	1·5866	·0108	1·38	7·60	·101	·767

In (1), (2), and (3) we observe that keeping the air-gap about 2 mms. the effect diminishes as the breadth of the needle increases.

In (2) and (6), where the breadth is practically the same, we see the diminishing effect of the air-gap as the gap is diminished. At the same time the gap could not practically be reduced to $\frac{1}{5}$ mm., and the variation, although reduced, is still important. I think that this investigation may be held to show that the air-gap as usually found in an electrometer is quite sufficient to account for the observed fact that there is a maximum sensibility depending on the potential of the needle ; and the conclusion is that the ordinary formula would be more nearly obeyed with a small air-gap.

It is of interest to note what difference results when, instead of the potential, the charge of the needle is kept constant. The method already used for finding the mechanical force on the needle will still be valid provided V_3 is reckoned for the displaced position.

If $_0V_3$ be the potential of the needle in the zero position, we get

$$V_3 = {}_0V_3 - \frac{a_1}{d_0}\theta(V_2 - V_1) \quad \text{to 1st power of } \theta.$$

Hence the force becomes

$$a_1(V_2-V_1)\{{}_0V_3-\tfrac{1}{2}(V_1+V_2)\}+\theta\left\{\left(a_2-\frac{a_1^2}{d_0}\right)(V_2-V_1)^2\right.$$
$$\left.-2b_2(V_3-V_1)(V_3-V_2)\right\}$$

a_1 appears to be about 10^{-21},

and d_0 ,, ,, ,, 10^{-19}.

So that as long as the potentials of the quadrants are kept small, no practical difference results.

It is important to point out that although diminishing the air-gap raises the potential of maximum sensibility, it does not follow that higher sensibility will be obtained by reducing the air-gap. This depends on a_1. The form of curve for c_{23} is shown.
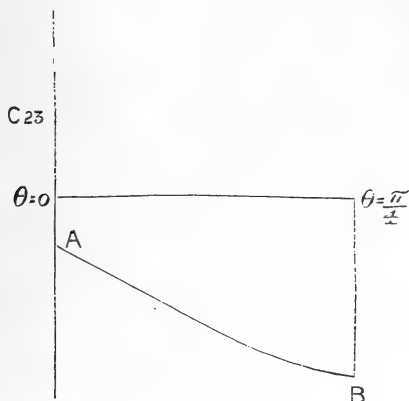
Diminishing the air-gap will move the whole curve down, and B probably more than A. a_1, which is the angle made by the tangent at A, probably increases numerically as the air-gap is reduced, but whereas b_2 tends to zero as the air-gap is reduced, a_1 tends to a finite limit. I therefore think it probable that higher sensibility will be attained by reducing the air-gap.

We have now to consider Ayrton and Sumpner's results with the White electrometer bifilar suspension. They found a maximum about 1200 volts, but after falling a little the sensibility curve rises as the potential of the needle increases. They show most satisfactorily that the effect is due to lack of symmetry, and particularly to lack of symmetry of the guard-tube. It is not, however, clear that this effect is the same as

that obtained by Hopkinson. In his experiments the maximum occurred at about 800 volts, and thereafter the sensibility went on decreasing much more rapidly than in Ayrton and Sumpner's experiments. Further, Hopkinson's instrument was much more sensitive. Again, Ayrton and Sumpner made some experiments with a single-fibre suspension, and although they obtained greater sensibilities as the potential of the needle was increased, they did not obtain a maximum.

Returning to their experiments with the bifilar suspension and the unsymmetrical guard-tube, it is almost obvious that moving the quadrants away from the guard-tube would reduce the disturbing effect.

I therefore think that their experiments apply only to an extremely unsymmetrical instrument of comparatively low sensibility, and that the maximum considered in this paper is beyond the range of their instrument.

XXIX. *Fluorescence of Crystals under the Röntgen Rays.* *By* Hon. R. J. Strutt, *Fellow of Trinity College, Cambridge*[*].

A FEW months ago M. Blondlot published an account of some experiments which led him to conclude that the Röntgen rays from a focus-tube are polarized. It has since appeared that he was (very naturally) misled by the presence of a new kind of refrangible rays from the tube, which are capable of brightening a minute electric spark.

The question of whether the Röntgen rays are really polarized remains untouched.

It is known[†] that the fluorescent crystals of magnesium platinocyanide light up with a different colour according to whether the plane of polarization of the incident light is perpendicular to the optic axis, or parallel to it. The change is from orange to scarlet. There seemed to be a possibility that a difference of the same kind might appear when the crystal was fluorescing under the action of the Röntgen rays.

I am indebted to Messrs. Johnson and Matthey for the loan of three large crystals of the salt. Röntgen rays from a focus-tube were allowed to fall normally on the prism faces, but the fluorescence was not found to depend on the angle between the optic axis of the crystal and the cathode stream in the tube. So that the experiment, for what it is worth, tells against the polarization of the Röntgen rays.

It was incidentally noticed that the prism faces were more

fluorescent than the basal planes, except in the case of one
crystal. One of the basal planes of this was very fluorescent,
the other not. It was thought at first that this was a proof
that the crystal possessed hemimorphic symmetry, like tour-
maline. But the other crystals showed no such difference
between the ends of the optic axis. It is probable that the
difference was due to some kind of decomposition which had
only affected one end of the crystal.

A number of other fluorescent crystals, including various
platinocyanides, apophyllite, calcite, felspar, and scheelite were
tried. But they did not exhibit any difference of fluorescence,
depending on which face was exposed to the rays, or on the
orientation of the face.

Another experiment may be briefly mentioned. It is known
that crystallized calcium tungstate, whether natural or arti-
ficial, is brightly fluorescent under the rays. The amorphous
precipitated tungstate is scarcely at all so. It was thought
worth trying whether prolonged boiling with water would
effect the change to the crystalline fluorescent modification.
Experience showed that this was so. Some of the precipi-
tated tungstate was divided into two parts. One of them
was boiled for four days with excess of water. On drying it
was found to be much more fluorescent than the untreated
portion, though less so than natural crystallized scheelite.

XXX. *On the Variation of Entropy as treated in Willard
Gibbs' 'Statistical Mechanics.'* By S. H. Burbury, *F.R.S.*[*]

1. IN a former paper in this Magazine (May 1900), I
considered the question whether there exist in nature
systems of which the motion is irreversible, and entropy tends
to a limit, although they consist of parts, the motion of each
of which is reversible separately. This was with especial
reference to Boltzmann's H theorem, the best known example
of such a process. In a later paper (February 1902), I con-
sidered Planck's treatise *Ueber irreversible Strahlungsvorgänge*
in the same relation. Boltzmann's and Planck's theories
depend ultimately on certain algebraic theorems, of which
we may take as a type that, given Σx, $\Sigma x \log x$ has its least
value when x is constant throughout the summation, or, in
another form, given $\int e^x \, dx$, $\int e^x x \, dx$ has its least value when
x is constant throughout the integration. But these theorems
of pure mathematics do not alone determine the question of
irreversibility, or the tendency of entropy to a limit. To

[*] Communicated by the Author.

determine that, a physical assumption has to be made with regard to the motion, by virtue of which assumption the algebraic theorem becomes applicable. This is done by both Boltzmann and Planck. Archimedes proposing to move the earth begins by assuming a *locus standi*.

2. In the present paper I propose to consider certain difficulties which to me at least present themselves in the late Professor Willard Gibbs' 'Principles of Statistical Mechanics,' Chapter XII. In this chapter he seeks to establish a general law for the variation of entropy in ensembles of systems, without, as I understand him, making any special physical assumptions, by the simple use of the typical algebraic theorem. To explain my difficulties it is necessary to set out the method developed in Chapter I. of the same work ; and this is fortunately easy, owing to the extreme simplicity and elegance in which he has himself presented it.

3. A material system is defined by the n generalized coordinates $q_1 \ldots q_n$, and corresponding momenta $p_1 \ldots p_n$. Each system is at all times free from the influence of any bodies external to itself, including all other systems, but with the exception of forces to fixed centres, such as gravitation. Its whole energy is therefore constant for all time. The " coordinates," however, including here in that term both p's and q's, vary. The system passes through a series of phases defined by the values of $q_1 \ldots p_n$. All phases in which the " coordinates " lie between the limits $q_1 \ldots q_1 + dq_1 \ldots p_n \ldots p_n + dp_n$ constitute the *extension in phase* $dq_1 \ldots dp_n$. And by Liouville's theorem $dq_1 \ldots dp_n$ is constant for the same group of systems.

4. Now suppose a very great number of such mutually independent systems. There will at a given instant be many systems within a given extension in phase. We are now to consider the extension in phase which lies between p_1' and $p_1'' \ldots q_n'$ and q_n''. The differences $p_1'' - p_1'$ &c. are infinitesimal ; and he assumes systems to be distributed continuously between these limits, a state of things which, as he points out, requires us to suppose the number of systems indefinitely great. In fact, since no system interferes with any other system, there is no limit to the number which may be in exactly the same phase.

5. The number of systems between the limits $p_1'' - p_1'$ &c. is now denoted by the continued product

$$\mathrm{D}(p_1'' - p_1')(p_2'' - p_2') \ldots \ldots (q_n'' - q_n'),$$

in which D is a function of $p_1 \ldots q_n$, and is called the *density in phase* at $p_1 - q_n$.

Let us call $q_1''\ldots p_n''$ the upper, and $q_1'\ldots p_n'$ the lower, limit. Then the number of systems which in time dt pass the lower limit by variation of p_1 alone is (p. 7)

$$\mathrm{D}\dot{p_1}'dt\,dp_2\ldots dq_n,$$

dp_2 being now written for $p_2''-p_2$ &c., and $\dot{p_1}'$ written for $\dfrac{dp_1}{dt}$ at the lower limit, and they pass in or out according as $\dot{p_1}'$ is positive or negative. In like manner the number of systems which in time dt pass the upper limit by variation of p_1 is

$$\mathrm{D}\dot{p_1}''dt\,dp_2\ldots dq_n,$$

and they pass out or in according as $\dot{p_1}''$ is positive or negative. Now D and p_1 have slightly different values at the two limits respectively. Therefore the increase in time dt of the number of systems between the limits due to the variation of p_1 alone is

$$-\frac{d}{dp_1}(\mathrm{D}\dot{p_1})dp_1\ldots dq_n\,dt.$$

This is of course on the assumption that $p_1''-p_1'$ is infinitesimal, and may be represented by dp_1.

6. What has been proved for $\dot{p_1}$ is equally true for each of the other p's and q's. Let then $\dfrac{\partial \mathrm{D}}{\partial t}$ denote the rate of increase of D per unit of time, for the particular extension in phase considered, due to this passing in and out of systems. Then

$$
\begin{aligned}
\frac{\partial \mathrm{D}}{\partial t} &= -\Sigma\left(\frac{d}{dq}(\mathrm{D}\dot{q}) + \frac{d}{dp}(\mathrm{D}\dot{p})\right)\\
&= -\Sigma\left(\dot{q}\frac{d\mathrm{D}}{dq} + \dot{p}\frac{d\mathrm{D}}{dp}\right)\Big|\\
&\quad -\Sigma\mathrm{D}\left(\frac{d\dot{q}}{dq} + \frac{d\dot{p}}{dp}\right)\Big\} \qquad \ldots \quad (1)
\end{aligned}
$$

But by Lagrange's equations, for each q and the corresponding p

$$\frac{d\dot{q}}{dq} + \frac{d\dot{p}}{dp} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

And therefore

$$\frac{\partial \mathrm{D}}{\partial t} = -\Sigma\left(\dot{q}\frac{d\mathrm{D}}{dq} + \dot{p}\frac{d\mathrm{D}}{dp}\right). \quad \ldots \quad \ldots \quad (3)$$

7. Again, if we follow the motion of a chosen system, and

D be the density in phase in which that system finds itself for the time being, we have a change of D with the time due to that motion alone. And denoting this by $\dfrac{\partial'D}{\partial t}$, we have

$$\frac{\partial'D}{\partial t} = +\Sigma\left(\dot{q}\frac{dD}{dq} + \dot{p}\frac{dD}{dp}\right). \quad \cdot \quad \cdot \quad \cdot \quad (4)$$

From (3) and (4) is deduced (p. 9) the theorem that the total change of D with the time for a moving system (call it $\dfrac{dD}{dt}$) is

$$\begin{aligned}
\frac{dD}{dt} &= -\Sigma\left(\dot{q}\frac{dD}{dq} + \dot{p}\frac{dD}{dp}\right) \\
&\quad + \Sigma\left(\dot{q}\frac{dD}{dq} + \dot{p}\frac{dD}{dp}\right) \\
&= 0,
\end{aligned} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (5)$$

or D remains constant for all time for the same moving system.

Subsequently Professor Gibbs writes (p. 16)

$$D = Ne^{\eta},$$

N being the whole number of systems, and η is a function afterwards used to express entropy. Evidently, since D is constant, η is also constant for each system for all time.

8. Professor Gibbs does not formulate *in extenso* the complete definition of D. But having regard to the foregoing, I shall not be misrepresenting him if I formulate it thus :—

Definition A.—D is the number of systems within the extension in phase $dq_1 \ldots dp_n$, or $d\sigma$, divided by $d\sigma$, in the limit when each of the factors $dq\ dp$ which constitute $d\sigma$ becomes infinitely small.

I think the result D = constant is not an approximation at all, but a rigorous deduction from the assumptions which have been made, and the definition of D. Whether it be consistent or not with the theory subsequently developed, we cannot, I think, escape from it, except by changing the definition of D.

9. I now come to Chapter XII., "On the Motion of Systems and Ensembles through Long Periods of Time." The object of this chapter is to show that, if time be only long enough, D, and by consequence η, may vary. Professor Gibbs begins by calling attention to the fact that η has been proved to be constant. But he says we must exercise

caution ; and he gives an illustration, by the study of which the reader is invited to the conclusion that the quantity η, rigorously proved (as appears to me) to be invariable with the time, will nevertheless be variable if the time be only long enough. I think we should exercise caution before we accept that conclusion. Of course, if the constancy of η depends on the fulfilment of certain conditions, it may be that with lapse of time the conditions will fail, and η cease to be constant. But we are not told what conditions, nor that any conditions, will fail.

10. Let us then consider the illustration. It will appear that it does point to a way of escape from the difficulty, but not, I think, exactly the way that Professor Gibbs recommends. An incompressible liquid has colouring-matter distributed continuously through it. If P be a point in the liquid, ρ, the density of colouring-matter at P, shall be defined in a way precisely analogous to the definition of D above given, namely—Definition A—ρ at P is the quantity of colouring-matter within a sphere of radius r described about P as centre, divided by $\dfrac{4\pi r^3}{3}$, in the limit when r becomes indefinitely small. Professor Gibbs, as we have seen, did not expressly formulate this definition for D, but he does expressly formulate it in the case of ρ (p. 145).

Now let u, v, w denote the component velocities of the liquid at any point, and consider the element of volume $dx\ dy\ dz$. By the motion of the liquid colouring-matter is passing into this element regarded as fixed in space through some faces, and out of it through other faces, and in exact analogy to equation (1) we have

$$\frac{\partial \rho}{\partial t} = -\left\{ \frac{d}{dx}(\rho u) + \frac{d}{dy}(\rho v) + \frac{d}{dz}(\rho w) \right\}. \quad (1\,a)$$

But in exact analogy to (2)

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \quad \quad (2\,a)$$

Therefore in exact analogy to (3)

$$\frac{\partial \rho}{\partial t} = -\left(u\frac{d\rho}{dx} + v\frac{d\rho}{dy} + w\frac{d\rho}{dz} \right). \quad (3\,a)$$

Let now $\dfrac{\partial' \rho}{\partial' t}$ denote the change of ρ for a particle moving with the liquid due to that motion alone. Then

$$\frac{\partial' \rho}{\partial' t} = +\left(u\frac{d\rho}{dx} + v\frac{d\rho}{dy} + w\frac{d\rho}{dz} \right). \quad (4\,a)$$

And for the total change of ρ with the time for a particle moving with the liquid

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial'\rho}{\partial't} = 0, \quad . \quad . \quad . \quad . \quad (5\,a)$$

or ρ is constant for the same particle for all time. That, I think, is a rigorous conclusion given the assumptions and the definition of ρ, not merely an approximation.

12. It follows from the proved constancy of ρ that

$$\iiint \rho^2\, dx\, dy\, dz, \text{ or } \overline{\rho^2}, \text{ is constant.}$$

Professor Gibbs then says, p. 145, " No fact is more familiar to us than that stirring tends to bring the liquid to a state of uniform mixture, or uniform densities of its components, which is characterized by minimum values of the average squares of those densities." There is then an apparent contradiction between our mathematical conclusion that $\overline{\rho^2}$ is constant, and our familiar experience that $\overline{\rho^2}$ can be made to vary. And "the contradiction," he says, " is to be traced to the notion of the density of the colouring-matter, and the process by which this quantity is evaluated." This " quantity," he further says, p. 145, " is the limiting ratio of the quantity of colouring-matter in an element to the volume of the element," which is my Definition A.

I quite agree with him as to the origin of the contradiction, but I think there is only one legitimate way of avoiding it, and that is to change the definition. The conclusion that he draws, p. 146, is, however, that " one might be allowed to say that a finite amount of stirring will not affect the mean square of the density of the colouring-matter, but an infinite amount of stirring may be regarded as producing a condition in which the mean square of the density has its minimum value, and the density is uniform." I cannot persuade myself to accept this conclusion. I appeal from Professor Gibbs the philosopher of Chapter XII. to Professor Gibbs the mathematician of Chapter I.

13. But the term " density at a point P " may mean more than one different thing, and may accordingly have more than one different definition. We have, every one of us, a rough general idea of what we mean by the term, though we seldom take the trouble to express our idea in the accurate language of mathematics. When asked to do so, we resort, as it were by instinct, to Definition A, which has the advantage of being mathematically irreproachable. Mathematically irreproachable indeed it is, but, for the present purpose, I think, useless. To show that it is not the only possible

definition, it is sufficient to substitute for the continuous liquid of Professor Gibbs' illustration, a medium consisting of discrete molecules, each of finite dimensions, so that if a space dS be small enough not more than one molecule can have its centre therein at the same time. As applied to such a medium Definition A is, not only useless, but wholly devoid of meaning. The "number of molecules within dS when dS becomes infinitely small" has no meaning whatever.

14. Nevertheless the term "density" as applied to this system of molecules has a meaning capable of definition. When we speak of the density at a point P, we evidently contemplate many molecules, but we should say that molecules very distant from P have nothing to do with the density at P. For a complete definition we should perhaps weight the molecules according to their nearness to P. But as this would be difficult, we might, as a rough makeshift, say ρ is the number of molecules within a sphere of defined radius r described about P as centre. That is unambiguous, and though open to certain objections, for some purposes sufficient. Returning now to our liquid we might define ρ the density of colouring-matter at a point P in the liquid, to be the quantity of colouring-matter within a sphere of a certain finite radius r described about P as centre divided by $\dfrac{4\pi r^3}{3}$,

or taking the volume of this sphere for the unit of volume, ρ is the quantity of colouring-matter in unit of volume at P. The proof of the constancy of ρ now fails. In the same way we might take a finite extension in phase including certain values q, p, of the variables in Professor Gibbs' ensemble, and define D as the number of systems in that finite extension in phase divided by the extension in phase. The proof of the constancy of D now fails. D and η will vary, and there is no necessity to resort to long periods of time.

15. But assuming the variability of D, or η, to be now by whatever means established, I find yet another difficulty in the consequences which Professor Gibbs deduces from it. So long as η remains constant for the same system, we may define η to be the entropy which that system has. That is not inconsistent with the definition of η above given: $D = Ne^\eta$. But η being now supposed variable for the same system, we require a definition. It is (p. 148) "an arbitrary function of the phase" in which the system finds itself. We are then to consider a certain finite extension in phase V which might be represented graphically by an area on the paper. This is divided into elements of extension in phase denoted by

DV, DV′, &c.   Each of these elements is supposed to be so
small that η may (he says) be regarded as sensibly constant
throughout any one of them at a certain instant t'.   But if
η is a function of the phase, this property of being sensibly
constant throughout any one of the elements DV, must be
true at every instant, and not at the particular instant t' only.
If η is not a function of the phase, what is the definition of η
for a system when η varies with the time?

16. Professor Gibbs then considers another instant t'',
which (so far as yet appears) may be either earlier or later
than t'.   A certain group of n systems are at t', all within
the same extension in phase DV′, and have η' for entropy.
Of these n systems most, but not all, are at time t'' within the
extension in phase DV″ and have entropy η''.   The remainder
are at t'' scattered through many other elements of extension
in phase, namely, DV_1'', DV_2'', &c., and have entropies η_1'',
η_2'', &c. respectively.   Let us say that the number which at
t'' are in DV″ is $Ne^{\eta''}$, the number which are in DV_1'' is
Ne_{η_1}'', the number which are in DV_2'' is Ne^{η_2} and so on.
Then since the number of the group is not altered we have

$$Ne^{\eta'} = \frac{N}{n}(e^{\eta''} + e^{\eta_1''} + e^{\eta_2''} + \&c.).$$

The mean value of η for the group is then

at t' $Ne^{\eta'}\eta'$, ... $\left.\begin{array}{l} \\ \\ \end{array}\right\}$ . . . (5)

at t'' it is $\dfrac{N}{n}(e_{\eta''}\eta'' + e^{\eta_1''}\eta_1'' + e^{\eta_2''}\eta_2'' + \&c.)$

But by the algebraic theorem of art. 1 in its second form,
or Theorem IX. of Chapter XI., since $ne^{\eta'} = e^{\eta''} + e^{\eta_1''} + \&c.$,
the first of the expressions (5) is less than the second, or
the mean value of η for the group is less at t' when the n
systems are all in the same element of extension in phase
DV″, than at t'' when they are scattered over different ele-
ments.   If t' is a later instant than t'', we have for this
group diminishing mean entropy.   If t'' is later than t' we
have increasing mean entropy.

17. But no reason is given, and no reason is apparent,
why a set of systems, which were in different extensions in
phase at an earlier instant, t'', should be in the same exten-
sion in phase at a later instant t', rather than *vice versa*.   All
that is proved is that the mean entropy of a group will in
certain cases either increase or diminish.   To determine
whether it tends in general to increase or to diminish, or to
prove that it tends to diminish, we require, as appears to me,

a further assumption, and that might take the form of assuming that the number of systems which in time ∂t pass out of the extension in phase $d\sigma$ and into $d\sigma'$ is proportional to the number that are in $d\sigma$ for the time being. Other assumptions might answer the purpose. But as Professor Gibbs has left it, η seems, according to the definition you give, either constant during all periods of time however long, or variable during any period of time however short.

[The above article was sent to press before the author was aware of the lamented death of Professor Willard Gibbs.]

XXXI. *The Electrical Resonance of Metal Particles for Light-Waves. Third Communication.* By R. W. WOOD, *Professor of Experimental Physics, Johns Hopkins University* *.

SINCE the appearance of the two preceding papers (Phil. Mag. vols. iii. & iv., 1902) on the above subject I have made further investigations, which appear to confirm the provisional hypothesis of electrical resonance, which was adopted to explain the very brilliant colours of films made up of metal granules of the order of magnitude of light-waves.

I have succeeded in obtaining the coloured films in prismatic form, and have established the fact that they exhibit anomalous dispersion for waves longer and shorter than the ones which are refused transmission. This was observed for electrical waves passing through a prism built up of tinfoil resonators by Garbasso and Aschkinass. I have obtained the coloured films on the walls of a tube in which I had fused a quantity of magnalium alloy *in vacuo*. The film was permanent in air, but the colours vanished as soon as the film was moistened with a little very dilute acid, owing to the conversion of the metal granules into a salt. I have also obtained coloured films by the cathode discharge from an electrode of selenium, though the colours in this case were not nearly as brilliant as those obtained by the distillation of the alkali metals. The microscope showed these films to be granular, and experiments with the quartz spectrograph demonstrated that they were very transparent to ultra-violet light, for which homogeneous films of selenium of the same thickness are absolutely opaque.

The silver films, red, purple, and blue, which I described in my second paper, seemed most suitable for continuing the experiments, owing to the comparative ease with which they

* Communicated by the Physical Society.

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are prepared and their permanency in air.   On attempting to repeat the experiment, I was somewhat annoyed to find that the silver solution dried on the plates with no trace of colour, a thin deposit of a black powder being the only result.   This powder, if rubbed with the finger, gave a blood-red streak on the plate, which resembled in every respect the coloured deposits obtained last year.   I tried cleaning the glass more carefully, using fresh chemicals and distilled water ; but all to no purpose.   It finally occurred to me, however, that I had formed the original deposits on some old lantern-slide plates, from which the gelatine had been removed with hot water.   Possible a film of gelatine of infinitesimal thickness might have been the necessary factor. To test this I wetted a sheet of glass and rubbed it over with a small thread of Nelson's photographic gelatine.   The plate being cold, only the slightest trace of gelatine could have dissolved in the film of water.   The plate was then drained off and dried on a hot metal plate.   On flowing this plate with the silver solution and drying it rapidly as before, I obtained a most wonderful film, as red as the densest ruby-glass over most of its extent, but with several patches of deep blue-violet, as rich in colour as dense cobalt-glass.   The presence of the gelatine seems in some way to keep the small particles from collecting into aggregates (the black powder). It will be remembered that in my first paper I alluded to an experiment in which the cooling of a bulb, and the condensation of the hydrocarbon vapour on the coloured film, sometimes destroyed the colour permanently, the minute sodium particles collecting into larger aggregates, which diffracted the light, and showed the pearly lustre, but exhibited no trace of colour by transmitted light.

As the formation of these films makes a most brilliant lantern experiment, I will give more explicit directions for their preparation.   The solution is one described by Carey Lea. Three solutions are prepared ; a 30-per-cent. one of ferrous sulphate, a 40-per-cent. one of sodium citrate, and a 10-per-cent. one of silver nitrate.   Fourteen c.cms. of the citrate solution are mixed with 10 c.cms. of the ferrous sulphate solution, to which is then added 10 c.cms. of the silver nitrate solution.   A dense black precipitate at once forms, and the whole is at once poured into a filter.   As soon as the liquid has entirely run through, the precipitate is washed with 10 cms. (*not more*) of distilled water.   This is to remove the salts which make the precipitate insoluble.   After the water has entirely passed through the filter, about 25 c.cms. of distilled water are poured into the filter, and the blood-red

solution which runs through collected. As it does not keep very well, it is best to prepare it on the day on which it is to be used.

A sheet of glass is washed clean, rinsed with fresh water, and the wet surface rubbed over with some shreads of gelatine. It is then drained for a few seconds and dried on a hot plate. A little of the silver solution is flowed over it, the surplus being drained off. If too much gelatine has been used, precipitation is apt to take place, the deposit taking the form of floating shreads of a reddish membrane. If no considerable precipitation occurs, the plate, which should have been quite warm when flowed, is placed once more on the hot plate until dry. The films formed in this way are usually deep red in colour, though sometimes patches of deep violet form, with sharply defined edges. As one may make a dozen trials without obtaining the violet patches, I endeavoured to find some way of producing them at will; and finally hit upon the right expedient. When the plate is about half dry, with a steaming film and a few small pools of the hot solution, it is removed from the hot plate, held at an angle, and treated with a few drops of alcohol, which are allowed to run down across the still damp portion of the plate. These portions speedily dry into a most gorgeous mosaic of red, purple, and violet patches, the experiment being especially striking in the lantern, as it occupies but a few seconds, and the colour-display spreads over the plate like the blaze of a sunset. So far as I have been able to find, these brilliantly coloured films were never obtained by Carey Lea, though the solution used is one that he employed in his experiments on allotropic silver.

Any desired depth of colour can be obtained with these films by merely allowing more or less of the solution to remain on the plate. I have films of such a deep red that they are almost opaque, a gas-flame being barely visible through them. The light which does get through is regularly transmitted, that is, the films are not turbid media. The spectroscope shows that the absorption-band is wider and less sharply defined than is the case with some of the purple potassium films, which have a rather narrow and very black band in the yellowish green. This can be explained by assuming that there is not a great regularity in the size of the particles, and consequently less sharp selective resonance. Inasmuch as some of the sodium films had a bright apple-green colour, in other words absorbed both the red and blue, it seemed worth while to see whether the silver films exhibited the same phenomenon. Red and violet films were formed on a quartz plate, and the absorption-spectra photographed with

a quartz spectrograph. It was found that they were transparent to the whole ultra-violet region, even the last two faint cadmium lines which almost all media cut off, appearing on the plate. If a second band is present it must be located in the infra-red region.

Before continuing the experiments with these highly coloured silver films, it seemed best to establish beyond any doubt that they were analogous in every respect to the films obtained with the alkali metals. One respect in which they differ is the absence of much effect when they are moistened with ligroin. It will be remembered that the films formed by distillation in exhausted bulbs showed most remarkable colour-changes when the hydrocarbon vapour was caused to condense on the wall by the local application of cold. This difference is not of serious moment; for it is quite possible that the silver particles are already immersed in some medium (possibly gelatine) which prevents the ligroin from entering. As I have mentioned in my previous paper, a very slight change of colour is produced by the ligroin; but only close examination makes it evident. The red films of silver can be changed to purple and blue by the local application of heat. This was also true of the alkali metal films. I mentioned in my first communication that where the sodium deposits were very scant the particles scattered a bright green light, the appearance being somewhat as if the inside of the bulb had been washed over with a solution of fluorescene. This light was of the wave-lengths of the light absorbed where the film was thicker. Only films too thin to show any colour by transmitted light exhibited this phenomenon. For convenience I spoke of this light as the "fluorescent light," without intending to convey the idea that it has any real connexion with fluorescence.

It remained only to be seen whether silver deposits could be obtained which showed this peculiarity. A plate was flowed with the silver solution, and then partly rinsed off with distilled water and dried.

By transmitted light there were traces of colour in patches, though most of the plate appeared colourless. On holding it in a beam of strong sunlight in a dark room there appeared many patches which exhibited a bottle-green diffused light, not quite as bright and pure as that shown by sodium, but in every way analogous to it.

Inasmuch as all of the peculiarities shown by the films of the alkali metals have been observed to a greater or less degree with the silver films, it seems safe to conclude that the cause of the colour is the same in the two cases.

An attempt was now made to determine whether the films show anomalous dispersion as we should expect them to do if the colour is due to resonance.

A sheet of thin plate-glass was prepared with gelatine and flowed with the silver solution. A number of glass tubes were then laid side by side on the plate, the fluid being drawn up by capillarity into prisms. The plate was then warmed until the prismatic strips of fluid had dried, when the glass tubes were removed. On examining the plate it was found that some very perfect prisms had formed, which resembled closely the cyanine prisms which I have described in a previous paper. The portion of the plate which seemed to carry the most perfect prism was cut from the glass sheet and mounted on the table of the spectrometer, a small portion of the glass having been previously cleaned off to enable a reading to be taken of the undeviated image of the slit. The prism was screened with black paper to cut off the light which did not pass through the most perfect portion, and the slit of the instrument illuminated with light from a monochromatic illuminator. By alternately exposing the clear glass and the prism, readings of the deviated and undeviated images could be taken. It was found that when blue-violet light was employed the prism did not deviate the image by any measurable amount, while with red light the deviation amounted to several times the width of the image of the slit. The latter was considerably broadened by diffraction, as is always the case with strongly absorbing prisms.

The prism was of a deep red colour, but transmitted violet freely, the absorption-band being in the green. It was found that the deviation was less for the extreme red of the spectrum than for the orange-red, which is just what we should expect. The absence of any deviation in the case of the violet light means of course that the refractive index does not differ much from unity for these waves. To determine the index for the red it is necessary to know the angle of the prism. This was determined by observing the images of the slit reflected from the strip of clear glass and the surface of the prism, the required angle being one-half of the observed angle. This angle was found to be 1′ 15″, while the deviation for red light was 2′ 42″. The refractive index calculated from these observations turns out to be 3·15, or about that of selenium for yellow light.

This is practically proof positive that the cause of colour is resonance, though the question is still open as to whether it is resonance of small metal particles in a manner analogous to the action of the tinfoil resonators on electro-magnetic

waves, or a resonance within the molecule, as in the case of the aniline dyes. Though I have not yet devised any crucial experiment which enables me to distinguish between the two, I have observed differences in the behaviour of these metal films from that of films formed of dye-stuffs, which makes it seem extremely probable that the action is different in the two cases.

To these arguments may be added the one that it seems unlikely that so many different metals should exhibit such similar molecular absorption, and that the granular condition should be essential to the production of colour, for I have succeeded in detecting the granules in every case by means of the microscope.

Let us now consider some of the differences between the action on light of these coloured films and films formed of aniline dyes. A film of granular silver can be formed which resembles very closely a film of cyanine, both having absorption-bands in the middle of the visible spectrum. If a very thin film of silver is prepared which exhibits a light purple colour by transmitted light, the colour is much less marked, *i. e.* the absorption is less, when the plate is held at an angle of say 30 degrees with the incident light than it is at normal incidence. Moreover, the colour changes from purple to red, the absorption-band appearing to move down the spectrum a trifle. Possibly a simple broadening of the band would produce this same colour change owing to the greater sensitiveness of the eye for red. A cyanine film does not exhibit this peculiarity, the colour of the transmitted light remaining nearly constant for varying angles of incidence.

The change in depth and hue of the colour when the incidence is made oblique is probably due to the fact that the absorption is different when the incident light has a component of the electric vector perpendicular to the surface, which, of course, is not the case when the incidence is normal. If the light is polarized to start with, this is found to be the case, the colour of the transmitted light being bluish-purple when the electric vector is perpendicular to the plane of incidence, and red when it is parallel to this plane in which case there is a component normal to the surface. These changes are not obtained with the film of cyanine. In the course of a recent discussion of the problem which I have had with Professor J. J. Thomson, he expressed the opinion that the capacity of a sheet of resonators would be greater, and the period of vibration consequently longer, when the electric vector was parallel to the sheet. This is in agreement

with the observations which I have made with the silver films, the absorption-band moving towards the red when the plane of polarization is so oriented that the electric vector has no component normal to the surface.

With sodium and potassium films I have obtained just the opposite effect, the absorption-band moving in the other direction. These results appeared at first to be very confusing, until I finally discovered what I believe to be the explanation.

I have studied the behaviour of the films at oblique incidence with polarized light with considerable care, and have obtained results which appear to be concordant among themselves.

If the plane of incidence is horizontal and the electric vector is parallel to this plane, *i. e.* with a component perpendicular to the surface, we find for example a heavy absorption-band in the green. As the plane of polarization is rotated through a right angle the band in some cases shifts its position in the spectrum (sometimes in one direction and sometimes in the other) and sometimes simply fades away entirely without undergoing any shift. On examining the spectrum with greater care, however, I found that in general other bands are present. If the band in the green moves towards the red, a band comes into the spectrum from the red end to meet it, and another band which was originally in the violet moves out of the spectrum in the direction of the ultra-violet. It is difficult to say at present whether the appearance and disappearance of these two fainter bands is due to their motion along the spectrum, or to the fact that one fades away and the other springs into existence, as in the case above cited. This difficulty is due to the fact that a part of the band lies outside of the visible spectrum, and a fading away can easily be mistaken for a drift. I am of the opinion that in some cases we are dealing with one phenomenon, and in other cases with the other.

I have examined several films in which a band in the yellow and one in the blue distinctly approached each other when the plane of polarization was rotated through a right angle. This explains why it is that a single band near the centre of the spectrum is sometimes seen to move in one direction and sometimes in the other.

In the one case it was doubtless the right-hand band of a pair which approached (the other lying in the invisible portion of the spectrum), in the other case it was the left-hand band of an approaching pair. Some typical cases of the changes which occur when the plane of polarization is rotated

are shown in the diagram.   The plane of incidence is horizontal in every case, *i. e.* the sheet of resonators stands vertical,
and at an angle of about 45 degrees with the incident light.
The direction of the electric vector is indicated by the double
arrow.   In general the band drifts without much change in
intensity, though in some cases it fades away almost to the
vanishing point when the plane of polarization is at 45 degrees
to the plane of incidence.   A case of this kind is shown in
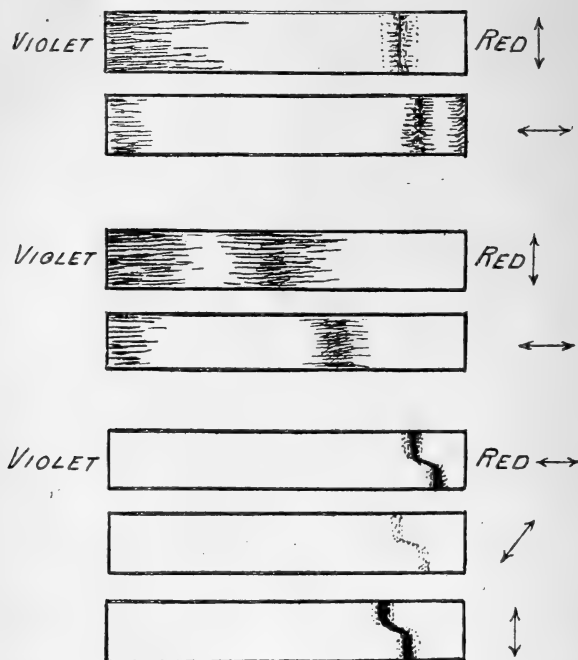
the last three spectra of the figure.   The band in this case
was crooked, as the slit of the spectroscope was placed at a
point where the colour of the film changed quite abruptly
from blue to purple.

The appearance of several bands in the spectrum suggests
that we are dealing with something of the nature of multiple
resonance, and it is to be hoped that some of the mathematical
physicists will work out the behaviour of a sheet of small
spherical resonators at normal and oblique incidence for
vibrations polarized in different planes.

XXXII. *The Ionization produced by Hot Platinum in Air.*

Trinity College, Cambridge.
July 4, 1903.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN his very interesting paper on " The Positive Ionization produced by Hot Platinum in Air at Low Pressures," in the July number of the Philosophical Magazine, Mr. Richardson comes to the conclusion that the positive leak from a hot platinum wire in air at low pressures is not a definite function of the temperature of the wire, but that it gradually dies away with time. He states further that " the phenomenon is exactly analogous to a monomolecular chemical reaction. The quantity which corresponds to the velocity of the reaction is probably a function of the pressure of the gas and of the temperature of the wire.

" In air at atmospheric pressure the falling off of the conductivity with time is much slower than at low pressures. Thus Mr. H. A. Wilson * found the following numbers . . ."

While I entirely agree with most of his conclusions, I do not think that the results which I have described in the paper from which Mr. Richardson quotes justify the conclusion that the rate of decay of the positive leak is less at high pressures than at low. I think also that only a part of the positive leak can be correctly described as analogous to a monomolecular reaction.

I have shown (*loc. cit.*) that on first heating platinum in air at atmospheric pressure a large leak is obtained, the greater part of which dies away in a few minutes. The temperatures at which I worked were higher than Mr. Richardson's, and the rate of decay of the initial temporary leak in my experiments was greater than the rates of decay which Mr. Richardson's experiments at much lower pressures indicate. The numbers which Mr. Richardson quotes as showing a slow rate of decay at atmospheric pressure are values of the more permanent part of the positive leak obtained after the temporary part of the leak had entirely disappeared.

It was shown in my paper (*loc. cit.*) that the comparatively permanent part of the leak which remains slowly diminishes with long continued heating of the platinum. In air, however, it does not fall to a zero value, but a permanent leak ultimately remains which does not decay with time. This

* Phil. Trans. A. vol. cxcvii. p. 415 (1901).

fact I have established more clearly in a paper recently read before the Royal Society. This permanent part of the leak is no doubt due to ionization of the air in contact with the hot platinum, and it is in no way analogous to a mono-molecular chemical reaction involving only a small quantity of material.

Mr. Richardson has also (*loc. cit.*) observed this permanent part of the leak as well as the temporary part which decays comparatively slowly. The positive leak from hot platinum can therefore be described as consisting of three parts :—

(1) A temporary part disappearing in a few minutes, and apparently independent of the air-pressure.

(2) A temporary part disappearing after many hours' heating. This part may be due to the same cause as (3), and its decay may be due to a gradual alteration in the state of molecular aggregation of the platinum by long continued heating.

(3) A permanent part which is zero in a vacuum and increases with the gas-pressure, and is at any pressure a definite function of the temperature. This part is due to ionization of the air in contact with the platinum.

I found (*loc. cit.*) that (1) reappears on leaving the platinum in air ; Mr. Richardson has confirmed this result.

It appears from the results obtained that (1) is due to the presence of some substance in the platinum which escapes at a rate proportional to the leak, so that (1) may be correctly described as analogous to a monomolecular reaction.

Yours very truly,
H. A. Wilson.

XXXIII. *Direct-Vision Spectroscope of one kind of Glass.*
By T. H. Blakesley, *M.A.*[*]

THE fundamental principle underlying the action of this instrument is the following :—Starting with the ordinary formula connecting μ the index, A the refracting angle of a prism, and D the minimum deviation—

$$\mu \cdot \sin \frac{A}{2} = \sin \frac{A + D}{2}$$

if the angle A is so chosen that the deviation shall be equal

to the refracting angle, *i.e.* D = A, then

$$\mu = 2 \cos \frac{A}{2}.$$

The glass being selected, and the ray of light which is to be directly transmitted being chosen, the refracting angle must be constructed in accordance with this simple formula. Since the refracting angle is thus to be equal to the angle of minimum deviation, it is clear that the light from the collimator must be directed upon the prism at right angles to the side from which the light is to emerge, and when the light emerges the ray selected will be at right angles to the side upon which incidence took place. If then the emergent light is received upon an obtuse-angled isosceles prism so placed that its base is parallel to the selected ray, the whole of the light will pass through, suffering reflexion once, and emerging in the following way:—The selected ray will be parallel to its course before incidence on the isosceles prism, and therefore still at right angles to the face on which the first incidence took place.

The rest of the rays will have changed sides as regards the selected ray, and therefore be in a condition to receive an extra dispersion by a prism of the same refracting angle as the original one, but having its point turned so that the selected ray is brought back to parallelism with its original course.

Now the first prism and the isosceles may be and are in fact merged into one right-angled prism, as shown in the accompanying plan, where also the hypothetical condition of the separation of these two is also shown, for explanation merely, in the right-hand top corner. In the plan the course of the ray selected is shown by *a b c d e f g h i k*, and then into the telescope (p. 270).

The collimator and the two first prisms form one rigid system, the telescope and the two last prisms another, and these move relatively round the symmetric point F.

It is clear that the telescope might have been fixed so as to receive the light emerging from *e*. In that case parallelism but not collinearity would have been secured. With the four prisms both conditions are reached.

When a ray which is not the selected one is brought into the centre of the field, it is clear that though between the second and third prisms the ray is not parallel to the collimator, it is still passing symmetrically through the four prisms as a whole, and therefore with minimum deviation.

The second and third prisms in the plan have been drawn of the same shape as the first and fourth, but as one of their three sides is not, in this instrument, employed, they may be much curtailed. I have thought it proper to draw them as shown that the parallelism of the various sides, where it exists, may be strongly brought out.
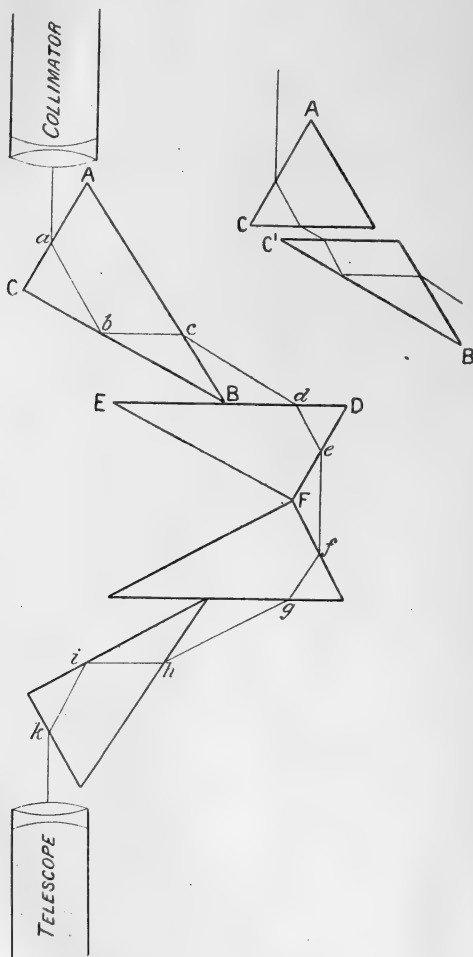
In conclusion it may be stated that in the actual instrument shown the glass has been selected so as to give admirable visibility from A to H, the line F very nearly corresponds with the selected ray, and there is a dispersion of 18° 20′ between A and G.

XXXIV. *Initial Acceleration in Chemical Change.*
By V. H. VELEY, *D.Sc., F.R.S.*[*]

DURING the past forty years the phenomenon of so-called initial acceleration has formed the subject of many observations in cases of very different types : the literature has been collated by van't Hoff[†], as also by myself[‡]. As more attention has been paid of recent years to investigations on the rate and conditions of chemical change and to the mathematical analysis of observations, it appeared of interest again to take up the subject from the stand-point of our more advanced knowledge. Though in most, if not all cases, the phenomenon in question can be referred to secondary actions, yet it does not appear possible to include all such cases in the same category. At the outset it is evident that when a chemical change starts slowly at first, then increases to a maximum and subsequently decreases according to the law of mass action, such a result might be the effect of two wholly different causes, namely, on the one hand of the gradual formation of some accelerating substance up to a constant amount, and on the other of a delay occasioned either by the formation of some intermediate substance or some retention up to a certain amount of one of the products of the reaction. The former would be a case of true acceleration, the latter of true retardation or reluctance.

In the present communication it is desired briefly to discuss examples of these two different categories, and more especially to deal with a consideration of cases of the latter.

First category, Case (1). The formation of an intermediate accelerator from one of the substances initially present.

(1) Formation of nitrous acid when certain metals are placed in dilute nitric acid, as first observed by Millon and more recently by myself[§].

(2) Formation of (probably) some persulphuric acid when

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[*] Communicated by the Author.

[†] Studies in Chemical Dynamics (Eng. trans.) pp. 91–101 (1896).

[‡] Phil. Mag. [5] xxxvii. p. 165 (1894). Ostwald, in alluding to this paper, writes (*Lehrbuch*, Band ii. p. 269) as follows :—" In der That sind sie " (Beispiele) " so häufig dass dadurch ein sonst ganz verdienstlicher Forscher V. H. Veley sich zu der ganz unhaltbaren Annahme verleiten liess, der geschilderte Reaktionsverlauf mit anfänglicher Beschleunigung sei der typische Fall der chemischen Vorgänge. . ."

Though I cannot but appreciate the compliment contained in the former portion of the paragraph, yet I must demur to the latter portion. It is true that in our present state of knowledge the assumption is untenable, but it is equally true that I have never maintained it.

[§] Phil. Trans. A. p. 279 (1891).

zinc is placed in dilute sulphuric acid ; the acceleration was noticed simultaneously by Spring and Aubin [*] in measurements of zinc dissolved per unit area per unit time $\left(\dfrac{d\mathrm{A}}{d\mathrm{T}}\right)$ and by myself in measurements of hydrogen gas evolved per unit volume per unit time $\dfrac{d\mathrm{U}}{d\mathrm{T}}$ [†]. This case is so far analogous to the one cited above in that if the zinc is cast into spheres and rolled about in the acid continually kept in agitation the reaction proceeds uniformly per diminution area per time, namely,

$$\frac{d(4\mathrm{M}r^2)}{d\mathrm{T}} = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or in relation to the concentration of acid C supposed in excess and volume of sphere

$$\frac{ds\mathrm{C}}{4/3\mathrm{M}r^3 + \mathrm{C}} = -l\,d\mathrm{T}. \quad . \quad . \quad . \quad . \quad (2)$$

Case 2. The formation of one of the products of the reaction which acts as an accelerator.

(1) The production of manganese sulphate [‡] in the reaction between potassium permanganate and oxalic acid, which was first investigated by Harcourt and Esson many years ago, but not fully dealt with as being of a too complicated type.

(2) The formation of mono-symmetric sulphur [§] in the conversion of rhombic sulphur into that modification.

(3) The formation of cyamelide in the polymerization of cyanic acid [‖].

In examples of Case (1) Ostwald gives the differential equation

$$\frac{d\mathrm{C}}{d\mathrm{T}} = (k_1 + k_2\mathrm{C})(\mathrm{A} - \mathrm{C})(\mathrm{B} - \mathrm{C}), \quad . \quad . \quad . \quad (3)$$

and for those of Case (2)

$$\frac{d\mathrm{C}}{d\mathrm{T}} = k_1 + k_2(\mathrm{A} - \mathrm{C})^2(\mathrm{B} - \mathrm{C}), \quad . \quad . \quad . \quad (4)$$

but it does not appear that at present any series of observations on the rate of chemical change of this type have been subjected to mathematical analysis.

[*] *Ann. Chim. Phys.* [6] xi. p. 505.
[†] Phil. Trans. A. p. 257 (1888).
[‡] Ibid. p. 201 (1866).
[§] Reicher, *Inaugural Dissertation*, Amsterdam, 1888.
[‖] Van't Hoff, *loc. cit. supra.*

Second Category. Case (1).—The formation of some third substance which occasions a delay.

(1) The formation of an unstable bromo-addition product in the bromination of organic acids, put forward by Hell and Urech as the most reasonable hypothesis, but not established by the isolation of such a compound in their investigations, however probable on *à priori* grounds.

(2) It was formerly supposed that the so-called induction period in the combination of hydrogen and chlorine, first observed by Bunsen and Roscoe, was due to the intermediate formation of chlorine monoxide (Pringsheim) or hypochlorous acid (Becquerel, Gautier and Helier, and myself), but since from more recent investigations * it appears that the presence of neither of these substances purposely introduced produces any measurable effect, another explanation has been proposed in accordance with the equation

$$Cl . Cl + HOH + H . H = H . Cl + HCl + HOH,$$
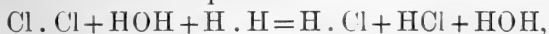
though the possibility of the formation of an intermediate compound $xCl_2 \, yH_2O . zH_2$, wherein x, y, and z are positive integers, is not excluded.

Other examples might be cited under this last head, but it is only intended to give a selection, while certain other examples might be discussed, such, for example, as the etherification of organic acids as observed by Berthelot and Péan de St. Gilles, which has as yet been explained only in a general and superficial manner.

Case (2). The solution or retention of one of the products of the reaction in the solvent. This affords a problem of chemical equilibrium of the second order which, though the subject of numerous observations, has not been the subject of mathematical analysis like more direct problems. It is therefore proposed to discuss this more fully, and though the observations were made for the most part several years ago, yet the more recent development of this branch of chemistry has induced me to recalculate my former results, and herein I have to express my indebtedness to Prof. Wm. Esson for kind assistance in the matter. More than forty years ago Harcourt observed that the evolution of nitrogen gas from aqueous solutions of ammonium nitrite starts at first slowly, then proceeds to a maximum from which it decreases in accordance with the law of mass action expressed by the equation

$$\log A/(A - x) = aT. \quad . \quad . \quad . \quad . \quad (5)$$

This phenomenon could be reproduced by a temporary

* Mellor, Journ. Chem. Soc. Trans. pp. 1292–1301 (1902).

cooling of the liquid. Fourteen years ago the same phenomenon was observed in other cases in which gases were evolved, as a product of chemical change, from homogeneous liquids, and, further, that the phenomenon could also be produced temporarily by suddenly increasing the superincumbent pressure and its converse by reducing the pressure ; also that the addition of finely-divided particles causes an increase in the rate of evolution of gas.

These phenomena resemble in general character those observed by Oersted, Schönbein, Gernez, Schröder, Tomlinson, &c., in the case of supersaturated solutions of gases, especially carbon dioxide, whether in artificial preparations such as so-called soda-water, or in natural waters such as those of Spa and Langen-Schwalbach.

In all cases the method of observation was that known as the twin U-tube method, which has previously been described ; in those in which the evolved gas is practically insoluble in water the U-tubes were inverted over it, but in others in which the gas is more soluble, then either a liquid other than water was used, or the gas was made to displace its own volume of water which had previously been saturated with it. This latter method of observation is not, however, so convenient, as the observations cannot be made quite continuously. The advantage of this method is that constant units of chemical change can be measured.

Example I. The evolution of carbonic oxide from formic acid by sulphuric acid. It was found that after the period of maximum constancy the carbonic oxide, taken as a measure of the decomposition, is evolved in accordance with the bimolecular law

$$dC/dT = -kC^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

but previously to that period the rate of evolution was not in accordance with that law. If then x is the amount of carbonic oxide formed after an interval T from the commencement, and C the total amount which can be formed, then

$$C - x = (C^{-1} + kT)^{-1}, \quad . \quad . \quad . \quad . \quad (7)$$

further, if a portion of the carbonic oxide is detained in the liquid (while a supersaturated solution is being formed) in continually diminishing quantity z, and y is the amount evolved after an interval T, then, during an unit interval, dz is detained while dy is evolved, and dz is a continually diminishing fraction of dy. The hypothesis which Esson suggests is expressed by the relation

$$dz/dT = nb \exp. (-ny) \, dy/dT, \quad . \quad . \quad . \quad (8)$$

which gives $\quad z = b\{1 - \exp. (-ny)\} = b(1 - m^{-y}), \quad . \quad . \quad (9)$

z being 0 when $y = 0$ and approaching a limit b as y increases.

The following table gives the results of a series of observations in which T is the time from start, C the original concentration $= 165\cdot5$, $k = 10^{-5}$, b taken as $4\cdot95$, and $m = 2$, or substituting in above equation

$$x = 165\cdot5 - \{(165\cdot5)^{-1} + 10^{-5}t\}^{-1}, \quad . \quad . \quad . \quad (10)$$

$$z = 4\cdot95\,(1 - 2^{-y}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Temperature $= 80\cdot5$.

(In these, and in subsequent cases, it will be understood that $\Delta C = $ constant, *i. e.* the mass of decomposing substance which should give 10 c. c. of gas under standard conditions of temperature and pressure unless otherwise stated.)

T found.	x calc.	y found.	$z = x - y.$	z calc.	$4\cdot95 - z.$
0	0	0	0	0	4·95
13·37	3·61	1	2·61	2·48	2·47
21·59	5·73	2	3·73	3·71	1·24
27·89	7·32	3	4·32	4·33	·62
33·27	8·66	4	4·66	4·64	·31
37·95	9·78	5	4·78	4·80	·15
42·45	10·87	6	4·87	4·87	·08
46·88	11·94	7	4·94	4·91	·04
51·16	12·94	8	4·94	4·93	·02
55·25	13·87	9	4·87	4·94	·01

The differences between the values of z as calculated from the differences of x and y and as calculated from equation (11) are, with the exception of the first value, concordant to within one per cent., which is within the limit of the experimental error involved in the found values of T.

The results obtained in other series of observations of a similar type are given below.

Series II. (Temperature 80·5).

$$C = 164, \quad b = 3\cdot22, \quad m = 1\cdot56, \quad k = 10^{-5}.$$

T found.	x calc.	y found.	$z = x - y.$	z calc.	$3\cdot22 - z.$
0	0	0	0	0	3·22
8·43	2·12	1	1·12	1·15	2·07
14·94	3·90	2	1·90	1·89	1·33
20·74	5·37	3	2·37	2·36	·86
25·84	6·64	4	2·64	2·67	·55
30·97	8·00	5	3·00	2·86	·36
35·72	9·06	6	3·06	2·99	·23
40·57	10·22	7	3·22	3·08	·14
44·92	11·23	8	3·23	3·22	·00
49·12	12·14	9	3·14	3·18	·04

Series III. (Temperature 80·8).
$C = 160·5$, $b = 7·23$, $m = 1·35$, $k = 11·86^{-5}$.

T found.	x calc.	y found.	$z = x - y$.	z calc.	$7·30 - z$.
0	0	0	0	0	7·30
10·82	3·44	1	2·44	1·88	5·44
18·15	5·57	2	3·57	3·29	4·01
24·35	7·40	3	4·40	4·32	2·98
29·80	8·98	4	4·98	5·10	2·20
35·05	10·46	5	5 46	5·66	1·64
40·23	11·88	6	5·88	6·09	1·21
45·49	13·31	7	6·31	6·40	·90
50·60	14·68	8	6·68	6·62	·68
55·60	16·20	9	7·20	7·25	·05
60·60	17·23	10	7·23	7·26	·04

Series IV. (Temperature 80·8).
$C = 162$, $b = 7·64$, $m = 1·74$, $k = 10·27^{-5}$.

T found.	x calc.	y found.	$z = x - y$.	z calc.	$764 - z$.
0	0	0	0	0	0
19·2	5·28	1	4·28	3·42	4·42
28·85	7·42	2	5·42	5·09	2·55
35·83	9·12	3	6·12	6·16	1·48
41·93	10·60	4	6·60	6·78	·86
47·70	11·90	5	6·90	7·15	·49
53·10	13·22	6	7·22	7·36	·28
58·08	14·32	7	7·32	7·53	·11
63·88	15·54	8	7·54	7·55	·09
68·40	16·64	9	7·64	7·58	·06
72·91	17·61	10	7·61	7·60	·04

In all these sets of results the differences between the values in the fourth and fifth columns respectively are within the limits of experimental error.

Unfortunately, though the solubility of carbonic oxide has been determined for a number of carbon compounds by Just [*] and others, no estimations have been made of its solubility in dilute sulphuric acid. But, as explained above, it is probable that a case is presented of supersaturation.

It is probable that the values of the constant z would be dependent not only upon obvious conditions, such as volume of liquid, temperature, and pressure, but also upon less obvious conditions, such as size and shape of vessel. If the hypothesis be correct, the state of affairs may be represented diagramatically as follows, taking Series I. as an example :—

ABC is the curve representing actual observations ($OM = T$,

[*] *Zeits. f. Physikal. Chemie,* vol. xxxvii. p. 361 (1901).

and $MP = C - y$} and ADE is the curve representing the course of decomposition if no gas were absorbed ($MQ = C - x$). Hence $PQ = z$ or gas absorbed in time T. If B is the point of saturation, $BD = b = 4\cdot95$, and after the point B the amount of gas made is the same per unit time as the amount of gas evolved, *i. e.* BC and DE are parallel.

Example II. The decomposition of ammonium nitrate into nitrous oxide and steam. This chemical change is in accordance with the monomolecular law, then if y is the gas evolved at time T and z the amount of absorption

$$C \times dT = \frac{1}{k} d(y - z), \quad . \quad . \quad . \quad . \quad (12)$$

also

$$C \times dT' = \frac{1}{k} dy, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

dT being the observed time of the evolution of dy, and dT′ being the calculated time. Therefore

$$dT/dT' = d(y - z)/dy = 1 + dz/dy. \quad . \quad . \quad (14)$$

In the table given below the times for each observation are given in the first column, the values of $1 + dz/dy$ in the second, the derived value of z in the third, the values of z calculated from equation (9) wherein

$$z = \cdot21\{1 - (1\cdot62)^{-y}\} \quad . \quad . \quad . \quad . \quad (15)$$

in the fourth, and the calculated values for $\cdot21 - z$ in the fifth column respectively.

Series **V.** (Temperature 190·7).

I. $\Delta T.$	II. $1 + \dfrac{dz}{dy}$.	III. z derived.	IV. z calc.	V. $\cdot 21 - z(\text{calc.}).$
4·33	1·09	·09	·08	·13
4·10	1·04	·13	·13	·08
4·08	1·03	·16	·16	·05
4·06	1·02	·18	·18	·03
4·03	1·01	·19	·19	·02
4·00	1·01	·20	·20	·01
3·99	1·01	·21	·20	·01
3·96	1·00	·21	·21	·00

In this series of observations the differences between the derived and calculated values of the factor z are quite insignificant.

The results obtained in two other series of observations are given below, the figures in the several columns having the same significance as above.

Series **VI.** (Temperature 182·5).

$$b = \cdot 89, \quad m = 1 \cdot 27.$$

I.	II.	III.	IV.	V.
5·32	1·18	·18	·19	·70
5·22	1·16	·34	·34	·55
5·03	1·12	·46	·45	·44
4·92	1·10	·56	·56	·33
4·85	1·09	68	·63	·26
4·76	1·06	·71	·69	·20
4·72	1·05	·76	·73	·16
4·76	1·06	·82	·77	·12
4·70	1·04	·86	·88	·01
4·63	1·03	·89	·89	nil.
4·49	1	·89	·89	nil.

Series **VII.** (Temperature 191·5).

$$b = \cdot 24, \quad m = 1 \cdot 62.$$

I.	II.	III.	IV.	V.
5·82	1·08	·08	·08	·16
5·75	1·07	·15	·15	·09
5·66	1·05	·20	·20	·04
5·56	1·03	·23	·21	·03
5·46	1·01	·24	·22	·02
5·40	1·00	·24	·24	nil.

In all these sets of results the differences between the values in the third and fourth columns respectively are also within the limits of experimental error.
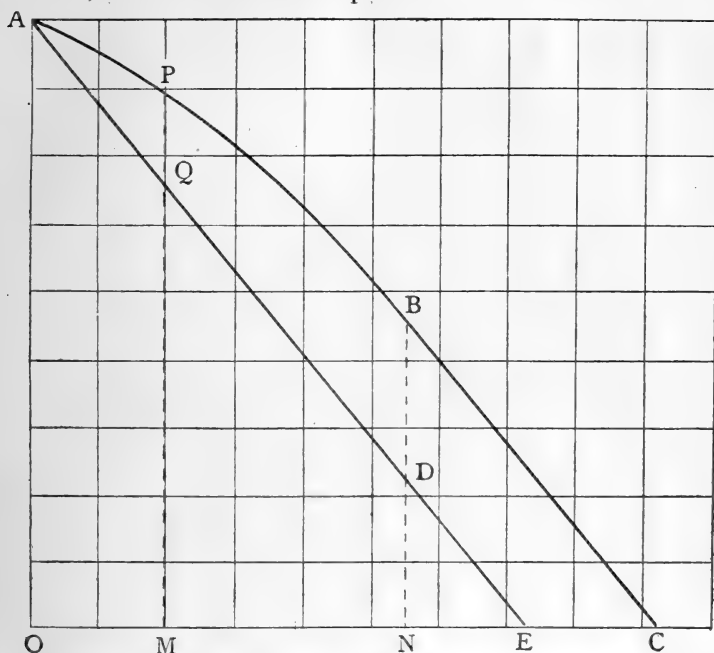
Example III. The decomposition of ammonium nitrite in aqueous solution into nitrogen and water, as being in accordance with monomolecular law, is also analogous to the case cited above, but at present difficulties have been experienced in fixing the conditions for obtaining a sufficient number of observations to apply the above equation. It has only been found possible to obtain at most two or three observations unless the volume of gas collected in each single observation is so small that the experimental error is thereby greatly increased.

Other examples of the above phenomena that have been observed are the decomposition of oxalic acid and of potassium ferrocyanide by concentrated sulphuric acid, but as regards both of these the formation of more than one gas within the solution renders the problem more complicated. The rate of evolution of oxygen from solutions of peroxides would probably present a similar and simple case, and others might also be cited, but in this communication it is only desired to point out a possible line of investigation upon the earlier stages of chemical reaction, which appear to be too often dismissed as inconsequent.

XXXV. *On the Vibrations set up in Molecules by Collisions.* By J. H. JEANS, *M.A., Isaac Newton Student and Fellow of Trinity College, Cambridge* *.

§ 1. A STEEL ball dropped on to a rigid steel plate will rebound perhaps half a dozen times before its energy is appreciably lessened; this is because of the great elasticity of steel. If the kinetic theory of gases is true, a system of molecules must rebound from one another and from rigid walls many billions of times before the total energy is appreciably lessened. The aim of the present paper is to show that, in so far as the data available enable us to judge, molecules will possess sufficient elasticity for this to occur.

§ 2. Let us suppose that a molecule possesses, in addition to its motions of translation and rotation, small vibratory motions of which the oscillations are, in the cases we have to consider, so small as to be isochronous. Then the kinetic and potential

energies of these vibrations will be expressible in the forms

$$2\mathrm{T} = \alpha_1 \dot{\phi}_1^2 + \alpha_2 \dot{\phi}_2^2 + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

$$2\mathrm{V} = \alpha_1 p_1^2 \phi_1^2 + \alpha_2 p_2^2 \phi_2^2 + \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

where $p_1, p_2, \ldots$ are the frequencies of vibration.

We call one molecule A, and suppose that a second molecule B approaches A, has its velocity altered in direction by the forces exerted upon it by A, and finally recedes from A. Let the instant at which the force between these molecules begins to be appreciable be taken to be $t = 0$, and the instant at which they are finally clear of each other's sphere of action be $t = \tau$. At any instant during the encounter let the forces exerted by B upon A be derivable from a force-function

$$\mathrm{V}_1 \delta\phi_1 + \mathrm{V}_2 \delta\phi_2 + \ldots \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (3)$$

Then the equations of motion are of the form (dropping suffixes)

$$\alpha\ddot{\phi} + \alpha p^2 \phi = \mathrm{V}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (4)$$

§ 3. Before $t = 0$ the molecule would be describing a free vibration, say

$$\phi = \mathrm{C}\cos pt + \mathrm{D}\sin pt. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (5)$$

The impulse $\mathrm{V}dt$ acting from $t = 0$ to $t = dt$ sets up an additional free vibration of initial displacement zero and velocity $\mathrm{V}dt/\alpha$; the displacement of this additional vibration at any subsequent time is therefore $\mathrm{V}dt \sin pt/\alpha p$. By compounding all these vibrations with the original vibration (5), we obtain for the displacement at any instant subsequent to $t = \tau$, the well-known solution

$$\phi = \mathrm{C}\cos pt + \mathrm{D}\sin pt + \frac{1}{\alpha p}\int_{t'=0}^{t'=t} \mathrm{V}_{t=t'} \sin p(t-t')dt'. \quad (6)$$

This may be written

$$\phi = (\mathrm{C}-\mathrm{Y})\cos pt + (\mathrm{D}+\mathrm{X})\sin pt, \quad \cdot \quad \cdot \quad (7)$$

where X and Y are obtained by equating real and imaginary parts in

$$\mathrm{X} + i\mathrm{Y} = \frac{1}{\alpha p}\int_0^\tau \mathrm{V}e^{ipt}dt. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (8)$$

§ 4. The square of the amplitude of vibration (5) is $\mathrm{C}^2 + \mathrm{D}^2$; that of (7) is $(\mathrm{C}-\mathrm{X})^2 + (\mathrm{D}+\mathrm{Y})^2$, so that the increase caused by collision is

$$\mathrm{X}^2 + \mathrm{Y}^2 - 2(\mathrm{C}\mathrm{X} - \mathrm{D}\mathrm{Y}). \quad \cdot \quad \cdot \quad \cdot \quad (9)$$

Now our final problem will be to find this quantity averaged over all the molecules and collisions in the gas. It is at once obvious that the last term vanishes on the average. Hence a correct result is obtained by ignoring the free vibration (5), and supposing that the collision simply sets up a vibration of amplitude $\sqrt{X^2 + Y^2}$, and therefore of energy $\frac{1}{2}\alpha p^2(X^2 + Y^2)$.

§ 5. In evaluating X and Y from equation (8), we shall suppose V divided into two parts, and shall consider at present only the part which is contributed by the motion of the centre of gravity of molecule B. A second part in V contributed by the rotation and small vibrations of the molecule B will be discussed later (§ 10).

If we integrate equation (8) by parts we get, since V vanishes at both limits,

$$X + iY = -\frac{1}{\alpha i p^2}\int_0^\tau \frac{dV}{dt} e^{ipt}dt. \quad . \quad . \quad . \quad (10)$$

Not only V but all its differential coefficients must vanish at both limits, so that we may repeat the integration by parts indefinitely; after n integrations we obtain

$$X + iY = \frac{(-1)^n}{\alpha p(ip)^n}\int_0^\tau \frac{d^nV}{dt^n} e^{ipt}dt. \quad . \quad . \quad (11)$$

The value of d^nV/dt^n will be comparable with unity provided that the unit of time selected is comparable with the scale of time-variation of V, and that n is not very great. Hence, if p is great when measured in these units, we see, from the presence of the factor $p^{-(n+1)}$ in (11), that X and Y will be very small.

Now for normal air the probable relative velocity of two molecules is of the order of 10^5 cm. per sec. From experiments on the viscosity &c. of gases, it is found that the distance apart of the centres of two molecules during an encounter—the distance which is usually described as the " diameter " of a molecule—is about 10^{-8} cm. The appropriate unit of time for that part of V which we are now discussing is therefore 10^{-13} sec. For p the value varies, so far as we know, from about 2×10^{15} per second in the case of ultra-red light to 8×10^{15} per second in the case of ultra-violet light. This gives for the value of p in the present units a range from 200 to 800. Since there is practically no limit to the value of n in (11), the factor $p^{-(n+1)}$ may be made very small. From this we conclude that practically no vibrations are set up in molecule A by the translational motion of

B, so long as the relative velocity is comparable with the average in normal air.

§ 6. A more convincing proof of the smallness of the integral in (8) is supplied by Cauchy's theorem. Imagine the function V evaluated for all values of t real and complex, then we have

$$\int V e^{ipt} dt = 2i\pi \Sigma R, \quad . \quad . \quad . \quad . \quad (12)$$

where the integral on the left-hand is taken from $t = -\infty$ to $t = +\infty$ along the real axis of t, and then back from $t = +\infty$ to $t = -\infty$ along a semicircle of infinite radius, having $t = 0$ for centre. On the right-hand ΣR is the sum of the residues of the function $V e^{ipt}$ inside this semicircle.

The first part of the integral is $X + iY$. The second may be written

$$\mathbf{L}_{R=\infty}^{t} \int_0^\pi V e^{ipR(\cos\theta + i\sin\theta)} R e^{i\theta} i\, d\theta,$$

and the integrand, in general, vanishes through the occurrence of the factor $e^{-pR\sin\theta}$, except near $\theta = 0$, $\theta = \pi$, and here the contributions to the integral vanish on account of the factor $V\, d\theta$. Lastly, if the function V has an infinity occurring at $t = a + i\beta$, with residue u, ΣR may be written $\Sigma e^{ip(a+i\beta)}u$, the summation extending over all infinities for which β is positive. Equation (12) accordingly becomes

$$X + iY = 2i\pi \Sigma e^{-p\beta}(\cos p\alpha + i\sin p\alpha)u. \quad . \quad . \quad (13)$$

By hypothesis $p\beta$ is large, and it is also positive. Thus the smallness of $X + iY$ is guaranteed by the occurrence of the factor $e^{-p\beta}$.

§ 7. To take a definite example, suppose

$$V = \frac{1}{a^2 + t^2}$$

$$X + iY = \frac{1}{\alpha p} \int_{-\infty}^{+\infty} \frac{e^{ipt}}{a^2 + t^2} dt = \frac{\pi}{a\alpha p} e^{-ap},$$

and the energy of vibration $\frac{1}{2}\alpha p^2(X^2 + Y^2)$, is

$$\frac{\pi^2}{2a} e^{-2ap}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The appropriate unit of time in this case is of course a. If p in these units has a value 200 we see from (14) that the "elasticity" of the molecules has introduced a factor e^{-400}.

To take a second example, which is of interest as being a case of failure of the general proof of § 6, suppose that

$$V = e^{-\kappa t^2}$$

$$X + iY = \frac{1}{\alpha p} \int_{-\infty}^{+\infty} e^{ipt - \kappa t^2} dt = \frac{1}{\alpha p} \sqrt{\frac{\pi}{\kappa}} \, e^{-\frac{p^2}{4\kappa}}.$$

In this case the "elasticity" is represented in the energy of vibration by a factor $e^{-p^2/2\kappa}$. The appropriate unit of time is $\kappa^{-\frac{1}{2}}$, and if p is, in these units, represented by 200, $p^2/2\kappa = 20,000$, and the energy gained by the vibrations at collision is reduced by the "elasticity" to $e^{-20,000}$ of what it would otherwise have been.

§ 8. These two instances have been selected at random, but it is obvious enough, from the general theorems of §§ 5 and 6, that any other form for V would give a very similar result. The smaller of the two factors introduced by the "elasticity" of the molecule has been e^{-400}. This means that if the molecules were all moving with average velocity the number of collisions required to dissipate a given fraction of the energy would be increased by the "elasticity" in a ratio of about $e^{400} : 1$. In other words, the "elasticity" could easily make the difference between dissipation of energy in a fraction of a second and dissipation in billions of years.

§ 9. It has, however, been seen (§ 5) that appreciable vibrations will be set up by a collision in which the relative velocity is comparable with 2×10^7 cms. per second. We must therefore examine the frequency of such collisions in gases at normal temperatures.

The number of collisions in which the molecules have velocities between c and $c + dc$, c' and $c' + dc'$ inclined at an angle between ϕ and $\phi + d\phi$ is proportional to *

$$e^{-hm(c^2 + c'^2)} \, c^2 c'^2 g \sin \phi \, d\phi \, dc \, dc', \quad . \quad . \quad (15)$$

where g is the relative velocity, given by

$$g^2 = c^2 + c'^2 - 2cc' \cos \phi. \quad . \quad . \quad . \quad (16)$$

Let us suppose that of the velocities c, c' the latter is the greater, and write

$$u = c' - c, \qquad v = c' + c.$$

From equation (16) it follows that g must lie between u and v. By differentiation, keeping u, v and therefore also c, c' constant, we have $g \, dg = cc' \sin \phi \, d\phi$. Expression (15) is now proportional to

$$e^{-\frac{1}{2}hm(u^2 + v^2)} \, (v^2 - u^2) g^2 \, dg \, du \, dv.$$

* Boltzmann, *Gastheorie*, i. p. 64, equation 61.

To obtain the number of collisions having values of g greater than G, this has to be integrated through all possible positive values of g, u, v, subject to

$$u < g < v, \qquad g > G.$$

Integration with respect to g gives

$$\tfrac{1}{3} e^{-\frac{1}{2}hm(u^2+v^2)} (v^2 - u^2)[g^3] \, du \, dv, \quad . \quad . \quad . \quad (17)$$

in which

$$[g^3] = 0 \text{ when } v < G,$$
$$[g^3] = v^3 - G^3 \text{ when } v > G, \, u < G,$$
$$[g^3] = v^3 - u^3 \text{ when } u > G.$$

On integrating (17) with respect to u, v we shall obtain the number required. When G is very great the only integrands which are of importance are those for which $u^2 + v^2$ is in the neighbourhood of its least possible value G^2, *i. e.* those in the neighbourhood of $u=0$, $v=G$. In (17) we may therefore replace $(v^2 - u^2)$ by G^2, and $[g^3]$ by $v^3 - G^3$, and therefore by $3G^2(v-G)$. We may also suppose the integration to extend from $v=G$ to $v=\infty$, and from $u=0$ to $u=\infty$. The integral is therefore

$$G^4 \int_0^\infty e^{-\frac{1}{2}hmu^2} \, du \int_G^\infty e^{-\frac{1}{2}hmv^2} (v-G) dv,$$

of which the value when G is large is found to be

$$\sqrt{\frac{\pi}{2h^5 m^5}} \, G^2 e^{-\frac{1}{2}hmG^2},$$

in which the last factor is obviously of preponderating importance.

For air under normal conditions we have

$$hm = 3/2\bar{c}^2 = 6 \times 10^{-10}$$

approximately. The value of G must, as we have seen, be comparable with 2×10^7. If we actually take this as the lowest value for which the vibrations are appreciable, we get $hmG^2 = 240,000$ and

$$e^{-\frac{1}{2}hmG^2} = e^{-120,000}.$$

If we take half the foregoing value for G,

$$e^{-\frac{1}{2}hmG^2} = e^{-30,000},$$

so that the dissipation of energy takes place $e^{90,000}$ times more slowly in the former case than in the latter. If G were comparable with the mean velocity in the gas, the energy of the gas would probably be reduced to half its value in a

fraction of a second, but the figures we have obtained show that under the actual conditions it is quite possible for the energy to remain appreciably constant throughout unthinkable ages.

§ 10. We now discuss the vibrations set up in A by the rotation and small vibrations of molecule B. If the molecule is rotating with angular velocity ω, and vibrating with frequency q, there will be terms in the force exerted by this molecule of frequencies *

$$\omega, \; q, \; q \pm \omega, \; q \pm 2\omega. \; \ldots \quad \ldots \quad \ldots \quad (18)$$

Any one of these frequencies, say p', will give rise to a term in V (equation 8) of the form

$$V_0 \cos p't + W_0 \sin p't,$$

where V_0, W_0 vary with the time as slowly as did the former V. The corresponding part of $X + iY$ may, from equation (8), be written in the form

$$\frac{1}{2\alpha p} \int_0^\tau \{ V_0(e^{i\,p+p'\,t} + e^{i\,p-p'\,t}) - iW_0(e^{i\,p+p\cdot t} - e^{i(p-p)t}) \}\,dt.$$

Each term can be treated in the same manner as before, and can be shown to be small, except when p is very nearly equal to p'; *i.e.* when p is very nearly equal to one of the values given in (18).

Now ω varies from molecule to molecule, the most probable value being of the order of 10^{13}†. It therefore appears that $p - \omega$, $p + q - \omega$, &c. will be very small only for a very small fraction of the molecules, and obviously even when this is so, the vibration set up will be very small. Thus perceptible vibrations in A will in general only be excited by vibrations of the same period in B, *i. e.* when $p = q$. Physically speaking the mechanism of the transfer of energy is found in the "absorption" of light-waves. It seems unnecessary to bring forward calculations to show that this is small.

§ 11. If E is the mean-energy of translation of the molecules of a gas, F that of a single vibration, the equation expressing the rate of change in F is

$$\frac{dF}{dt} = -\epsilon F + \phi(E)\,F + f(E), \quad \ldots \quad (19)$$

where $-\epsilon F$ represents dissipation into the æther, $\phi(E)F$ represents the amount regained by "absorption," and $f(E)$ is the gain to F through collisions, which we have been

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* "The Mechanism of Radiation." Phil. Mag. [6] ii. p. 242.
† L. c. p. 424.

estimating numerically. The gas assumes an approximately steady state *, given by $d\mathrm{F}/dt = 0$, or

$$\mathrm{F} = \frac{f(\mathrm{E})}{\epsilon - \phi(\mathrm{E})}. \qquad \cdots \quad (20)$$

Here $\epsilon - \phi(\mathrm{E})$ must be positive for physical reasons, and $f(\mathrm{E})$ has been seen to be very small. The whole question at this point fits on to the theory which I have attempted to develop in earlier papers, especially in that just referred to.

Collision of a Molecule with a Free Ion.

§ 12. If the body B is not a molecule but a free ion, two changes must be made.

Firstly, since the ion is of less mass than a molecule, its probable velocity will be greater. There are about 700 ions in the hydrogen atom, and therefore about 1400 in the molecule, so that the most probable value for the velocity of an ion at normal temperatures will be about $\sqrt{1400}$ times that for the hydrogen molecule, and therefore about 7×10^6 cms. per sec. This kinetic theory velocity is of course quite distinct from the " velocity of diffusion " under an electric force which is capable of direct measurement : it is probably more analogous to the velocity of the Becquerel rays.

Secondly, as the ion moves with greater velocity than the molecule, it will approach to within a shorter distance of the molecule A. Instead of 10^{-8} cm., the value found from kinetic theory experiments for the closest approach of two molecules, we must use 10^{-9} cm., the value found by Townsend, Rutherford and others from ionization experiments for the closest approach of ions and molecules.

The ratio of these two quantities—7×10^6 cms. per sec. and 10^{-9} cm.—gives $(7 \times 10^{15})^{-1}$ second as the appropriate unit of time when discussing collisions between ions and molecules. In these units the frequency of vibration of the molecules is comparable with unity, so that appreciable vibrations will be set up by collisions with ions, and these vibrations may occur in all parts of the spectrum. The actual number 7×10^{15} might suggest that violet light would predominate, but the calculation is too rough for any stress to be laid on this result.

§ 13. To sum up briefly, we may say that the " elasticity " of molecules has been found to be amply sufficient to resist the slow attacks of other molecules, but not sufficient to resist the faster attacks of free ions.

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* " The Distribution of Molecular Energy," Phil. Trans. cxcvi. p. 397.

XXXVI. *Notices respecting New Books.*

Vorlesungen über Experimentalphysik. Von AUGUST KUNDT.
Herausgegeben von KARL SCHEEL. Braunschweig : Friedrich
Vieweg und Sohn. 1903. Pp. xxiv + 852.

THIS volume, which is intended as a memorial to the late Pro-
fessor Kundt, contains the courses of lectures on experimental
physics delivered by him during the Session 1888–89. The book
consists of the lectures as they were then delivered, and no attempt
has been made to modernize them by references to recent scientific
discoveries. The work forms an excellent introduction to physics.
The language is clear and simple, the treatment interesting and
original in its freshness. The text is beautifully illustrated by
534 excellent diagrams and pictures of apparatus. Few text-books
would probably prove more attractive to either the general reader
or the elementary student. A fine portrait of the author forms a
suitable frontispiece to this attractive volume.

*A Treatise on the Theory of Solution, including the Phenomena of
Electrolysis.* By W. C. D. WHETHAM, *M.A., F.R.S.* Cambridge :
at the University Press, 1902. Pp. x + 488.

THIS important work is a most noteworthy contribution to the
literature of physical chemistry, and is bound to rank as a classical
treatise on the subject. Both on account of its thorough and
exhaustive treatment of the subject, and its remarkably clear and
cautious exposition of it, the book is one of unsurpassed excellence.

Mr. Whetham is well known to those interested in physical
chemistry both as an original investigator and as the author of an
excellent little treatise on *Solution and Electrolysis* published about
eight years ago. The work under review is an expansion of this
treatise, and furnishes a good proof of the rapid progress made
during the last few years. It is practically a new work on the
subject.

The book consists of fourteen chapters. Chapter I. contains an
account of thermodynamics, and deserves special commendation
for its crisp clearness : no better account of this difficult subject
has, in our judgement, ever appeared. The next two chapters
deal with the Phase Rule. The problems connected with solubility,
osmotic pressure, vapour-pressures, and freezing-points are considered
in chapters IV.–VI. and in chapter VII. we have a most interesting
critical account (remarkable for its unbiassed judgment) of the
various rival theories of solution. Chapters VIII.–XII. deal with
electrolysis, the conductivity of electrolytes, primary cells, contact
electricity, polarization and the theory of electrolytic dissociation.
The last two chapters are concerned with diffusion in solutions and
solutions of colloids. A most useful and comprehensive table of
the electrochemical properties of aqueous solutions is added at the
end of the book.

We have been unable to notice any errors or slips (with the exception of one on p. 4, where a pressure of one atmosphere is stated to be equal to $1·013$—instead of $1·013 \times 10^6$—dynes per sq. cm.), and the book bears traces of exceptionally careful revision throughout.

The work is sure to meet with a very warm welcome from all students of physical chemistry.

HERMANN VON HELMHOLTZ. *By* LEO KOENIGSBERGER. Erster Band. Braunschweig: Friedrich Vieweg und Sohn. 1892.

THIS first volume of the life of Helmholtz brings us to the period when, as Professor of Physiology in Heidelberg, he was engaged on the second part of his Physiological Optics and writing on Klangfarbe, musical temperament and other subjects.

The earliest chapters give a profoundly interesting account of the extraordinary boy growing up in the metaphysical atmosphere of his home-life but shaking off the trammels of metaphysics and later the bondage of professional routine. Linguist, musician, chemist, physicist, physiologist, and self-taught mathematician, the young army doctor advanced to fame by a succession of memoirs, teaching his contemporaries and the outgoing generation in every domain of Science. A scientific post was found for the Militärarzt so early as 1848 (his 27th year) before his Dienstjahre were completed. The record of the early years preceding this event and immediately succeeding it is, in some respects, the most interesting part of this interesting book. His position as supporter of Liebig in his fight against vitalism, which led to his early paper on the nature of decomposition and fermentation, is not very clearly given. On the other hand the account of the circumstances attending the appearance of his memoir on the Erhaltung der Kraft is excellent. Poggendorff refused it! It was crowded out of the *Annalen* not being exactly experimental in its nature " and you yourself know how great is the bulk of experimental work now-a-days " wrote the apologetic Editor to Magnus in 1847. And when it did appear Helmholtz received from high military quarters the warmest praise " for the practical use to which he had applied his studies." Would such happen to an R.A.M.C. " Student on Probation " even to-day? The full acknowledgement by Helmholtz of the claims of Mayer and Joule, when subsequently he became aware of their writings, is pleasant reading.

In this country the account of Helmholtz's impressions on meeting for the first time with W. Thomson (Lord Kelvin) will be read with special interest. This was in 1855. The meeting led to forty years of friendship.

To realise the sustained interest of the work the reader must apply to the book itself. The Daguerreotype of Helmholtz at the age of 27 is a most precious addition. The other portraits are also good.

J. J.

1.

FC

L   N   S

Red.

Fig. 6.

m'

s'

Fig. 7.

B   O   C

FIG. 2.

FIG. 1.

FIG. 3.

Air bands.

Violet.

Red.

Fuchsin bands.
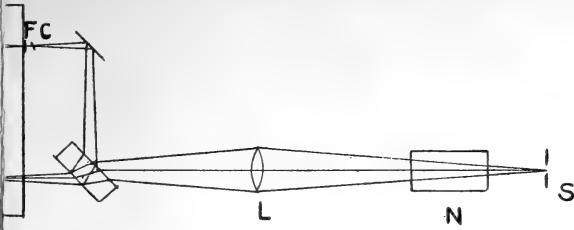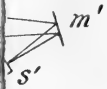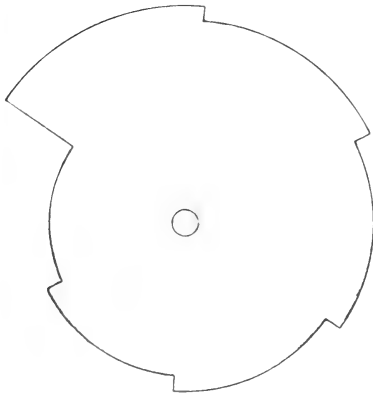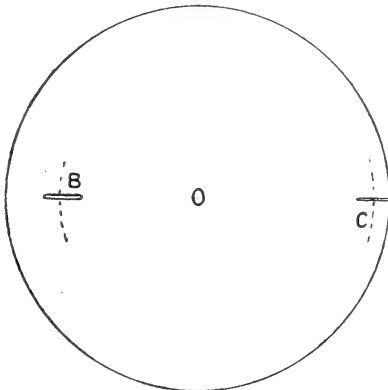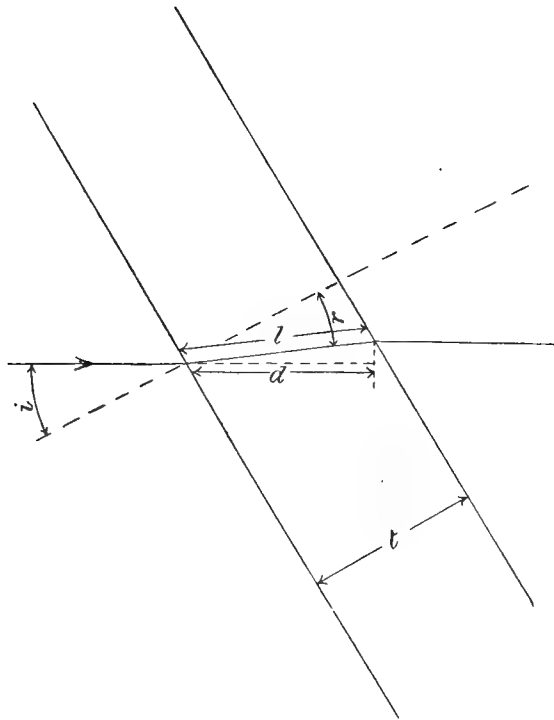
FIG. 6.

FIG. 4.
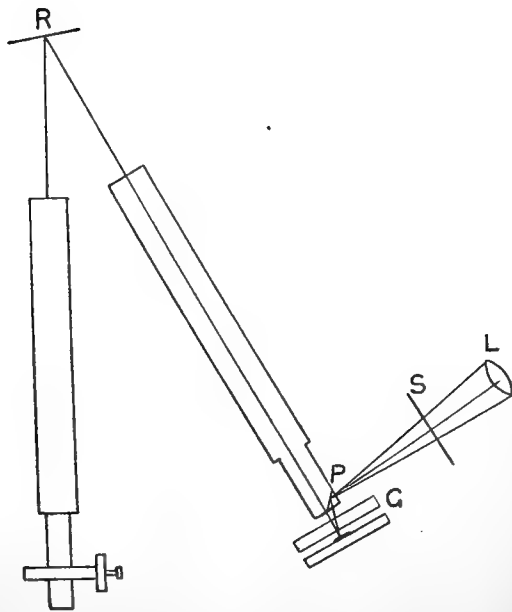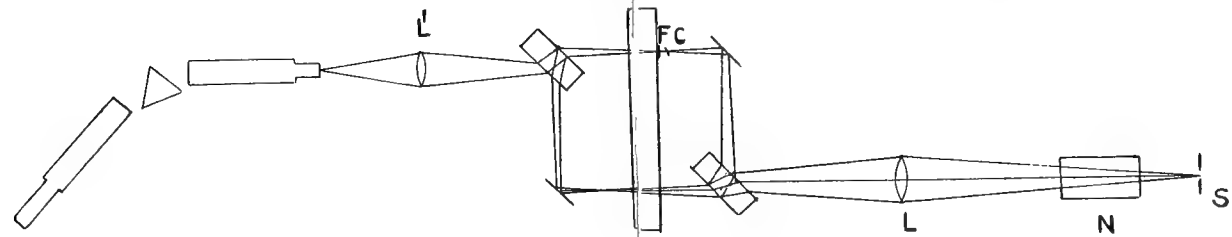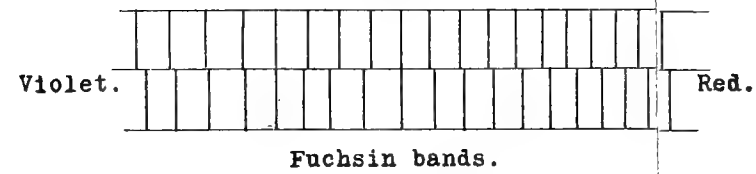
FIG. 5.
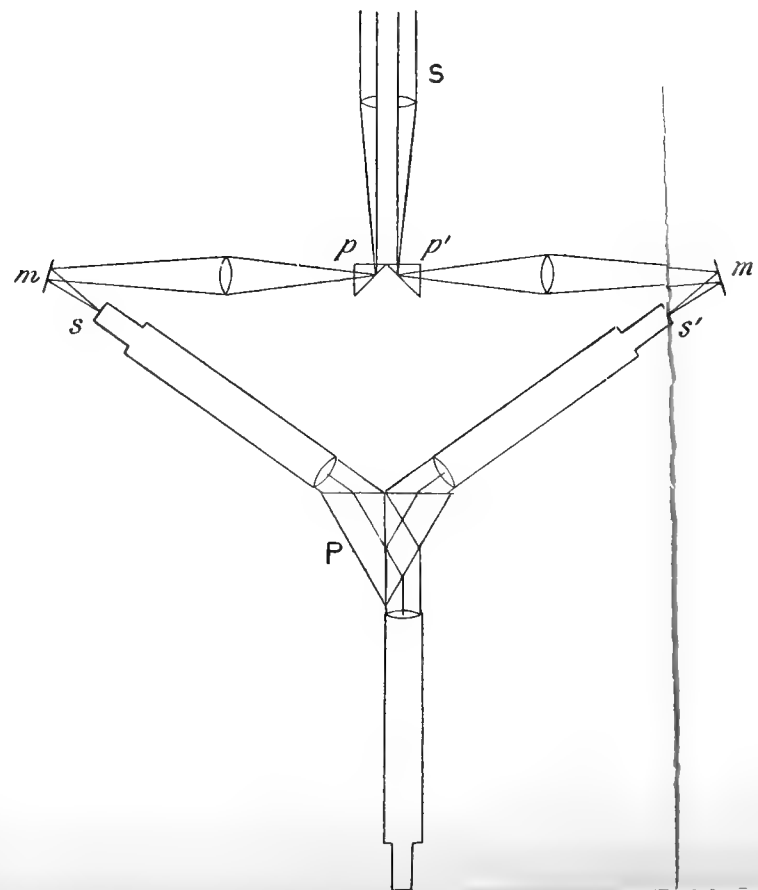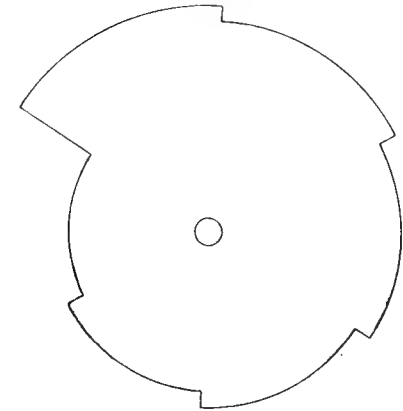
FIG. 7.
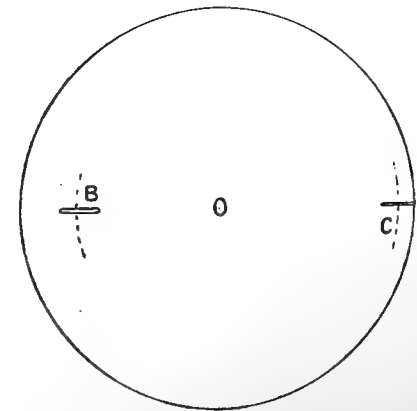
Fig. 1.

gths in Micro-milli

.600

Fig. 2.
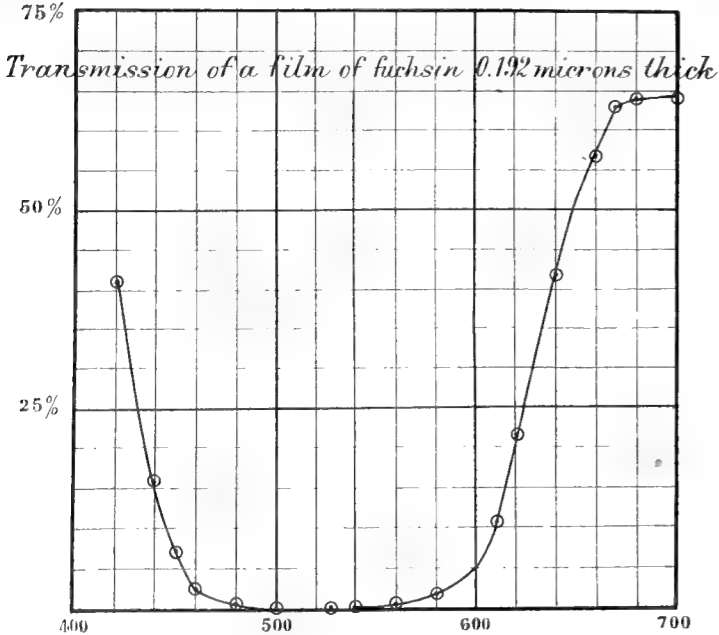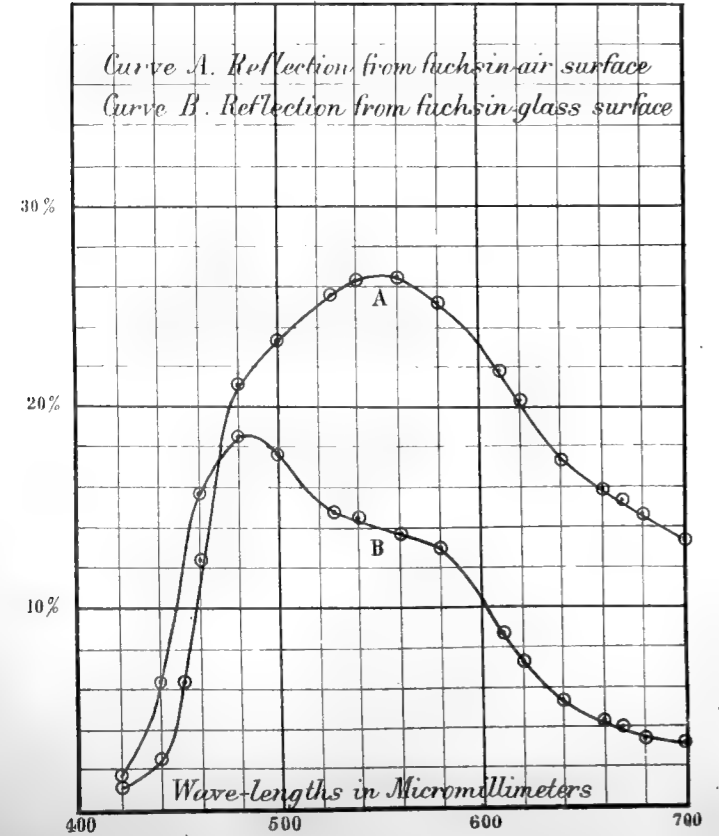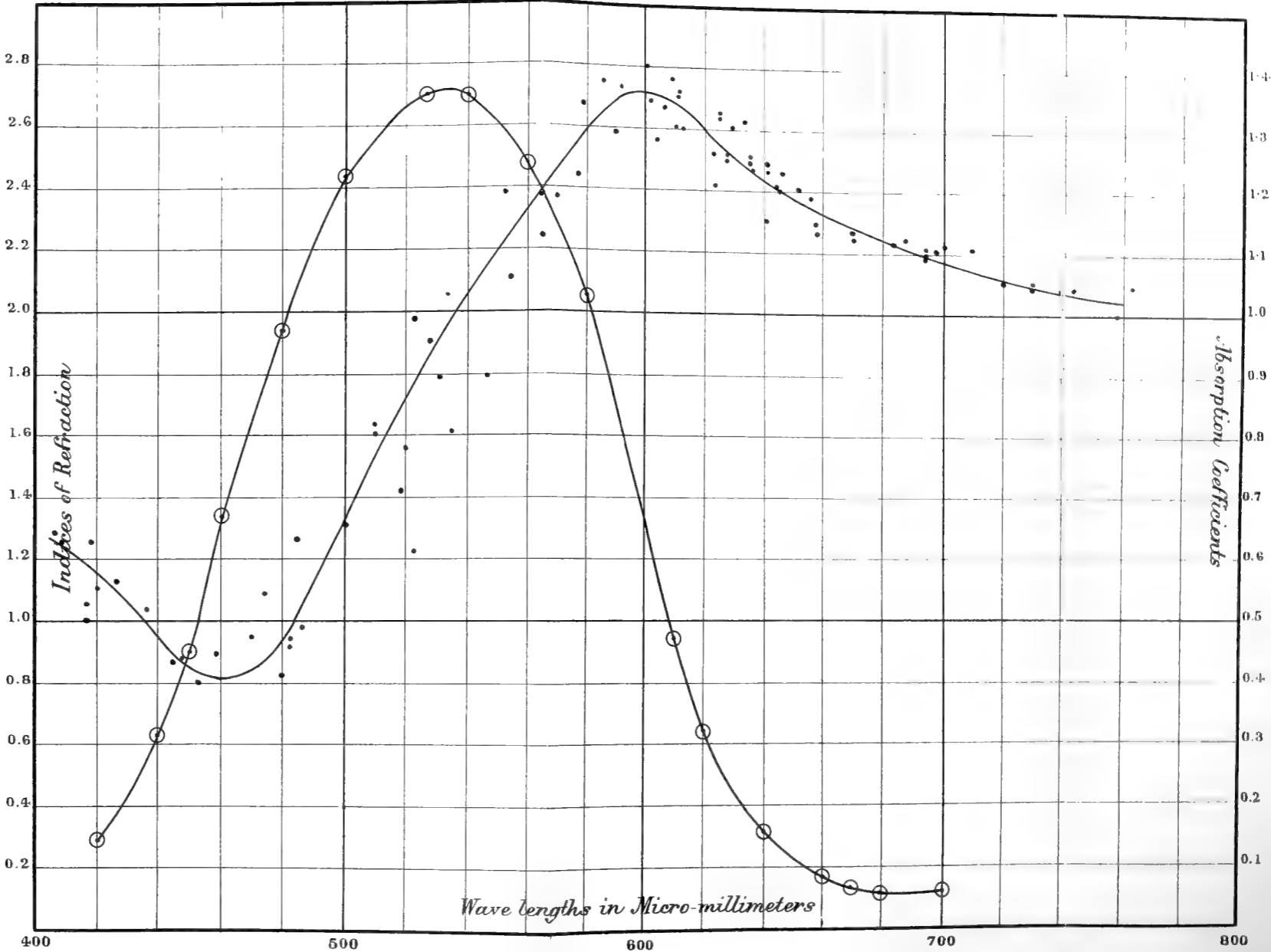
Transmission of a film of fuchsin 0.192 microns thick

Fig. 1.

Fig. 3.

Curve A. Reflection from fuchsin-air surface

Curve B. Reflection from fuchsin-glass surface

Wave-lengths in Micromillimeters

Indices of Refraction

Absorption Coefficients

Wave lengths in Micro-millimeters

THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

◆

[SIXTH SERIES.]

SEPTEMBER 1903.

XXXVII. *On the Production and Distribution of Sound. By* Lord Rayleigh, *O.M., F.R.S., Scientific Adviser to the Trinity House*[*].

Theory of Conical Trumpet.

THE theory of small periodic vibrations having their origin at a single point of a gas and thence spreading symmetrically has long been known. The following statement is from 'Theory of Sound'[†], § 280. In it a denotes the velocity of sound, and $k = 2\pi/\lambda$, λ being the wavelength.

"If the velocity-potential be

$$\phi = -\frac{A}{4\pi r} \cos k(at - r), \quad \ldots \quad (1)$$

we have for the total current crossing a sphere of radius r,

$$4\pi r^2 \frac{d\phi}{dr} = A \{\cos k(at - r) - kr \sin k (at - r)\}$$
$$= A \cos kat,$$

when r is small enough. If the maximum rate of introduction of fluid be denoted by A, the corresponding potential is given by (1).

"It will be observed that when the source, as measured by A, is finite, the potential and the pressure-variation (proportional to $d\phi/dt$) are infinite at the pole. But this does

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not, as might for a moment be supposed, imply an infinite emission of energy. If the pressure be divided into two parts, one of which has the same phase as the velocity, and the other the same phase as the acceleration, it will be found that the former part, on which the work depends, is finite. The infinite part of the pressure does no work on the whole, but merely keeps up the vibration of the air immediately round the source, whose effective inertia is indefinitely great.

" We will now investigate the energy emitted from a simple source of given magnitude, supposing for the sake of greater generality that the source is situated at the vertex of a rigid cone of solid angle ω. If the rate of introduction of fluid at the source be $A \cos kat$, we have

$$\omega r^2 d\phi/dr = A \cos kat$$

ultimately, corresponding to

$$\phi = - \frac{A}{\omega r} \cos k (at - r), \quad . \quad . \quad . \quad . \quad (2)$$

whence

$$\frac{d\phi}{dt} = \frac{kaA}{\omega r} \sin k (at - r), \quad . \quad . \quad . \quad (3)$$

and

$$\omega r^2 d\phi/dr = A\{\cos k (at - r) - kr \sin k (at - r)\}. \quad . \quad (4)$$

Thus if $d\mathrm{W}$ be the work transmitted in time dt, we get, since $\delta p = -\rho \, d\phi/dt$,

$$\frac{d\mathrm{W}}{dt} = - \frac{\rho kaA^2}{\omega r} \sin k(at - r) \cos k(at - r)$$
$$+ \rho \frac{k^2 aA^2}{\omega} \sin^2 k(at - r).$$

Of the right-hand member the first term is entirely periodic, and in the second the mean value of $\sin^2 k(at - r)$ is $\frac{1}{2}$. Thus in the long run

$$\mathrm{W} = \frac{\rho k^2 aA^2}{2\omega} t. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

" It will be remarked that when the source is given, the amplitude varies inversely as ω, and therefore the intensity inversely as ω^2. For an acute cone the intensity is greater, not only on account of the diminution in the solid angle through which the sound is distributed, but also because the total energy emitted from the source is itself increased.

" When the source is in the open, we have only to put $\omega = 4\pi$, and when it is close to a rigid plane, $\omega = 2\pi$.

" These results find an interesting application in the theory

of the speaking-trumpet, or (by the law of reciprocity, §§ 109, 294) hearing-trumpet. If the diameter of the large open end be small in comparison with the wave-length $(2\pi/k)$, the waves on arrival suffer copious reflexion, and the ultimate result, which must depend largely on the precise relative lengths of the tube and of the wave, requires to be determined by a different process. But by sufficiently prolonging the cone, this reflexion may be diminished, and it will tend to cease when the diameter of the open end includes a large number of wave-lengths. Apart from friction it would therefore be possible by diminishing ω to obtain from a given source any desired amount of energy, and at the same time by lengthening the cone to secure the unimpeded transference of this energy from the tube to the surrounding air.

"From the theory of diffraction it appears that the sound will not fall off to any great extent in a lateral direction, unless the diameter at the large end exceed half a wave-length. The ordinary explanation of the effect of a common trumpet, depending upon a supposed concentration of *rays* in the axial direction, is thus untenable."

Data respecting Fog-Signals.

The above theory should throw light upon the production of sound in " fog-signals," where sirens, or vibrating reeds, are associated with long conical trumpets. In the practice of the Trinity House these are actuated by air compressed to a pressure (above atmosphere) of 25 lbs. per square inch, or 1760 gms. per sq. cm., a pressure which appears rather high. According to Stone the highest pressure used in orchestral wind-instruments is 40 inches (102 cm.) of water.

As might be expected from the high pressure, the energy consumed during the sounding of the signal is very considerable. The high note of the St. Catherine's Service signal takes 130 horse-power, and the corresponding note of a Scottish signal (tested at St. Catherine's in 1901) requires as much as 600 horse-power. The question obtrudes itself whether these enormous powers are really utilized for the production of sound, or whether from some cause, possibly unavoidable, a large proportion may not be wasted.

Comparison with Musical Instruments &c.

These statements as to horse-power may be better appreciated if I record for comparison the results of some rough measurements, made in 1901, upon the power absorbed by smaller instruments. In the calculation it will suffice to regard the compressions and rarefactions as *isothermal.*

Thus if v_0, p_0 represent the volume and pressure of air in its natural (atmospheric) condition, v, p the corresponding quantities under compression, so that according to Boyle's law $pv = p_0 v_0$, then the work (W) of compression is given by

$$W = p_0 v_0 \log (p/p_0), \quad \cdots \quad (6)$$

or, if the compression be small,

$$W = p_0 v_0 \frac{p - p_0}{p_0}. \quad \cdots \quad (7)$$

In c.g.s. measure p_0 (the atmospheric pressure) will be 10^6, and if v_0 be measured in cubic centimetres, W will be expressed in ergs. If in (7) v_0 be understood to mean the volume compressed *per second of time*, W will be given in ergs per second, of which $7 \cdot 46 \times 10^9$ go to the horse-power.

The first example is that of a small horn (without valves) blown by the lips. It resonates to e' of my harmonium, and the pitch when sounded is about e' flat. From one inspiration I can blow it for about 30 seconds with a pressure (in the mouth) of $1\frac{1}{2}$ inch (3·8 cm.) of mercury. The contents of the lungs may be taken at 1200 c.c., giving 40 c.c. per second as the wind consumption. This is the value of v_0, and $(p - p_0)/p_0$ is $\frac{1}{20}$. Hence W in ergs per second will be 2×10^6, or in horse-power

$$W = \cdot 00027 \text{ H.P.}$$

The sound from this very small horse-power is unpleasantly loud when given in a room of moderate dimensions.

In the case of the harmonium reed e' the wind consumption was 220 c.c. per second, and the pressure 2 inches of water, so that $(p - p_0)/p_0 = \frac{1}{200}$. Hence

$$W = \cdot 00015 \text{ H.P.}$$

A small hand fog-signal of Holmes' pattern, known as the "Little Squeaker," consumed a horse-power calculated on the basis of similar measurements to be ·03. For the very effective "Manual" of the Trinity House Service the horse-power was about 3·0.

These examples may all be classed under the head of reeds, the harmonium reed being "free" and probably in consequence less efficient, and the others "striking." To them may be added the case of a whistle of high pitch[*], for which the wind consumption represented $1 \cdot 8 \times 10^6$ ergs per second, or ·00024 horse-power, practically the same as for the small horn above. The latter was certainly the more powerful of the two, considered as a source of audible sound.

[*] Proc. Roy. Soc. xxvi. p. 248 (1877); Scientific Papers, i. p. 329.

It may now be instructive to consider the case of a large siren, such as the 7-inch disk siren experimented upon at St. Catherine's in 1901. The wind consumption here was 29 cubic feet, or 810 litres, per second. This average current, for the purposes of a rough calculation, may be analysed into a steady current of the same amount and an alternating current whose extremes are represented by ± 810 litres per second, the latter being alone effective for the production of sound. The first question that arises is to what pressure does this correspond, and is it a reasonable fraction of the actual pressure employed?

The answer must depend upon the other circumstances of the case, such as the character of the cone or other tubular resonator associated with the siren. We shall begin by supposing that there is nothing of this kind, so that the above alternating flow takes place at the surface of a sphere of radius r situated in the open. The velocity-potential and the rate of total flow being given by (1) and (2) with ω equal to 4π, we have for the maximum rate of that flow $A\sqrt{\{1 + k^2r^2\}}$, or with sufficient approximation for our purpose A simply. If s be the "condensation,"

$$\delta p = a^2 \rho s = -\rho \, d\phi/dt,$$

so that by (1)

$$s_{\max} = \frac{kA}{4\pi ar} = \frac{A}{2\lambda ar}. \quad \cdots \quad (8)$$

To obtain a numerical result we must make some supposition as to the magnitude of r. Let us take $kr = \frac{1}{4}$. Then

$$s_{\max} = \frac{4\pi A}{\lambda^2 a}, \quad \cdots \quad (9)$$

in which A is the maximum flow, λ the wave-length, and a the velocity of sound, *i.e.* 3×10^4 cm. per second. In the experiments referred to, the pitch was low and such that

$$\lambda = 8 \text{ feet} = 240 \text{ cm.},$$

whence with $A = 8 \cdot 1 \times 10^5$ c.c. per second we find

$$s_{\max} = \frac{1}{180}.$$

The maximum condensation corresponding to the assumed introduction of air is thus only $\frac{1}{180}$ of an atmosphere. The pressure is in the same proportion, and we see that it is but an insignificant fraction of the pressure actually employed (25 lbs. per square inch). We infer that no moderate pressure can be utilized in this way, and that some cone or

resonating tube is a necessity. It may be remarked that the radius r of the sphere, on which the introduction of air is supposed to take place, is $1/4k$ or $\lambda/8\pi$, that is in the case taken 4 inches or 10 cms.

Cones and Resonators.

The next question is what improvement in the direction of utilizing higher pressures can be attained by the association of cones and resonators ? But to this it is at present difficult to give a satisfactory answer. Theory shows that, apart from friction and other complications perhaps not very important, the efficiency of a *small* source may by these means be increased to any extent. Thus, in the case of the cone, if u be the maximum velocity of a progressive wave at a point where the section is σ, conservation of energy requires that σu^2 be constant. The maximum total flow (σu) is therefore proportional to $\sigma^{\frac{1}{2}}$, *i. e.* to the linear dimension of the section. If the vibrations are infinitesimal, we may begin with as small a diameter as we please and end with a large one, and thus obtain any desired multiplication of the source. For it is the total flow at the *open* end of the cone which measures the power of the source for external purposes. If, however, the quantity of air periodically introduced at the small end can no longer be treated as infinitesimal, this argument fails ; and it is probable that the advantage derivable from the cone diminishes. In an extreme case we can easily recognize that this must be so. For the most that the cone could do would be to add its own contents to that of the air forcibly introduced. As the latter increases without limit, the addition must at last become relatively unimportant, and then the cone might as well be dispensed with. Similar considerations apply to the use of a resonator.

There is no reason to doubt that great advantage accrues from the use of the conical trumpet in existing fog-signalling apparatus, although probably it falls short of what would be expected according to the theory of infinitely small vibrations. If it be a question of striving to augment still further the force of the sound, we must remember that the application of power has already been carried to great lengths. The utilization of more power might demand an increase in the scale of the apparatus. This in itself would present no particular difficulty, but we must not forget that everything has relation to the wave-length of the sound, and that this is to a great extent fixed for us by the nature of the ear. It may well be that we are trying to do more than the conditions allow, and that further advance would require a

different kind of apparatus. As matters stand, it seems to be generally admitted that the instruments using great power are not proportionally effective.

If, as I incline to believe, a large proportion of the power applied to important instruments is not converted into sound, there should be an opening for reducing the very large demands now made. We have to consider what becomes of the power wasted. I have long thought that it is spent in the eddies consequent upon the passage of the air through the comparatively narrow ports of the siren, and in this opinion my friend Sir O. Lodge, with whom I have recently had an opportunity of discussing the matter, concurs. If indeed it were a question of steady flow, one might pronounce with certainty that a great improvement would ensue from a better shaping of the passages, which on the downstream-side should cone out gradually from the narrowest place. And although the intermittent character of the stream is an important element, this conclusion can hardly be altogether disturbed. The advantage of an enlargement of the ports themselves should also be kept in sight.

The conical trumpets at present employed must act to some extent as resonators, so that the precise relation of the pitch or speed of the siren to the trumpet cannot be a matter of indifference. Although the relation in question is liable to be disturbed by changes of temperature, it would appear that a better adjustment than is feasible with the present governors should be arrived at. To effect this an instrument capable of indicating the vigour of the vibration within the trumpet, as the speed of the siren varies, would be useful.

Vibration Indicator.

Experiments that I have tried appear to prove that the problem above proposed can be solved in a very simple manner. The principle is that of the unsymmetrical formation of jets when an alternating air-current flows through an aperture coned *upon one side*. An experiment given in 'Theory of Sound,' § 322, may be quoted in illustration :—
" When experimenting with one of König's brass resonators of pitch c', I noticed that when the corresponding fork, strongly excited, was held to the mouth, a wind of considerable force issued from the nipple at the opposite side. This effect may rise to such intensity as to blow out a candle upon whose wick the stream is directed . . . . Closer examination revealed the fact that at the sides of the nipple the outward flowing stream was replaced by one in the opposite

direction, so that a tongue of flame from a suitably placed candle appeared to enter the nipple at the same time that another candle situated immediately in front was blown away. The two effects are of course in reality alternating, and only appear to be simultaneous in consequence of the inability of the eye to follow such rapid changes."

The application of the principle for the present purpose is very simple. The candle is replaced by a U pressure-gauge (fig. 1), the jet from a contracted nozzle playing into one limb. The whole is inclosed airtight in a test-tube, so that no permanent pressure or suction has any effect upon the gauge. The nozzle communicates by means of a flexible tube with the space where the vibration is to be measured. Some throttling to check the vibration of the liquid may be convenient.

The small Holmes apparatus gave an indication of about 3 inches of water, and the Trinity House "Manual" one inch of *mercury*. The sensitiveness may be lessened by contracting the nozzle or probably by insertion of water to diminish the air-space within the test-tube.

Fig. 1.
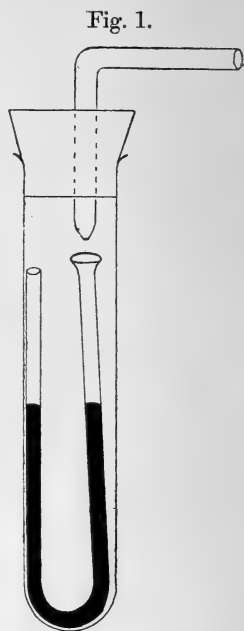
Reeds.

Reeds have the advantage of working without a governor, and the pitch once properly fixed is more likely to be maintained. I do not know whether reeds have been tried for very large scale instruments. In the Barker apparatus * *three* reeds are combined with one trumpet. At first sight it may seem doubtful whether the tongues would vibrate in the same phase, but upon examination I think it will appear that this is the only way in which they could vibrate. On a large scale either the reeds must be multiplied, or an entirely different shape must be adopted, out of all proportion *broader* than at present. Some experiments that I have tried seem to show that the latter alternative is not impracticable.

* Report of Trinity House Fog-Signal Committee on Experiments conducted at St. Catherine's Point, Isle of Wight, 1901.

Trumpets of Elongated Section.

In the trumpets at present employed the section is of circular form and the greater part of the axis is vertical. This disposition has its conveniences, but it entails bending the axis at the wide end of the cone if the mouth is to face horizontally. The effect of such a bending upon the propagation of the wave within the trumpet is hard to estimate. When, as in the case of certain rock-stations, the sound is required to be heard in all directions, a symmetrical form is adopted in the Trinity House Service, the mouth of the trumpet which faces vertically being partially stopped by an obstacle known as the "mushroom." The intention is to cut off the sound in a vertical direction while allowing it to spread in horizontal directions through the annular aperture between the bell-mouth of the trumpet and the mushroom.

Considering the case of the axis horizontal throughout, we may inquire into the probable distribution of the sound. The ratio between the diameter of the mouth and the wave-length is here of essential importance. If the diameter much exceed the half wave-length, the sound is concentrated in the prolongation of the axis. If on the other hand the diameter do not exceed the above-mentioned quantity, we may expect a tolerably equable distribution of sound, at any rate through angles with the axis less than 80°. It follows that the behaviour of the various components of a compound sound may be quite different. The fundamental tone may spread fairly well, while the octave and higher elements are unduly concentrated in the neighbourhood of the axis.

It appears then that a limitation must be imposed upon the size of the mouth, if it be desired that the sound should spread. But since the spreading is required only in the horizontal plane, the limitation applies only to the *horizontal* dimension of the mouth. There is no corresponding limitation upon the vertical diameter. We are thus led to prefer an elongated form of section, the horizontal dimension being limited to the half wave-length, while the vertical dimension may amount if desired to many wave-lengths. This subject was explained and illustrated in a lecture before the Royal Institution[*], the source of (inaudible) sound being a "birdcall" giving waves of 3 cms. length, which issued from a flattened trumpet whose mouth measured 5 cms. by $1\frac{1}{2}$ cm. The indicator was a high-pressure sensitive flame, and it appeared very clearly that when the long dimension of the section stood vertical the sound was approximately limited

[*] Proc. Roy. Inst. Jan. 1902; Nature, 66. p. 42 (1902).

to the horizontal plane, but within that plane spread without much loss through all directions less inclined than 80°.

In order to carry the demonstration a little nearer to what would be required in practice, I have lately experimented with the sound from a reed organ-pipe, giving waves of length equal to 8 inches (20 cms.) and thus easily audible. The trumpet is of wood, pyramidal in form, and the section at the mouth is 36×4 inches (91×10 cms.). The length (OB) is 6 feet (183 cms.). These dimensions were chosen so that OA, OC should exceed OB by $\frac{1}{4}\lambda$. A larger difference might entail
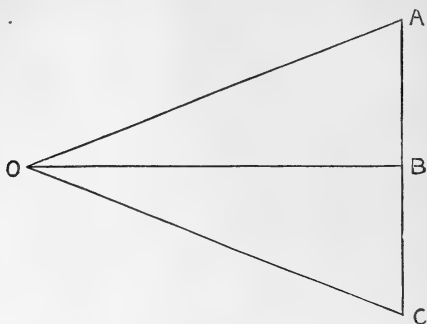
too great a discrepancy of phase in the waves at A and B ; a less difference might lead to an unnecessary prolongation of the cone along the axis. The trumpet was so mounted that its mouth just projected from an open window, and that it could be readily turned round OB as horizontal axis so as to allow the length of the section (AC) to be either horizontal or vertical. The observers took up various positions on a lawn at a moderate distance from the window.

To observers in the line of the axis OB it should make no difference how the trumpet is rotated. On the whole this expectation was confirmed, but a little precaution is required. As usual the phenomenon was complicated by reflected sounds (doubtless from the ground). It was well in every case to make sure, by slightly raising or lowering the head, that the maximum sound was being heard.

When the point of observation deviated (in the horizontal plane) from the axis, the difference due to rotation was soon apparent. At 30° obliquity the sound appears greatly increased as AC passes from the horizontal to the vertical position. At higher obliquities with AC horizontal the sound falls off greatly, but recovers when AC is made vertical. Altogether the effects are very striking, and carry conviction to the mind more fully than experiments with sensitive flames where one is more or less in doubt as to the *magnitude* of the

differences indicated by the flame. It will be remarked that to carry out this experiment upon a practical scale will mean a very large structure, the linear magnification being that (6 times) required to pass from an 8-inch wave-length to one (say) of 48 inches.

Work done by Detached Sources.

In the case of a single source the pressures to be overcome are proportional to the magnitude of the source, and thus the work done is proportional to the *square* of the magnitude of the source, as is indeed otherwise evident. If, as is usually the case in practice, the object be to emit sound in one (horizontal) plane only, an economy may be effected by *distributing* the source. If sources all in one phase be distributed along a vertical line, the effect is the same at distant points in the horizontal plane as if they were all concentrated in one point, but the *work* required to be done may be much less, the saving corresponding to the fact that in directions other than horizontal the sound is now diminished. We will begin by considering two unit sources in the same phase.

If ϕ, ψ be the potentials of these sources at a point whose distances are r, r', we have, as in (1),

$$\phi = -\frac{\cos k\,(at-r)}{4\pi r}, \qquad \psi = -\frac{\cos k\,(at-r')}{4\pi r'}. \quad . \quad (10)$$

Thus, when $r=0$,

$$4\pi r^2 \frac{d\phi}{dr} = \cos kat - kr \sin kat,$$

$$\frac{d\phi}{dt} + \frac{d\psi}{dt} = \frac{ka \sin kat}{4\pi r} + \frac{ka}{4\pi \mathrm{D}} \sin k\,(at-\mathrm{D}),$$

if D denote the distance between the sources. The work done by the source at $r=0$ is accordingly proportional to

$$1 + \frac{\sin k\mathrm{D}}{k\mathrm{D}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and an equal work is done by the source at $r'=0$. If D be infinitely great, the sources act independently, and thus the scale of measurement in (11) is such that unity represents the work done by each source when isolated. If $\mathrm{D}=0$, the work done by *each* source is doubled, and the two sources become equivalent to one of doubled magnitude.

If D be equal to $\frac{1}{2}\lambda$, or to any multiple thereof, $\sin k\mathrm{D}=0$, and we see from (11) that the work done by each source is unaffected by the presence of the other. This conclusion may be generalized. If any number (n) of equal sources

in the same phase be arranged in a vertical line so that the distance between immediate neighbours is $\frac{1}{2}\lambda$, the work done by each is the same as if the others did not exist. The whole work accordingly is n, whereas the work to be done at a single source of magnitude n would be n^2. Thus if the sound be only wanted in the horizontal plane, the distribution into n parts effects an economy in the proportion of $n:1$. It is not necessary that all the possible places between the outer limits be actually occupied. All that is necessary is that there be n equal sources altogether, and that the distance between any pair of them be a multiple of $\frac{1}{2}\lambda$.

Returning for the moment to the case of two sources only, we may be interested to estimate the work consumed by following the law of emission of sound to a distance in the various directions. If μ be the cosine of the angle between any direction and the vertical, the relative retardation due to the difference of situation is μD. If the potential at any great distance due to one source is $\cos kat$ *, that due to the other may be represented by $\cos k(at - \mu\text{D})$. For the aggregate potential we have

$$\cos kat(1 + \cos \mu k\text{D}) + \sin kat \sin \mu k\text{D},$$

or for the *intensity*

$$2(1 + \cos \mu k\text{D}). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This is in direction μ. For the total intensity over angular space we must integrate with respect to μ from -1 to $+1$. The *mean* intensity is thus

$$2\int_0^1 (1 + \cos \mu k\text{D})\,d\mu = 2\left(1 + \frac{\sin k\text{D}}{k\text{D}}\right). \quad . \quad . \quad (13)$$

The scale of measurement is at once recovered by supposing $\text{D}=0$, in which case the intensity in various directions would be uniform. The ratio of the mean intensities, which is also that of the work done, is thus

$$\tfrac{1}{2}\left(1 + \frac{\sin k\text{D}}{k\text{D}}\right). \quad . \quad . \quad . \quad . \quad (14)$$

This is the ratio in which the work done is diminished when a source is divided into two parts and these parts separated to a distance D.

While from the theoretical point of view there is no doubt as to the saving that might arise from the use of a number of separated sources, it is to be noticed that the saving is in the *pressure*. Since at the present time most of the pressure

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* It is not necessary to exhibit the dependence on r.

employed with a single source appears to be wasted, we are left in doubt whether with the existing arrangements economy would be attained by breaking up the source.

We will now investigate the expression for the energy radiated from any number of sources of the same pitch situated at finitely distant points in the neighbourhood of the origin O. The velocity-potential ϕ of the motion due to one of the sources at (x, y, z) is at Q

$$\phi = -\frac{A}{4\pi R}\cos(nt + \epsilon - kR), \quad . \quad . \quad . \quad (15)$$

where R is the distance between Q and (x, y, z). At a great distance from the origin we may identify R in the denominator with OQ, or ρ; while under the cosine we write

$$R = \rho - (\lambda x + \mu y + \nu z), \quad . \quad . \quad . \quad (16)$$

λ, μ, ν being the direction-cosines of OQ. On the whole

$$-4\pi\rho\phi = \Sigma A \cos\{nt + \epsilon - k\rho + k(\lambda x + \mu y + \nu z)\}, \quad . \quad (17)$$

in which ρ is a constant for all the sources, but A, ϵ, x, y, z vary from one source to another. The *intensity* in the direction λ, μ, ν is thus represented by

$$[\Sigma A \cos\{\epsilon + k(\lambda x + \mu y + \nu z)\}]^2 + [\Sigma A \sin\{\epsilon + k(\lambda x + \mu y + \nu z)\}]^2,$$

or by

$$\Sigma A^2 + 2\Sigma A_1 A_2 \cos[\epsilon_1 - \epsilon_2 + k\{\lambda(x_1 - x_2) + \mu(y_1 - y_2) + \nu(z_1 - z_2)\}],$$
$$. \quad . \quad . \quad (18)$$

the second summation being for every pair of sources of which A_1, A_2 are specimens. We have now to integrate (18) over angular space.

It will suffice if we effect the integration for the specimen term; and this we shall do most easily if we take the line through the points (x_1, y_1, z_1), (x_2, y_2, z_2) as axis of reference, the distance between them being denoted by D. If λ, μ, ν make an angle with D whose cosine is μ,

$$D\mu = \lambda(x_1 - x_2) + \mu(y_1 - y_2) + \nu(z_1 - z_2), \quad . \quad . \quad (19)$$

and the mean value of the specimen term is

$$A_1 A_2 \int_{-1}^{+1} \cos\{\epsilon_1 - \epsilon_2 + kD\mu\}d\mu,$$

that is

$$\frac{2A_1 A_2}{kD}\sin kD \cos(\epsilon_1 - \epsilon_2). \quad . \quad . \quad . \quad (20)$$

The mean value of (18) over angular space is thus

$$\Sigma A^2 + 2\Sigma \frac{A_1 A_2 \cos (\epsilon_1 - \epsilon_2) \sin k\mathrm{D}}{k\mathrm{D}}, \quad . \quad . \quad . \quad (21)$$

where D denotes the distance between the specimen pair of sources. If all the sources are in the same phase $\cos (\epsilon_1 - \epsilon_2) = 1$. If the distance between every pair of sources is a multiple of $\frac{1}{2}\lambda$, $\sin k\mathrm{D} = 0$ and (21) reduces to its first term.

We fall back upon a former particular case if we suppose that there are only two sources, that these are units, and are in the same phase. (21) then becomes

$$2 + 2 \frac{\sin k\mathrm{D}}{k\mathrm{D}},$$

agreeing with (11), which represents the work done by each source.

If the question of the phases of the two unit sources be left open (21) gives

$$2 + 2 \cos (\epsilon_1 - \epsilon_2) \frac{\sin k\mathrm{D}}{k\mathrm{D}} . \quad . \quad . \quad . \quad (22)$$

If D be small, this reduces to

$$2 + 2 \cos (\epsilon_1 - \epsilon_2),$$

which is zero if the sources be in opposite phases, and is equal to 4 if the phases be the same.

If, however, $\sin k\mathrm{D}$ be equal to -1, the case is altered. Thus when $\mathrm{D} = \frac{3}{2}\lambda$, we get

$$2 - \frac{4}{3\pi} \cos (\epsilon_1 - \epsilon_2),$$

and this is a minimum (and not a maximum) when the phases are the same.

In (22) if the phases are 90° apart, the cosine vanishes. The work done is then simply the double of what would be done by each source acting alone, and this whatever the distance D may be.

Continuous Distributions.

If the distribution of a source be continuous, the sum in (21) is to be replaced by a double integral. As an example, consider the case of a source all in one phase and uniformly distributed over a complete circular arc of radius c. If D be the distance of two elements $d\theta$, $d\theta'$, we have to consider the integral

$$\iint \frac{\sin k\mathrm{D}}{k\mathrm{D}} d\theta \, d\theta', \quad . \quad . \quad . \quad . \quad (23)$$

where

$$\mathrm{D} = 2c \sin \tfrac{1}{2}(\theta - \theta') \quad . \quad . \quad \quad . \quad (24)$$

Since every element $d\theta'$ contributes equally, it suffices to take $\theta' = 0$, so that the integral to be evaluated is

$$\int^{\pi} \frac{\sin\,(2kc\sin\frac{1}{2}\theta)}{2kc\sin\frac{1}{2}\theta}\,d\theta, \quad . \quad . \quad . \quad (25)$$

or, if $\frac{1}{2}\theta = \phi$, $2kc = x$,

$$2\int_0^{\frac{1}{2}\pi} \frac{\sin\,(x\sin\phi)}{x\sin\phi}\,d\phi. \quad . \quad . \quad . \quad (26)$$

The integral (26) may be expressed by means of Bessel's function J_0, for

$$J_0(x) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\cos\,(x\sin\phi)\,d\phi,$$

so that

$$\int_0^x J_0(x)\,dx = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin\,(x\sin\phi)}{\sin\phi}\,d\phi.$$

Thus, if constant factors be disregarded, we get

$$\frac{1}{x}\int_0^x J_0(x)\,dx, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

in which $x = 2kc$. Since (27) reduces to unity when c, or x, vanishes, it represents the ratio in which the work done is diminished when a source, originally concentrated at the centre, is distributed over a circular arc of radius c.

The case of a source uniformly distributed over a circular *disk* of radius c is investigated in my book on the 'Theory of Sound' *. According to what is there proved, the factor, analogous to (27), expressing the ratio in which the work done is diminished when a source originally concentrated at the centre is expanded over the disk, has the form

$$\frac{2}{k^2c^2}\left\{1 - \frac{J_1(2kc)}{kc}\right\}, \quad . \quad . \quad . \quad (28)$$

where, as usual,

$$J_1(z) = \frac{z}{2} - \frac{z^3}{2^2.4} + \frac{z^5}{2^2.4^2.6} - . \quad . \quad . \quad (29)$$

Another case of interest is when the distribution takes place over the surface of a *sphere* of radius c. In (25) we have merely to introduce the factor $\sin\theta$, equal to $2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$, so that we get instead of (26)

$$4\int_0^{\frac{1}{2}\pi}\frac{\sin\,(x\sin\phi)}{x}\cos\phi\,d\phi = \frac{4}{x^2}(1 - \cos x).$$

* MacMillan, 1st edition, 1878; 2nd edition, 1896, § 302.

The factor, corresponding to (27), is therefore simply

$$\frac{\sin^2 kc}{k^2 c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

No work at all is done if c be such that kc is a multiple of π, or $2c$ a multiple of λ.

By the method of the 'Theory of Sound' (*loc. cit.*) we might in like manner investigate the effect of distributing a source of sound uniformly throughout the *volume* of a sphere, but the above examples will suffice for our purpose.

Experimental Illustrations.

There is no difficulty in illustrating upon a small scale the results above deduced from theory. The simplest experiment is with an ordinary open organ-pipe, gently blown, so as to exclude overtones as much as may be. The open ends act as two equal sources of sound in the same phase. Connected by a long flexible tube with a well-regulated bellows, the pipe can be held in any position and be observed in the open air from a moderate distance. When the length of the pipe is perpendicular to the line of observation, the two sources are at the same distance and the effects conspire. But if the pipe point toward the observer, the two sources, being at about $\frac{1}{2}\lambda$ apart, are in antagonism and the sound is much enfeebled.

In order to exemplify the principle further, a multiple pipe was constructed (fig. 2). This consisted of a straight lead tube 31 inches long and ·35 inch bore*, open at the ends A, B. At four points D, E, F, G, distant $6\frac{1}{2}$ inches from the ends and 6 inches from one another, the tube was perforated, and the holes were blown by four streams of wind from the branched supply-tube C. The whole was cemented to a framework of wood, so that it could be turned round without relative displacement. The intention was that all six apertures A, D, E, F, G, B should act as sources of sound in the same phase, but one could hardly be sure *à priori* that this behaviour would be observed. Would the simultaneous motions of the air-column on the two sides of E (for example) be both towards E or both from E? Might it not rather be that the motions would take the same absolute direction, in which case E, F, &c., would fail to act as sources? A little observation, however, sufficed to prove that the apparatus really acted as intended. By listening through a rubber tube whose outer end was brought into proximity with the apertures, it was easy to satisfy oneself

* Inch $= 2.54$ cms.

that D, E, F, G were effective sources of sound, and were in
fact more powerful than the open ends A, B, as was to be

Fig. 2.
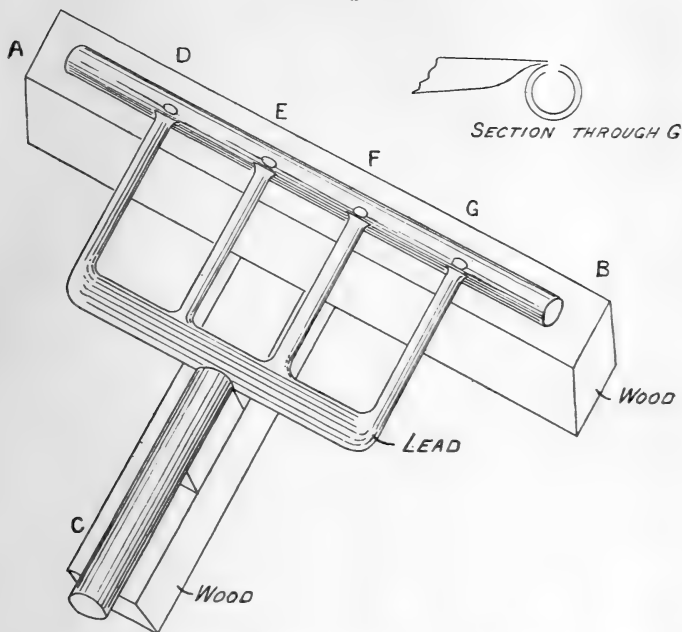
expected. The half wave-length of the actual sound was
8 inches, showing that the " openness " of the pipe at D, E,
F, G was rather imperfect, owing to the smallness of the
holes. Still the apparatus afforded a combination of six
sources of sound, all in the same phase and at about half a
wave-length apart.

The observations were made upon a lawn ; and, as the
sound was rather feeble, a very moderate distance sufficed.
When A B was vertical, the sources cooperated and the
maximum sound was heard. But when A B was moved
away, so as no longer to be perpendicular to the (horizontal)
line of observation, the sound was less, a deviation of 30°
causing a very great falling off. The effect was as if the
sound had suddenly gone away to a great distance. The
success of the experiment no doubt depended a good deal
upon the absence of over-tones, a condition of things favoured
by the feebleness of the sound and also by the high pitch.

Terling Place, Witham,
July 21, 1903.

XXXVIII. *On the Condensation Nuclei produced in Air and Hydrogen by heating a Platinum Wire.* By GWILYM OWEN, *B.Sc. (Vict.),* 1851 *Exhibition Science Scholar of University College, Liverpool; Christ's College, Cambridge**.

[Plate XI.]

Introduction.

IF ordinary air saturated with water-vapour be suddenly expanded, a dense fog is produced, owing to the condensation of the water-vapour on the numerous dust-particles presnt in the air.

These particles of dust can be removed from the air by filtering the latter through cotton-wool, or by repeated expansions, by which process the dust-particles are carried down to the bottom of the vessel.

Aitken † found that by heating a glass tube containing pieces of iron or brass wire, or by making a platinum wire red-hot by means of an electric current, he could produce sufficient nuclei to give dense fogs with a very small supersaturation of the air, which was originally dust-free. He was unable to detect any decrease in the weight of the wires, even after nuclei sufficiently numerous to give many dense fogs had been produced by the heating. Aitken did not come to any definite conclusion as to the nature of the nuclei, but suggested that they were very small particles of matter emitted by the wire, also that some of them might be formed by the condensation, during the expansion, of some vapour given off from the wire when heated.

The present investigation was undertaken at the suggestion of Prof. Thomson in the hope of obtaining some further information with regard to the production of condensation-nuclei by the heating of a platinum wire.

It will be convenient to mention here some results got by C. T. R. Wilson ‡, as reference will be made to them in the paper. Wilson found that when dust-free air, originally saturated with water-vapour, is suddenly expanded, then condensation takes place if the maximum degree of supersaturation thus produced exceeds a certain limit. Measuring the expansion by $\frac{v_2}{v_1}$, the ratio of the final to the initial volume, he found that when $\frac{v_2}{v_1}$ is less than $1\cdot25$, no condensation takes place in dust-free air. When $\frac{v_2}{v_1}$ is greater than $1\cdot25$

* Communicated by Prof. J. J. Thomson, F.R.S.
† Trans. Roy. Soc. Edin. xxx. p. 348 (1883).
‡ Phil. Trans. A. vol. clxxxix. p. 265 (1897).

and less than 1·38, condensation takes place in the form of a shower, called by him "Rain-like Condensation." If $\frac{v_2}{v_1}$ exceeds 1·38, then dense fogs are produced.

Description of Apparatus.

In the course of the experiments, platinum wires of various thicknesses were experimented with. In the earlier experiments wires 0·06 mm. in diameter were used, but as these were so liable to break, owing to the violence of the air-movements when an expansion was made, thicker wires (0·2 mm.) were used in the later experiments. Usually the wire was stretched across a glass bulb, being fastened to thicker pieces of platinum which were fused through the ends of glass tubes joined to the bulb.

In other cases the thicker leads were both fused through the same tube, and the wire was fastened to them in the form of a loop or spiral. This second method was superior to the first, in that it made it much easier to take out the old wire when necessary and to put in a new one; but it could only be used for the thicker wires, as a loop of thin wire soon got twisted up by the rush of air during the expansions.

The bulb containing the platinum wire was connected to the expansion apparatus by means of a sealing-wax joint. The expansion apparatus used was constructed on the plan adopted by C. T. R. Wilson [*].

It consists of a cylindrical glass tube (Pl. XI. fig. 1), 25 cms. long and 5·5 cms. diameter, in which slides freely a thin-walled glass plunger made out of a Welsbach chimney. The open end of the plunger has been ground smooth and perpendicular to the axis. The outer tube A is tightly wired down on a large rubber stopper, through which passes the tube C, of about 1 cm. internal diameter. The greater part of the space beneath the plunger is filled with distilled water, so that the gas in A is completely cut off from the space between the water-surface and the top of the plunger. The small-bore tube passing through the stopper and closed at the outer end with a pinch-cock is introduced as affording a convenient inlet or outlet for the distilled water. By opening the tap D the plunger rises until the pressure of the gas in the bulb is atmospheric (neglecting the weight of the piston). By closing D, and pulling the valve P, the space beneath the plunger is put into communication with the vacuum V, and the plunger descends with great speed upon the rubber stopper, and is there held by the pressure of the gas above it.

* Phil. Trans. A. vol. cxcii. p. 405 (1899).

The gas in the bulb thus suffers a sudden expansion. By opening D the plunger will rise again to the same height. The expansion can be increased by exhausting a little air out of the bottle F through the tube E, the tap D being open at the time. Smaller expansions can be obtained by allowing some more gas to enter the apparatus through the U-tube S, or (and this method was very often employed) by manipulating the tap D so that the plunger does not ascend as far as it would go. In the latter case, by noting the pressures of the gas in the bulb before and after expanding, the value of the expansion is easily calculated.

If B be the height of the barometer, and π the pressure of water-vapour at the temperature of the experiment, then, allowing the plunger to ascend as far as it will, the pressure of the gas before expansion is

$$P_1 = B - \pi,$$

neglecting weight of plunger; the pressure after expansion is

$$P_2 = B - \pi - p,$$

where p is the height of the mercury in the tube M above the surface of the mercury in F.

If v_1 = initial volume of the gas, and v_2 = its final volume, then the expansion is

$$\frac{v_2}{v_1} = \frac{P_1}{P_2} = \frac{B - \pi}{B - \pi - p}.$$

If the plunger be not allowed to rise as far as it would, and

p_1 = height of mercury in M before expanding
p_2 =   ,,   ,,   ,,   after   ,,

then

$$\frac{v_2}{v_1} = \frac{P_1}{P_2} = \frac{B - \pi - p_1}{B - \pi - p_2}.$$

The U-tube S was used, as it afforded a very convenient method of cutting off communication between the experimental bulb and the atmosphere or the gas-generating apparatus.

To prevent the water-vapour condensing on the glass in small drops, and so obstructing a clear view of the interior of the cloud-chamber, the latter, after the insertion of the wire in it, was well washed with nitric acid, or with chromic acid, and then rinsed out with distilled water. This washing would also remove any dirt that might be on the wire. But the latter was subjected to a further cleaning process by being raised to a high temperature, yellow heat, for some hours after the bulb had been fixed to the expansion apparatus.

A very convenient method of keeping the wire for a long time at this high temperature was to place the wire in series with a few turns of thick wire wound on a "choking" coil connected to the alternating town mains.   To make visible any condensation resulting from the expansion, the light from a fish-tail burner placed edgeways was focussed on the bulb, and by looking against a dark background in a direction making a small angle with the light, one or two drops could be easily detected.

With the first platinum wire tried it was found that (in air) heating the wire by means of a very small electric current for two or three seconds produced no condensation nuclei, for no condensation was obtained on expanding a few seconds after the heating-current had been cut off.   The value of the expansion used in this case was about 1·1.

On gradually increasing the current through the wire, and therefore also its temperature, a point was reached at which condensation was obtained in the form of a small shower on expanding, after the wire had been raised to this temperature for one or two seconds.   On increasing the temperature still further, the density of the shower rapidly increased, and soon dense fogs were obtained long before the wire was sufficiently hot to be luminous.

This result indicated that condensation nuclei large enough to be detected with the expansion used were only produced when the temperature of the wire reached a certain value.

In order to make quantitative measurements on this point, an arrangement was adopted in which the experimental wire formed one arm of a Wheatstone's bridge, the corresponding arm being a spiral of thick german-silver wire (1·16 ohms resistance) immersed in a bath of paraffin oil.   The two other arms of the bridge were high resistances of the order of 1000 ohms, got from a Post-Office Box.

The arrangement is represented diagramatically in fig. 2. In the battery-circuit a rheostat was inserted, by means of which the current through the wire and so the temperature of the wire could be varied.

In the diagram *ab* is the experimental wire, and **BD** the thick wire standard-resistance.   Balance is obtained by varying the resistance of the arm AC.

By means of this arrangement measurements were made of the resistance of the wire when condensation resulted on expanding after the heating of the wire.   To reduce the determinations of resistance to centigrade temperatures, the resistance of the wire was determined first in melting ice, then in steam, and lastly when a very small grain of potassium

sulphate placed on it just melted.   From these three determinations the coefficient δ in Callendar's * formula

$$\theta - \pi = \delta \frac{\theta}{100}\left(\frac{\theta}{100} - 1\right)$$

was calculated.   Knowing δ, the value of the centigrade temperature θ corresponding to any value of the platinum temperature π could be calculated from the above formula.

Owing to the conditions of the experiment, however, the temperatures could not be measured to any very great degree of accuracy.   For it was found that when a current was sent through the wire, the latter did not attain a steady temperature for some time, ranging from a few seconds at the lower temperatures to two or three minutes at temperatures of the order of 1000 degrees, this being due to the air in the cloud-chamber gradually getting heated.   To wait after cutting off the current until the air in the cloud-chamber cooled down would involve a delay of some minutes before the expansion could be made, and in this interval the nuclei produced by the heating of the wire would have all disappeared by diffusion to the walls of the vessel.

By using, however, a dead-beat galvanometer, it was found possible after some practice to so manipulate the resistances in the variable arm that the bridge was balanced after the current had been passing for two or three seconds.   At the moment of balance the current was cut off, and the expansion made a few seconds later.   In this way the wire could be heated to different temperatures, which could be fairly accurately determined without the air in the cloud-chamber getting heated to any appreciable extent.

Results for Platinum Wire in Air.

With the method described above it was found that the wire must be raised to a fairly definite temperature in order that condensation may result on expanding, this minimum temperature depending upon the value of the expansion used to catch the condensation-nuclei produced, the relation being that the lower the temperature of the wire, the greater the value of the expansion required to get any condensation.   In other words, the smaller the expansion used, the higher must the temperature of the wire be raised in order that the production of condensation-nuclei may be observed.

This also means that the higher the temperature to which the wire is raised, the bigger are the condensation-nuclei resulting from the heating.

* H. L. Callendar, Phil. Mag. vol. xlviii. p. 519 (1899).

For a given expansion the number of nuclei caught increases rapidly with the temperature of the wire after reaching the minimum temperature at which the nuclei are caught with that expansion, the condensation changing from the form of a small shower, through denser and denser showers, into a fog. Similarly, for a given temperature of the wire, the number of nuclei caught increases with the expansion.

This shows that the effect of heating the platinum wire in air is to produce condensation-nuclei of all sizes up to a certain size—the maximum size depending upon the temperature of the wire.

The curves shown in Pl. XI. diagrams 1, 2, 3 were got for platinum wire in air by proceeding in the following way.

An expansion of a definite amount was arranged, and the wire heated for two or three seconds up to a certain temperature ; the expansion was then made and the result of the same noted. If no condensation resulted, the same process was repeated several times, with the wire raised each time to a slightly higher temperature, until condensation took place in the form of a small shower. A similar series of observations were made for a different value of the expansion. The curves show the relation between the lowest temperature to which it is necessary to raise the wire in order that condensation may commence for expansions of different amounts.

The points on the curves corresponding to expansions greater than 1·25 were got by noting at what temperature Wilson's rain-like condensation was increased in density by the heating of the wire.

It is seen from the curves that the production of condensation-nuclei by the heating of platinum wire in air to about 160° C. can be detected with large expansions. When the temperature of the wire is as high as 300° C., condensation-nuclei are detected with expansions as small as 1·1.

If the temperature of the wire be raised to about 400° C., the nuclei are so large and numerous that dense fogs can be obtained with the smallest expansions.

Results for Platinum Wire in Hydrogen.

When hydrogen was first substituted for air, no special precautions were taken to ensure its purity. The result obtained, however, is interesting. At the first heating of the wire nuclei were produced at a temperature not very different from what was obtained in the case of air ; but after making the wire red-hot for one minute it was found necessary to raise the wire to a much higher temperature (to about

400° C.) than before in order that condensation might result for an expansion of the same amount.

That this effect was not due to the wire having on it some dirt which was driven off when the wire was raised to red heat, is disproved by the fact that the wire had been made red-hot before the hydrogen was passed into the apparatus. In the light of later experiments this effect is very probably due to the hydrogen having been rendered purer by the high temperature of the wire causing the combination with hydrogen of the small quantity of oxygen in the impure gas. For in later experiments, when every precaution was taken to obtain hydrogen free from oxygen, it was found that the platinum wire must be raised to temperatures above a red heat before condensation-nuclei were produced.

Pure hydrogen was obtained from pure zinc and dilute pure HCl. The gas on its way to the expansion-apparatus was passed through wash-bottles containing boiled distilled water and a solution of potassium permanganate, then through tubes containing red-hot copper, calcium chloride, and tightly-packed cotton-wool.

The distilled water in the expansion-apparatus and in the cloud-chamber had been well boiled, and allowed to cool in an atmosphere of hydrogen. Before every experiment in hydrogen the wire was always heated for some minutes to a bright yellow heat. This was found to be necessary in order to obtain fairly consistent results.

The results obtained in pure hydrogen are of a nature similar to those obtained in air.

As in the case of air, a relation was found to exist between the expansion and the minimum temperature at which nuclei are produced which are caught with that expansion; and, as before, the nuclei were found to increase in size and number as the temperature was increased. But in the case of hydrogen the results are obtained at temperatures some six or seven hundred degrees higher than in air—in fact, not until the wire is luminous.

The curves giving the relation between the expansion and minimum temperature are roughly straight lines, and are shown in diagrams 4, 5. The fact that some points are a good deal off the curve is not so serious when one considers the experimental difficulties, such as the difficulty of measuring the temperature accurately when so high, and of distinguishing between the density of one shower and another. The extreme delicacy of the expansion method of detecting nuclei is another source of difficulty. Owing to the results not being very regular, a large number of experiments were made to

find out how the temperature to which the wire must be raised in order that condensation may commence did depend upon the expansion. The curves given represent the main features of the results obtained. The results plotted on diagram 6 were obtained in a different way from the others. Each point here represents the minimum expansion which gives condensation when the wire is at the corresponding temperature.

The condensation-nuclei produced in hydrogen persist for a much shorter time than they do in air. In hydrogen the nuclei (which would have given a dense shower had the expansion been made immediately after the heating-current was cut off) will have all disappeared in five minutes after the heating. It was often noticed though, that if an expansion be made immediately after the heating, those nuclei not carried down by the first shower will persist for a much longer time than would have been the case had the expansion not been made.

The following effect in hydrogen may also be mentioned. If a large number of nuclei have been produced by heating the wire to a bright yellow heat, then raising the wire to a dull red heat for a few seconds causes the nuclei to disappear entirely. This effect is possibly due to the property which a hot body has of removing the dust-particles from a vessel, which has been explained by Lodge [*] as due to the bombardment of the dust out of the air upon the cool wall-surfaces.

New Form of Cloud-Chamber.

It has been already mentioned that one of the experimental difficulties was to measure the temperature of the wire before it had attained a steady state, this being necessary to prevent the heating of the gas in the apparatus.

To be able to pass the current through the wire until it attained a steady state would be advantageous in two ways. In the first place the temperature could be more accurately determined, and secondly, the time during which the current was passing through the wire would be longer. To meet these ends the apparatus shown in fig. 3 was constructed.

The wire was fixed inside a separate glass bulb B, connected with the cloud-chamber A by means of a hollow ground-glass tap, the internal diameter of which at the narrower end was one centimetre.

The diameters of the bulbs B and A were respectively 5 cms. and $4\frac{1}{2}$ cms. The short tube connecting the larger bulb and the tap was 3 cms. long and 1·2 cm. internal diameter.

[*] 'Nature,' vol. xxxi. p. 265 (1885).

The apparatus was used in the following way :—The piston was allowed to ascend while the two bulbs A and B were in communication. The tap was then closed, and the heating-current sent through the wire, the resistance of which was measured when a steady state had been attained. The effect of the heating is to increase the pressure in B, so that if connexion be suddenly made between the bulbs by opening the tap, a puff of gas carrying with it condensation-nuclei will enter A out of B. The tap is then closed and the expansion made. The small puff of heated air will not appreciably heat the gas in A.

A further advantage possessed by this apparatus was that the wire not being in the cloud-chamber, no drops fell on the wire. As the current passed through the wire for a longer time than in the other experiments, it was thought that nuclei might possibly be detected at still lower temperatures. The reverse, however, was found to be the case, the expansion and minimum temperature curve for air lying on one occasion between the limits 300° C. and 500° C. This is probably due to the nuclei disappearing as fast as produced, owing to the presence of the hot wire as mentioned before. For this reason the experiments with this method were discontinued. The results got with this apparatus, however, go to prove that the production of nuclei depends not so much upon the length of time the wire is heated as upon the maximum temperature attained.

Effect of an Electric Field.

As a platinum wire at a dull red heat is known to radiate positive ions, and at higher temperatures negative corpuscles, the effect of putting an electric field on the wire was tried. The cloud-chamber used is shown connected to the expansion-apparatus in Pl. XI. fig. 1. The platinum spiral, connected to earth, was fixed at a mean distance of about 1·5 cm. from the water-surface. By means of a battery of small storage-cells the potential of the water could be raised in steps of 2 volts from ±2 volts to +120 volts.

It was thought that putting on an electric field between the wire and water-surface while the wire was being heated might lower (or raise) the minimum temperature at which nuclei are caught with any given expansion. It would follow from this that the condensation got (when there is no field) by raising the wire to a temperature a little higher than the minimum temperature would be altered in density if the wire were heated to that temperature with the field on.

No such effects, however, were observed in air or in

hydrogen, either with expansions below or above the value 1·25 necessary to catch negative ions. The potentials tried in this experiment were 2, 10, 40, 80, 120 volts positive and negative.

From this result, therefore, the nuclei produced by the heating of platinum wire in air and hydrogen are uncharged.

What these nuclei actually are it is impossible to say with absolute confidence. It is highly improbable that they are due to dirt on the wire. On one occasion a wire was kept red-hot for thirteen hours with a stream of filtered air flowing along it, and yet afterwards fogs were got by heating it to a temperature below 200° C.

There is an interesting analogy between the results described in this paper and those got by Dr. Walter Stewart * on the disintegration of platinum wires at high temperatures.

He found that a platinum wire at a clear red heat in air lost 0·84 per cent. of its weight in two hours ; at a white heat it lost 3·89 per cent.

In hydrogen he found that the wire does not disintegrate, even at a clear white heat.

The fact that Aitken, by heating a platinum wire in air to a red heat, got enough nuclei to give many dense fogs, without the wire appreciably losing weight, is not surprising when we consider the excessively small dimensions of the nuclei under consideration. For from C. T. R. Wilson's † calculation the radius in cms. of waterdrops equivalent in their action to the nuclei which are caught with expansions of 1·25 is about $8·6 \times 10^{-8}$, so that the nuclei are not much larger than molecules.

In view of the extreme delicacy of the expansion method of detecting nuclei, the production of nuclei by the heating of platinum wires *may* be looked upon as a case of disintegration of the wire. If this view of the results be taken, it is interesting to notice that a platinum wire disintegrates in air at a temperature below 300° C., and, contrary to Dr. Stewart's result, it does disintegrate in hydrogen at temperatures of the order of 1000° C.

In conclusion, I wish to acknowledge my indebtedness and sincere thanks to Prof. Thomson for his kind encouragement and valuable suggestions while these experiments were in progress.

* Phil. Mag. vol. xlviii. p. 481 (1899).
† Phil. Trans. A. vol. clxxxix. p. 306 (1897).

XXXIX. *An Automatic Mercury Vacuum-pump.* *By* S. R.
Milner, *D.Sc.* (*Lond.*), *Lecturer in Physics, University
College, Sheffield* *.

[Plate XII.]

THE vacuum-pump described in the following pages is
the outcome of an attempt made to modify the ordinary
form of Töpler pump into a simple and portable automatic
instrument of small size. It possesses an improvement in the
form of pump-head, and in the method of expelling the last
traces of air from the pump-bulb, which I believe conduces
to a greater efficiency in working than is the case with the
ordinary form, and may therefore make a description of the
instrument of interest.

The pump is of the shortened type worked by a water
vacuum-pump, and its general form may be seen by reference
to fig. 1 (Pl. XII.). (The separate parts are spread out in the
figure for the sake of clearness—in reality the whole apparatus
is contained on the two sides of a board 80 cms. high by 25 wide.)
The reservoir containing the mercury, A, is about 30 cms.
only below the pump-bulb B, and an electrically worked
three-way tap, T, connects the air space of A alternately to
the atmosphere and to a water vacuum-pump. This causes
the mercury to rise and fall in the pump-bulb, and to expel
the air from it into an " external vacuum " chamber E E',
which has been previously evacuated by the water-pump.

The tap T is caused to turn at the proper times by means
of contacts, made by the mercury of the pump, actuating
either of the pairs of coils K or K' (of which only one coil
each is shown in the figure). A current through K causes
the lever *l l'* attached to the plug of T to be pulled down on
the right-hand side to the correct position for the tap to
make the connexion *u* to *z*. This opens A to the atmosphere
through a drying-tube not shown in the figure. During the
motion of the lever a small screw *h* attached to it, by knocking
down the bent lever *i*, breaks the contact *ij* in series with K,
while it makes a contact *ij'* in series with the other coils K'.
The current through K which has produced the motion is
thus immediately cut off, but the *lever* remains in position
through the friction of the tap, until a current passes through
K'; this causes a similar action on the left-hand side and
a connexion *uv* of A to the water-pump to be made. All the
breakings of contact thus take place outside the pump, so
that no sparking ever occurs in the mercury, and a further

* Communicated by the Author.

advantage is that the current is only on while the tap is being turned, and there is thus no waste of electrical energy.

As is well known, for an efficient action of the Töpler pump the air from the pump-bulb must first be expressed into a fairly highly exhausted " auxiliary chamber," D, and from D finally, when a sufficient amount of air has been accumulated there, into the not very high external vacuum E E′ which is all that can be produced by a water-pump. Without some arrangement of this kind the impossibility of driving the very minute air-bubbles, trapped when a high stage of exhaustion is reached, down a mercury column necessarily several centimetres long soon puts a limit to the degree of vacuum obtainable. Hence results the necessity of arranging for two classes of pump-stroke in an automatic instrument, one—which we may call a " partial " stroke—expelling the air from B into D only, and the other—a " complete " stroke and requiring a longer time—in which the mercury goes right over into E, and expels into it the air previously accumulated in D. The necessity for complete strokes, which is continual at first, falls off very rapidly with the exhaustion, and the loss of time produced by allowing complete strokes to occur unnecessarily becomes considerable in the long run. The theoretical requirement for the least waste of time is clearly that the air in the auxiliary chamber should be expelled when, and only when, it has reached a certain limiting pressure—say half a centimetre of mercury ; this condition, which it is difficult to satisfy in the usual form of automatic pump, can be exactly satisfied in an electrically controlled apparatus. It is accomplished in the instrument by the gauge G attached to D, working in connexion with a small contact-breaker Br.

The gauge consists of the two tubes p and q joined by a short tube r at the bottom ; in q two electrodes Y Y′ are sealed about $\frac{1}{2}$ cm. above the entrance of r. The limb q contains just so much air as will extend down as far as the entrance of r, when it is under the pressure of the mercury column pr only, *i. e.* when there is a perfect vacuum in D. When the pump is set up and used for the first time it is clear that, after two or three complete strokes have been carried out, the gauge will automatically empty itself until just this quantity of air remains permanently trapped behind. The action of the gauge is that, whenever the contact Y Y′ is made by the rise of the mercury in q, a current passes through the coils of the contact-breaker Br, which is thus caused to break a contact Q in a circuit through the two electrodes X X′ in D and the tap coils K′. The effect of

this, and also of the modified form of pump-head adopted, will be best made clear by considering the whole action of the pump when set into operation. The modification in the form of pump-head, as is seen by a reference to the figure, consists in the entrance of the capillary $d\,d'$ into the side instead of the bottom of the auxiliary chamber, while this latter is connected below by a tube $f\,f'$ containing a mercury-trap F, with the tube $b\,b'$ leading from the reservoir to the pump-bulb.

Method of Action.

To set the pump in action the lever of the three-way tap is turned down on the left-hand side so as to connect u to v, and a preliminary exhaustion to about 10 cms. of mercury pressure is given by the water-pump to the vessel to be evacuated, to B, D, E E', and the air space of A simultaneously. The stopcock M is then turned off and the battery contact P closed, when the pumping will go on automatically.

A contact between the wires U and V in the reservoir is already made by the mercury there, so that a current flows round the circuit P U V K $j\,i$. (The electrical connexions are indicated by the dotted lines in the diagram.) Thus the tap is turned so as to connect the air space of A to the atmosphere; at the same time the lever $l\,l'$ breaks the circuit at $j\,i$, and prevents the sparking inside the reservoir which would otherwise occur as the contact U V is broken by the mercury leaving the reservoir. The speed of the mercury as it passes into the pump-bulb—and also up the side tube $f'\,f$ as far as the point f, where it is stopped by the mercury trap there—can be adjusted to have any value by a screw clip on a rubber tube connected to z, which regulates the rate of admission of air.

Meanwhile, there being air at a pressure of a few centimetres in D, the mercury of the gauge is pushed up in the tube g so as to make contact between the terminals Y Y', and, the contact $j''\,i$ of the tap-coil being now also made, a current passes round the circuit P U Y Y' Br K' $j'\,i$. By a suitable resistance R inserted in series with the coil Br, this current is made small enough to keep the contact Q broken without turning the tap T. Thus the latter is not turned until the mercury from the pump has expelled the air into E and reached the electrode Z, thus completing the circuit P U Z K' $j''\,i$. This turning puts the reservoir in connexion with the water-pump, and again immediately breaks the circuit of the battery at $j''\,i$ outside the pump. The mercury sinks in

B, and that in E and D comes back to the reservoir by the path ff'' (sealing columns being left in ee' and dd'), until the contact U V is made when the tap is again turned.

In the subsequent strokes the action depends on the state of the gauge G at the time when the mercury is passing through dd'. So long as the pressure of the air in D is under $\frac{1}{2}$ cm. of mercury the air in q will reach below the level of the electrodes Y Y', and break the circuit through Br. The contact Q is consequently unaffected, and the circuit P U X X' Q K' j' i will become closed, and the three-way tap turned, as soon as the mercury has filled up D to the level of X X'. With this state of the gauge the air is thus expelled only into D. As soon, however, as by the accumulation of air in D the pressure there has risen above the limiting amount, the contact Y Y' is made by the rise of the mercury in q, Q is broken, and the three-way tap is consequently not turned until the mercury from the pump-bulb has expelled the accumulated air in D into the external vacuum-chamber, and has closed the circuit P U Z K' j' i.

On stoppage of the water-pump the mercury simply rises in the apparatus to some point of E depending on the pressure of the air in EE', and remains there permanently, sealing off the vacuum-tube against any leakage of air through the pump. With the form of stopcock M, of which a description has already appeared in this Magazine *, leakage into EE' is practically nonexistent, and a partially exhausted tube can thus be left any length of time in connexion with the pump without its vacuum being affected.

Rate of Working.

The rate of working of an instrument of this kind of course depends chiefly on the efficiency of the water-pump. Apart from this the best effect is obviously obtained when the connexion c to the vacuum-tube is as high as possible above the reservoir A, as the latter then requires exhausting to the least extent. A limit to the height is, however, set by the fact that the height from the mercury in A up to the top of d or e (whichever is the higher) *plus* the vertical height of ee' must be under 76 cms. for pumping to go on at all stages of the exhaustion. The addition to the apparatus of a winchester W, acting as an air-reservoir to the water-pump, has also a beneficial effect on the efficiency. On the downward stroke the air in A now empties itself quickly into W, whence it is carried off more slowly by the water-pump. The result, with a reservoir of suitable size, is that the mercury in B falls very

* Phil. Mag. July 1903, p. 78.

quickly to the level of c, and from that point slowly until the contact UV is made. It is true that the rate of removal of the air from the apparatus by the water-pump is diminished by this addition, since the air has now to be drawn away at the lower pressure of W, but this effect is counterbalanced by the water-pump being now kept in practical operation during the up-stroke, as well as the down-stroke, of the mercury, and there is no appreciable increase in the period of a stroke. Thus the net result of the addition of W is that the period of contact of the vacuum-tube with the pump-bulb is materially increased without affecting the period of a stroke, which is exactly what is wanted at high exhaustions.

In the instrument, bd is 58, ee' 17, and bc 39 cms., and the pump-bulb has a capacity of about 250 c.c. A water-pump on the ordinary town supply gives a partial stroke in 25 seconds, and a complete one in 33, when a 5 seconds' interval of contact is allowed of the vacuum-tube with the pump-bulb. The interval may be given any value by altering the position of the wire V, which is adjustable.

It may also be worth noting that the use of a small pump-bulb, with its consequent saving in mercury, although of course it would be intolerable in the case of a hand-worked instrument, is advantageous in all respects when the action is automatic. For with an automatic instrument the time of a stroke is generally roughly proportional to the volume of the pump-bulb, and under these conditions the rate of exhaustion is *greater* with a small than with a large bulb. In fact, if we put V for the volume of the vacuum-tube, $v = x$V for that of the pump-bulb, and kv be the time occupied by a stroke (proportional to v), the theoretical ratio in which the pressure is reduced after pumping for the time t is

$$(1 + x)^{-\frac{t}{kx\mathrm{V}}}.$$

This expression has its minimum value $e^{-\frac{t}{k\mathrm{V}}}$ when $x = 0$, and tends towards unity as x increases indefinitely. The result of this reasoning is unaffected by the fact that at high exhaustions it is necessary to allow between the strokes a considerable interval of contact of the vacuum-tube with the pump-bulb, to let the air drain into the latter, as this interval should also clearly be made roughly proportional to the volume of the pump-bulb. Moreover, with instruments worked by a water-pump I find that in practice the time required for the stroke with a bulb of 250 c.c. can be made *less* than half that necessary with a bulb of 500 c.c., in consequence of the greater height bc obtainable, and the

resulting increased efficiency of the water-pump. Probably a reduction in size to 100 c.c. would be attended with similar advantage, but a limit to an advantageous decrease is of course soon reached, when the time taken by the mercury to stream through the capillary tubes becomes an appreciable portion of the whole.

Comparison of Pump-head with Ordinary Form.

In the ordinary form of pump-head, of which a diagram is given in Pl. XII. fig. 2, it is necessary to have the capillary *dd′* not only short, but bent so that the free surface of the mercury sealing-column remains permanently in the wider auxiliary chamber D: otherwise on the downward stroke the short column would be entirely sucked back into the pump-bulb.

This arrangement, however, possesses serious disadvantages. A very obvious one is the violent shock to the apparatus which necessarily accompanies the impact of the mercury from B against the extension into the capillary of the large mass of mercury in D *. Further, the very minute air-bubbles that are trapped at high exhaustions are in a mercury column continuous from B to D, and there is a tendency for them to stick, either onto the side of the capillary, or more particularly at the opening into the wider tube D. In this case, of course, they are liable to be drawn back into the pump-bulb on the downward stroke. It is impossible to ensure their expulsion from the capillary by much increase in the velocity of the mercury from B, both by reason of the increase in the shock which this produces, and of the fact that a high speed of the mercury as it comes up from B into the capillary is probably detrimental to the due escape of air into the latter. Finally, although one does not know, it is quite possible that traces of air are compressed into a film on the surface of the mercury thread by the sudden pressure which accompanies the impact, and if so, this air also will be largely returned to the pump-bulb.

In the modified form the mercury in the capillary is but a short sealing-column at the most two or three centimetres long (the lower bend of *dd′* is about 1 cm. high). The rate of entrance of air into the reservoir is adjusted so that the mercury is rising quietly by the time it has reached the

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* The inertia of the mercury in D made itself evident in an interesting, though at the time very undesirable, way in an earlier pump I endeavoured to construct, in which the electrode X (fig. 2) was in permanent connexion with the mercury in D, and an electrical contact was made when the mercury from the pump-bulb reached that in the capillary. At high vacua, on making contact in this way, sparking in the capillary invariably occurred, indicating that the violence of the impact produces a rebound of the two mercury masses from each other.

entrance to the capillary. Thus the air from B is pressed quietly into the capillary, and the mercury also enters it without shock. Then, on coming down the 6 or 7 cms. of the capillary, the mercury attains a high velocity by the time it strikes the short sealing-column at d'; and this, having so small a mass, is shot out without any perceptible shock into D, and opens up as it were a clear path for the trapped air-bubble to follow it—at any rate, the freer character of the motion has an obvious effect in diminishing its tendency to stick. The mercury is, moreover, projected in a thin stream through the fairly highly exhausted space of D, a process which is likely to considerably assist it to give up any air it may have condensed on it. On the downward stroke of the pump, the column in the capillary breaks with certainty at d, and the mercury in it drops wholly into D with the exception of the short sealing-column left in the lower bend of the tube. And, finally, all the mercury which has passed over into D (which may still have air entangled in it) passes back to the reservoir by the path ff', by the insertion in which of a series of bulbs on a capillary tube the dragging of trapped air back into the pump-bulb can be rendered practically impossible.

The smoothness of action of this form of pump-head is remarkable. The pump works quite silently with the exception of the mercury hammer effect of the mercury streaming into D. As regards its efficiency, I have not been able to make a direct comparison between the two forms under the same conditions, but the following results obtained with the pump show generally the time and number of strokes required with it to evacuate a space of 250 c.c. to certain stages.

On starting from a vacuum of about 8 cms. pressure produced by the water-pump, the first 10 strokes were complete, each occupying 33 secs., of which the vacuum-tube was open to the pump-bulb for 5 secs. By this time the dark space round the cathode had become 1·5 cm. long. The next stroke was partial (25 secs.), and the 12th complete. On now altering the position of the wire V so as to allow about a 20 seconds' interval of connexion between the tube and the pump-bulb, the next 10 strokes were all partial, and, 14 minutes from starting the pump, sufficed to make the dark space extend the whole length of the tube (16 cms.). The rate of exhaustion beyond this point is largely dependent on the rate at which air comes off from the glass of the tube, as well as on the efficiency of the pump. I have, however, obtained in a comparatively short time (*e. g.* for a constricted tube in about $\frac{3}{4}$ hr.) a vacuum through which the induction-coil would not discharge (16 cm. spark).

XL. *A General Theorem on the Transfinite Cardinal Numbers of Aggregates of Functions.* By Philip E. B. Jourdain, *B.A., Trinity College, Cambridge* *.

THE (transfinite) cardinal numbers of certain aggregates of functions have been stated by Cantor † and Borel ‡. The question can now be treated much more shortly on the basis of the rules of calculation for transfinite numbers introduced by Cantor § and completed by Whitehead ||. It forms the most general question that can be asked respecting aggregates of relations between elements whose *cardinal* number alone is given (*i. e.* in which we abstract from their ordinal types, if these exist).

In the following the letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{e}$ denote any finite or transfinite cardinal numbers, and $\aleph_o$ has the same meaning as it has with Cantor.

Consider a function determined by the following determinations:—

Calling, for shortness, every system of values of $\mathfrak{e}$ independent variables a " point " (of an arithmetic space ¶ of $\mathfrak{e}$ dimensions, $R_{\mathfrak{e}}$); suppose that to each such point belong $\mathfrak{b}$ values of a function, and each of these values can be chosen from an aggregate of cardinal number $\mathfrak{a}$. Then to each point of $R_{\mathfrak{e}}$ case belongs an aggregate of cardinal number

$$\mathfrak{a}^{\mathfrak{b}}.$$

Further, suppose that each variable need only take an aggregate of cardinal number $\mathfrak{d}$ of values in order that the function be completely determined for the *whole* of $R_{\mathfrak{e}}$. Then the cardinal number of all the " argument-points " required is

$$\mathfrak{d}^{\mathfrak{e}}.$$

Consequently, the cardinal number of all the functions considered is

$$(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{d}^{\mathfrak{e}}}. \quad \cdots \quad (1)$$

This general formula may now be applied to the cases hitherto known.

* Communicated by the Author.
† Math. Ann. xxi. p. 590 (1883).
‡ *Leçons sur la Théorie des Fonctions,* pp. 125, 126 (1898).
§ Math. Ann. xlvi. (1895).
|| Amer. Journ. of Math. xiv. (1902).
¶ We do not suppose that the domain of each variable in $R_{\mathfrak{e}}$ is necessarily of the cardinal number $\mathfrak{c}$ of the continuum.

Suppose, firstly, that we have to deal with real one-valued functions of one real variable, and both the argument and function values are certain continua ; say

$$a \leqq x \leqq b, \qquad c \leqq y \leqq d,$$

where a, b, c, d are any real numbers, which may be $\pm \infty$. Then $\mathfrak{c} = \mathfrak{b} = 1$, $\mathfrak{a} = \mathfrak{d} = 2^{\aleph_0} = \mathfrak{c}$, and (1) becomes

$$\mathfrak{c}^{\mathfrak{c}} = 2^{\aleph_0 \cdot \mathfrak{c}} = 2^{\mathfrak{c}} = 2^{2^{\aleph_0}}.$$

If, however (as in the case with continuous, analytic, ... functions), the $\mathfrak{d} = \aleph_0$ —since the function is completely determined for *every* argument x, by its datum for a certain *enumerable* aggregate among these argument values—then (1) becomes

$$\mathfrak{c}^{\aleph_0} = (2_0)^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0} = \mathfrak{c}.$$

Thus the aggregate of all real continuous* or analytic functions of one real (or complex) variable is at most of the cardinal number of the continuum.

That, further, these aggregates are also *at least* of cardinal $\mathfrak{c}$ follows from the fact that there is a partial aggregate of this cardinal (*e. g.*, linear functions). Hence by the Schröder-Bernstein theorem, the cardinal is exactly $\mathfrak{c}$.

These two results have been known, but their derivation from the general formula (1) appears interesting.

2.

The property that their cardinal number is $\mathfrak{c}$ is possessed by all functions to which an " existence-theorem " is applicable. For such functions, when they cannot be constructed by a finite number of operations on the variables and undetermined constants (*i. e.* when they are not rational functions), must be representable as the limiting function of a fundamental sequence of functions which are either rational or limits of a sequence of rational functions. Now the cardinal number of the aggregate of all functions defined by fundamental series (an enumerable aggregate) is

$$\mathfrak{c}^{\aleph_0} = \mathfrak{c}.$$

This result includes all Cantor's results, except the one treated above—the case $\mathfrak{c}^{\mathfrak{c}}$,—and that on the aggregate of *integrable* functions. This last we shall now consider.

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* Or even of functions continuous except for an aggregate of cardinal $\aleph_0$.

3.

That a function should be integrable, it is necessary and sufficient that the aggregate of its discontinuities should be a content-less aggregate. Two functions which coincide except at the points of such an aggregate have the same integral. Thus the values of the function at these points are arbitrary within finite limits), and may thus independently take values of an aggregate of cardinal number $\mathfrak{c}$ without altering the integral.

Now, from the investigations of Harnack and others, we know that content-less aggregates of cardinal number $\mathfrak{c}$ exist. Hence to every single definite integral belong an aggregate of, so to speak, *equivalent* integrable functions of cardinal number

$$\mathfrak{c}^{\mathfrak{c}}.$$

Since every integral is a continuous function of its upper limit, the cardinal number of the totality of integrable functions is at most $\mathfrak{c}$; and since also

$$\mathfrak{c} \cdot \mathfrak{c}^{\mathfrak{c}} = \mathfrak{c}^{\mathfrak{c}+1} = \mathfrak{c}^{\mathfrak{c}},$$

we conclude that the cardinal number of the aggregate of all integrable functions (of one real variable) is

$$\mathfrak{c}^{\mathfrak{c}} = 2^{\aleph_0 \cdot \mathfrak{c}} = 2^{\mathfrak{c}} = 2^{2^{\aleph_0}}.$$

Further, we know that

$$\mathfrak{c}^{\mathfrak{c}} > \mathfrak{c} \, *.$$

So this aggregate has a greater cardinal number than Cantor† appeared to think.

4.

By application of the formula (1) we see at once that the cardinal number of all functions (whether one-, finitely-, $\aleph_0$-, or $\mathfrak{c}$-valued) of $\mathfrak{c}$ variables, which are defined in a continuum, is

$$\mathfrak{c}^{\mathfrak{c}^{\mathfrak{c}}} = 2^{2^{2^{\aleph_0}}};$$

and

$$\mathfrak{c}^{\mathfrak{c}^{\mathfrak{c}}} = (\mathfrak{c}^{\mathfrak{c}})^{\mathfrak{c}^{\mathfrak{c}}} > \mathfrak{c}^{\mathfrak{c}}.$$

* Cantor, *Jahresber. d. d. M.-V.* i. (1894).
† Math. Ann. xxi. p. 590 (1883).

5.

We may finally notice the fact that the property of an analytic function of not being more than $\aleph_0$-valued (Cantor, Vivanti, Volterra, Poincaré) has no influence on the cardinal number of the aggregate if the function has $\aleph$ variables. In fact, if the function has ν variables (where ν is any finite cardinal number), the cardinal number is

$$\left(\mathfrak{c}^{\aleph_0}\right)^{\aleph_0^{\nu}} = \mathfrak{c}^{\aleph_0} = \mathfrak{c} \; ;$$

but if the function has $\aleph_0$ variables, the number is

$$\mathfrak{c}^{\aleph_0^{\aleph_0}} = \mathfrak{c}^{\mathfrak{c}} \; ;$$

and this would not be altered if $\mathfrak{b}$ were equal to $\mathfrak{c}$.

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XLI. *The Axial Dioptric System.* By R. F. MUIRHEAD, *M.A., B.Sc.**

Preliminary.

THE subject of this article is one which has received much attention from mathematicians both before and after the publication of Gauss's *Dioptrische Untersuchungen,* in which the idea of characterizing the axial dioptric system by means of four Cardinal Points was first stated and applied. Most of the papers on the subject which have come after Gauss's have been based on his fundamental conception, and have aimed at simplification or further development.

The present essay adds one to the number of these. If it has merit that will be found, I believe, chiefly in the simplified geometrical treatment of the fundamental theory given in the first section, which contains in a short space a direct geometrical proof of the existence and main properties of the Cardinal Points. The later Sections which aim at greater generality of treatment may be found interesting from the mathematical point of view.

By an *Axial Dioptric System* we mean a set of homogeneous media whose mutual boundaries are spherical surfaces having all their centres on a straight line called the *axis* of the system.

We take the conceptions of Ray and Focus in their usual

sense, as abstract geometrical conceptions suggested by a very narrow beam, and a small region towards or from which rays converge or diverge; and further we shall keep the usual restrictions as to the distances of points from the axis and the inclinations of rays to it being so small that we may reject the squares of these small quantities. In order to avoid infinitesimals in our diagrams we shall suppose these to be distorted by having the measurements at right angles to the axis depicted on an indefinitely larger scale than those parallel to the axis. Thus the whole theory serves as a first approximation to the representation of the actual phenomena.

Several convenient new terms will be introduced in the course of the article; for the present we may remark that we shall avoid the use of the word *conjugate*, which is apt to be misleading on account of its use in other branches of mathematics to denote a relationship which is reciprocal in its character. We shall speak of object-points and image-points and their correspondence; also of in-rays and out-rays when we need brief terms for incident and emergent rays.

We shall suppose throughout that a strict convention as to the *signs* of the various quantities is adhered to. We shall understand that all lines are continued indefinitely in both directions, even in the case of rays belonging to media of limited extent, so that for geometrical purposes we may suppose the whole of space occupied by as many different media, each with its points and rays, as there are separate media in the dioptric system.

Before entering on our special method we shall state a proposition with regard to refraction at a single spherical surface, omitting for brevity the proof, which is easy and well known.

Proposition A.—When a thin pencil of rays is incident on a single refracting surface, each ray meeting the surface nearly at right angles, it emerges as another pencil, the focus of the emergent pencil being collinear with that of the incident pencil and the centre of the sphere; and the distances of these two foci from the centre having a relation depending only on the refractivity of the surface and the radius of the sphere.

Section I.

From Proposition A it follows that if a pair of points be equally distant from the centre so are their images.

Hence if $A_1 B_1$ (fig. 1, p. 328) be two points equally distant

from O, the centre of the spherical surface, and $A_2 B_2$ their images, then, in the limit, $A_1 B_1$ and $A_2 B_2$ are both at right angles to $A_1 O A_2$ and coplanar with it.
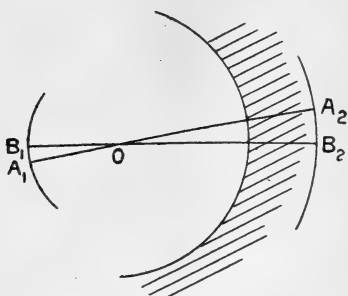
Fig. 1.   Fig. 2.
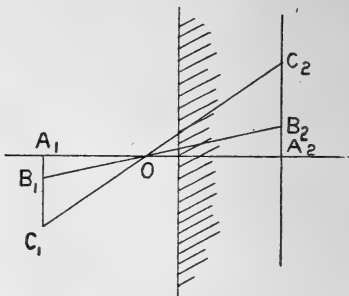
Also if $A_1 B_1 C_1$ (fig. 2) are three points in a straight line perpendicular to $A_1 O A_2$, then in the limit $A_2 B_2 C_2$ are in a line perpendicular to $A_1 O A_2$ and coplanar with $A_1 B_1 C_1$, and we have by similar triangles

$$\frac{A_1 B_1}{A_2 B_2} = \frac{A_1 O}{A_2 O} = \frac{A_1 C_1}{A_2 C_2}$$

$$\therefore \frac{A_1 B_1}{A_1 C_1} = \frac{A_2 B_2}{A_2 C_2}.$$

By symmetry it is clear that the ratio $\dfrac{A_1 B_1}{A_2 B_2}$ is unaltered when the line $A_1 B_1$ turns round about $A_1 O$, keeping at right angles to it, while A_1 remains fixed.

Thus if we have any number of points in the plane through A perpendicular to $A_1 O$, they have for their images a set of points forming a similar and similarly situated figure in the plane through A_2 perpendicular to $A_1 O A_2$.

Now consider an axial system of spherical refracting surfaces. Let $A_1 A_2 A_3 \dots A_n$ be a point on the axis and its successive images. The points in the plane of A_1 (the plane through A_1 perpendicular to the axis) form a figure which is similar and similarly placed to its first image in the plane of A_2, therefore also to its second image in the plane of A_3, and so on. Finally, we deduce that the last image in the plane of A_n is similar and similarly situated to the original object consisting of a plane figure in the plane of A_1. Hence:

Proposition B.—The lines joining the object-points in the plane of A_1 to their image-points in the plane of A_n are all concurrent in a point a on the axis, which we shall call the *vertex* for the plane of A_1 (or for any point in the plane of A_1).

Again, all points on a given ray incident on one spherical refracting surface have images through which the corresponding emergent ray passes. Hence, the image of a point is known when we know the emergent rays corresponding to two incident rays through the object-point; and the line of an emergent ray is known when we know the images of two points on the incident ray. This is obviously at once extensible to the in-rays and out-rays of a dioptric system.

The properties arrived at in the two preceding paragraphs form the basis of the geometrical treatment adopted in this article.

Given the axis of a dioptric system, and two extra-axial points P, Q, *with their images* P', Q', *to construct the out-ray corresponding to any given in-ray.*

Let PA, QB, P'A', Q'B' (fig. 3) be perpendiculars on the axis. Let PP', QQ' cut the axis in α, β. Then α is the vertex for A and β the vertex for B.
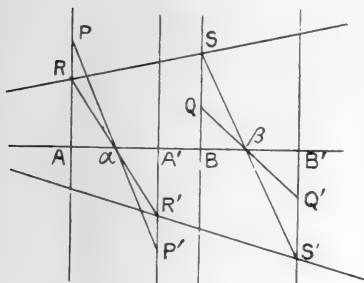
Fig. 3.   Fig. 4.
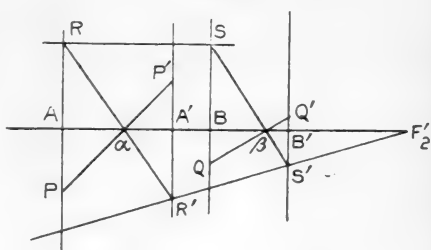
Let any in-ray cut the planes of A and B in R and S. Let Rα meet the plane of A' in R' and Sβ meet the plane of B' in S'. Then R' is the image of R and S' that of S.

Hence R' S' is the out-ray for the in-ray RS.

To construct the image for any object-point.—Take any two in-rays passing through it. The intersection of their out-rays gives the required image.

Now modify the above construction by supposing the object-point at infinity. Draw RS parallel to the axis (fig. 4) and find R' S' as before. Let it cut the axis in F_2'. All in-rays parallel to the axis pass through F_2' after emergence.

Thus F_2' is the *second focal point* of the system. Starting with an out-ray parallel to the axis, and reversing this construction, we get F_1 the *first focal point* of the system.

Next let P′ (fig. 5) be the image of any point P. Draw PU′ parallel to the axis to meet P′ F_2' in U′. Draw P′ V parallel to the axis to meet P F_1 in V. Let the perpendiculars from V, U′ on the axis meet the axis in H, H′ and PU′, P′ V in U, V′.
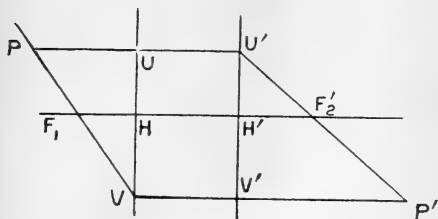
Fig. 5.   Fig. 6.
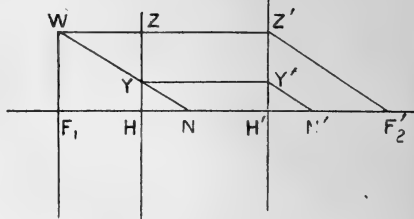
Since the in-ray P F_1 V has V V′ P′ for out-ray, it follows that the image of V is in VP′, and hence that the *vertex* for H is at infinity. Hence the image of U is in PUU′. But it also lies in U′ F_2' P′. Hence U′ is the image of U. Hence H′ is the image of H.

The *vertex* of H being at infinity, the *magnification* for the plane of H is unity. Hence H and H′ are the *first and second principal points*.

From W (fig. 6), a point in the first focal plane, draw W Z Z′ parallel to the axis to meet the planes of H and H′ in Z and Z′. Z′ is the image of Z, and Z′ F_2' is the out-ray for W Z, since W Z is parallel to the axis. Draw W N parallel to Z′ F_2' to meet the axis in N. Since W is on the focal plane, all in-rays through it emerge parallel to one another, and therefore to Z′ F_2'. Hence the out-ray for W N is parallel to it.

To find the position of its out-ray let W N cut the plane of H in Y, and draw Y Y′ parallel to the axis to meet the plane of H′ in Y′, and draw Y′ N′ parallel to W N to meet the axis in N′. Then Y′ is the image of Y, and therefore the out-ray for W N passes through Y′, and is in the line N′ Y′.

Since F N = H′ F′ and H N = H′ N′ it follows that the positions of N and N′ are independent of that of W, so long as W is in the first focal plane.

Hence all in-rays through N have their out-rays through N′ and unaltered in direction. Thus N and N′ are the *first and second nodal points*.

We have seen that when the axis of the system is given the whole system becomes determinate when two extra-axial object-points (not in the same normal plane) and their images are given. The following data are also sufficient to enable

us to construct the image of any object: the axis, one extra-axial point A with its image A′, and two other *axial* points K. L with their images K′ L′ (fig. 7).

Fig. 7.
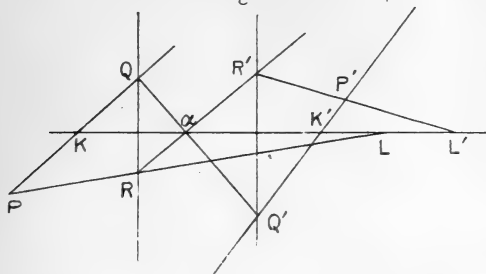
The intersection of A A′ with the axis gives α the *vertex* for A. Let P be any object-point. Draw K P Q meeting the plane of A in Q. Draw Q α Q′ meeting the plane of A′ in Q′. Then K′ Q′ contains the image of P.

Draw L R P cutting the plane of A in R, and R α R′ cutting the plane of A′ in R′. Then L′ R′ contains the image of P.

Hence P′ the intersection of K′ Q′ and L′ R′ is the required image.

Now if we send α to infinity A and A′ lie in the first and second principal planes. If we send K and L′ to infinity, L and K′ become the first and second focal points.

The ordinary construction for the image of a point by the aid of focal points and principal points is obviously a particular case of that just given.

SECTION II.

Correlative Fundamental Theory.

Corresponding to the whole of the theory of the first section based on Proposition B there can be developed another having a dualistic relation to it, based on the theorem that all in-rays through an axial point meet their corresponding out-rays in a certain plane normal to the axis.

Let A P (fig. 8) be a line perpendicular to the axis A A′, and A′ P′ its image, and let P P′ cut the axis in α, so that α is the vertex for A. Let α′ be the image for α. Then the in-ray α P meets its out-ray α′ P′ in P′. So long as P is in the plane of A the positions of α and of A′ are fixed. The plane of A′ we shall call the *base-plane for* α, or for any in-ray passing through α. Thus we have :—

Proposition C.—All in-rays through a point α on the axis intersect their out-rays in a fixed plane normal to the axis, the *base-plane for* α.
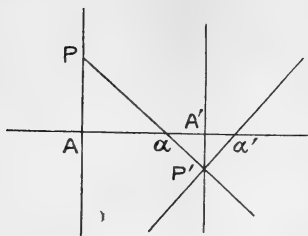
Fig. 8.   Fig. 9.
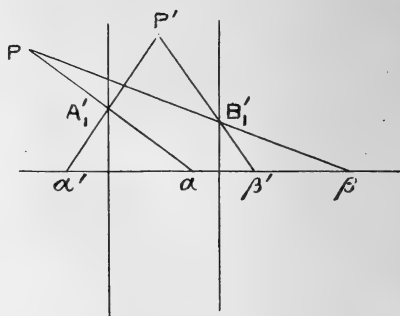
Hence the following construction (fig. 9) for the image of a given point P, having given two points α, β on the axis, their images α' and β', and their base planes. Draw P α, P β meeting the base-planes of α, β, in A_1' B_1'. Then P', the intersection of α' A_1' and β' B_1' is the image of P.

Nodal Points.—If the base-plane of α is at infinity it is clear that α and α' coincide with N and N', the first and second nodal points of the system.

Thus the property given in Proposition C may be looked on as a generalization of the property of Nodal Points.

We may note that if two axial object-points A and B with their images A' and B' and their *vertices* α and β are given, as well as the planes of A, B, A' and B', then, as shown in the last section, the out-ray for any given in-ray can be constructed by drawing *three straight lines*, each joining a pair of points, the third being the out-ray itself.

On the other hand, if A and B, their images A' and B', and their base-planes a and b are given, we can construct the image of any given object-point by drawing *four straight lines* as shown in Proposition C.

We may collect the results of our generalization as to the cardinal points into a single figure as follows :—

Let A_0 be a point on the axis of which A is the image, and let A' be the image of A. Let E be the *vertex* for A_0, and E' its image. Let B be the point for which A is vertex, and B' its image.

Then if A goes to infinity A_0 and A' become the first and second focal points, E and E' the first and second nodal points, and B, B' the first and second principal points.

Double Points and Planes, or Bravais Points and Planes.

It was Bravais who pointed out in 1851 the existence of two points on the axis, each of which is its own image. The easiest method of determining these points is probably by expressing algebraically the relation between the abscissæ of an axial point and its image (which can easily be deduced from our fundamental construction) and then equating these abscissæ, which gives a quadratic equation for the abscissa of a double point. But for the sake of uniformity of treatment we shall investigate the double points without using algebra (fig. 10).

Fig. 10.
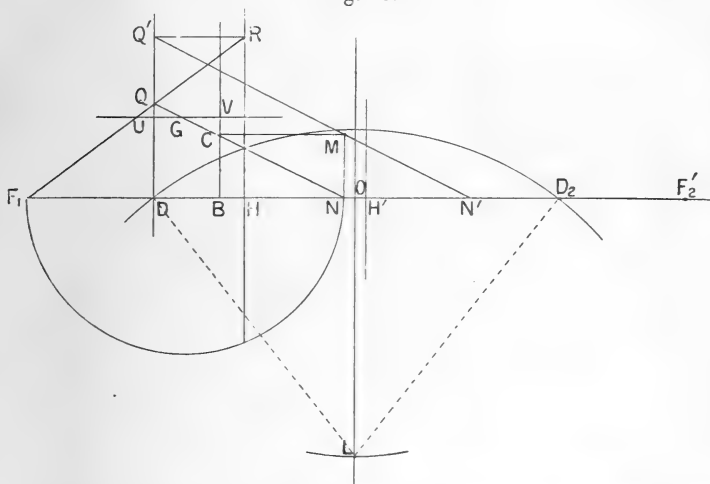
Let F_1, F_2', H, H', N, N' have the same significations as before. Take B on the axis so that $NB = H'H$. Let F_1R be any ray through F_1, meeting the plane of H in R.

Let a line parallel to the axis, equally distant from H and R, meet F_1R in U, and the plane of B in V.

Through N draw a line meeting F_1R in Q, U V in G, and B V in C, such that $CG = GQ$. (In Thomas Simpson's 'Geometry,' a construction is given for a problem including this as a particular case. It has two solutions, real or imaginary.)

Draw $QQ'D$ perpendicular to the axis, meeting the axis in D and a line through R parallel to the axis in Q'. D is a double point.

For $QQ' = BC$. Hence if we complete the parallelogram $Q'QNM$, we have $NM = CB$.

Hence the triangles CBN, MNN' are equal in all respects.

Hence $M N'$ is parallel to $C N$. Hence Q', M, N' are collinear.

Thus the out-ray for $F_1 Q$ is $R Q'$ and the out-ray for $Q N$ is $Q' N'$. Hence Q' is the image of Q. Hence all points in $Q D$ have their images in the same line, and D is its own image.

Having proved the existence of two Double-Points, which we shall name D_1 and D_2, we may proceed as follows:—

If Q be any point in a double-plane its image Q' may be got by drawing $N' Q'$ parallel to $N Q$ to meet the double-plane in Q'.

It follows that $D Q' : D Q = D N' : D N$, a ratio independent of the position of Q in the plane of D.

This of course is a special case of the theorem that a set of points in a plane perpendicular to the axis form a figure similar to its own image.

The general construction for the image of a given object-point when the axis and two extra-axial object-points with their images are given, requires modification when a given point and image have the same plane, *i. e.* when they lie in a double-plane. The *vertex* of the point then coincides with the double-point. But the constancy of the ratio $D Q' : D Q$ gives an obvious construction. If P and P' be the given point and image in the plane of D, and the image of Q (also in D) be required, we have only to make $D Q' : D Q = D P' : D P$.

Again, taking
$$f = F_1 H = N' F_2', f' \equiv F_1 N = H' F_2', \ h \equiv H H' = N N',$$
$$x \equiv F_1 D,$$
we have, since
$$F_1 D : F_1 H = D Q : H R = D Q : D Q' = D N : D N'$$
$$\frac{x}{f} = \frac{F_1 N - x}{F_1 N' - x} = \frac{f' - x}{f' + h - x}$$
$$\therefore \ x^2 - x(f + f' + h) + ff' = 0.$$

Hence the midpoint between D_1 and D_2 also bisects $N H'$, $H N'$, and $F_1' F_2'$.

Let this point be called O. It may be named the *midpoint of the dioptric system.*

We have $\qquad O D_1{}^2 = O D_2{}^2 = O F^2 - F H . F N.$

This affords another construction for D_1 and D_2 when the cardinal points are given.

Bisect $N H'$ in O and draw $O L$ perpendicular to the axis having its length a mean proportional between f and f'. With L as centre and $O F_1$ as radius, describe a circle which will cut the axis in D_1 and D_2.

Section III.

In Section I. we showed that when the axis and two extra-axial points with their images are given we can in general determine the image of any point, and the cardinal points of the system. The question arises as to whether any necessary relation exists between four such points and the axis, or whether we can always find a dioptric system for which a line and four points taken at random are axis, object- and image-points. The latter alternative would be proved true if we could show that the fundamental construction given in Section I. gives a transformation of object-points into image-points in such a way that any two pairs of these being taken for the given points, the same transformation as before would be arrived at. For in that case we should know that, starting from four cardinal points, the same transformation is arrived at, and it is easy to show that by means of three media and two refracting surfaces we can form a dioptric system having any four arbitrarily chosen cardinal points on an axis.

Let C A, C A′, C B, C B′ be four lines concurrent in C, and α, β two fixed points. Let P Q, any line in the plane of the figure, cut C A in P, C B in Q. Let P α meet C A′ in P′ and Q β meet C B′ in Q′.

Thus the position of the line P′ Q′ is uniquely determined by that of the line P Q. To denote this relationship we shall call P′ Q′ the *new line* corresponding to the *old line* P Q.

The only old lines in the plane for which the construction fails to determine the new lines are those passing through C. For such lines the points P Q P′ Q′ all coincide with C, so that the new line is only partially determinate. We might by applying the method of limits extend our definition so as to make such new lines fully determinate, but we prefer to complete the theory without having recourse to that method.

We proceed to prove that (subject to the exception above mentioned) all old lines through a given point have new lines which are concurrent.

Let L P_0 Q_0, L P_1 Q_1, L P_2 Q_2 be three straight lines through L cutting C A in P_0, P_1, P_2 and C B in Q_0, Q_1, Q_2. Draw P_0 α $P_0′$, P_1 α $P_1′$, P_2 α $P_2′$ meeting C A′ in $P_0′$, $P_1′$, $P_2′$, and draw Q_0 β $Q_0′$, Q_1 β $Q_1′$, Q_2 β $Q_2′$ meeting C B′ in $Q_0′$, $Q_1′$, $Q_2′$. Denoting by O$\{$A B C D$\}$ the anharmonic ratio of the pencil O A, O B, O C, O D, we have

$$\alpha\{C P_0′ P_1′ P_2′\} = \alpha\{C P_0 P_1 P_2\} = L\{C P_0 P_1 P_2\}$$
$$= L\{C Q_0 Q_1 Q_2\} = \beta\{C Q_0 Q_1 Q_2\} = \beta\{C Q_0′ Q_1′ Q_2′\}.$$

Thus the equi-anharmonic ranges C $P_0′$ $P_1′$ $P_2′$ and

$C\,Q_0'\,Q_1'\,Q_2'$ have a common point C.   Hence $P_0'\,Q_0'$, $P_1'\,Q_1'$ and $P_2'\,Q_2'$ are concurrent in a point which we shall call L′.

Thus all old lines through L have their new lines through L′.   The point L′ thus uniquely determined when L is given we define to be the *new* point corresponding to the *old* point L.

It is clear that (subject to the exception previously mentioned) every old line in the plane has a new line, and every old point has a new point, and that all the old points in an old line have their new points in the new line corresponding to it.

Take now the following construction :—

Let L and l be points collinear with C.

Draw $L\,P_1\,Q_1$ to meet $C\,A$ in P_1, $C\,B$ in Q_1.

   ,, $l\,P_2\,Q_1$,, $C\,A$ in P_2.

   ,, $L\,P_2\,Q_2$,, $C\,B$ in Q_2.

   ,, $l\,P_1\,q_1$,, $C\,B$ in q_1.

Let $L_2\,P_2\,Q_2$ and $l\,P_1\,q_1$ intersect in O.   Thus $C\,l\,P_1\,Q_1\,P_2$ is a complete quadrilateral.   Hence $L\,O\,P_2\,Q_2$ and $l\,O\,P_1\,q_1$ are harmonic ranges.

Now draw $P_1\,\alpha\,P_1'$, $P_2\,\alpha\,P_2'$, $Q_1\,\beta\,Q_1'$, $Q_2\,\beta\,Q_2'$, $q_1\,\beta\,q_1'$ to meet $C\,A'$ in P_1', P_2' and $C\,B'$ in Q_1', Q_2', q_1'.

Let $P_1'\,Q_1'$ and $P_2'\,Q_2'$ intersect in L′, $P_2'\,Q_1'$ and $P_1'\,q_1'$ in l', and $L_1'\,P_2'\,Q_2'$ and $l'\,P_1'\,q_1'$ in O′.

$$Now\ O'\{Cq_1'\,Q_1'\,Q_2'\}=\beta\{Cq_1'\,Q_1'\,Q_2'\}=\beta\{Cq_1\,Q_1\,Q_2\}$$
$$=L\{Cq_1\,Q_1\,Q_2\}=L\{l\,O\,P_1\,q_1\}=-1.$$

Thus $Cq_1'\,Q_1'\,Q_2'$ is a harmonic range.

Hence by the converse of the quadrilateral property, L′ l' C are collinear [*].

Thus we have proved that for the old points in $C\,L$, any line through C, the new points all lie in a certain line $C\,L'$. This construction fails when $C\,L$ coincides with $C\,A$ or $C\,B$, but it is obvious, by the former construction, that the new point for P, any point in $C\,A$, is P′ in $C'\,A'$, where $P\,\alpha\,P'$ are collinear.

We can now complete the definition of *new lines* in the cases previously excepted, by laying down the principle that

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[*] To prove this formally.   Let $C\,P_1'\,P_2'$ cut $l'\,Q_2'$ in p_2'.   Then

$$P_1'\{L'\,O'\,P_2'\,Q_2'\}=P_1'\{Q_1'\,q_1'\,C\,Q_2'\}=-1\;;$$

and $Q_2'\{P_1'\,P_2'\,p_2'\,C\}=l'\{P_1'\,P_2'\,p_2'\,C\}=l'\{q_1'\,Q_1'\,Q_2'\,C\}=-1\;;$

$$\therefore\ Q_2'\{P_2'\,p_2'\,C\,P_1'\}=-1.$$

Thus the harmonic pencils $P_1'\,(L'\,O'\,P_2'\,Q_2')$ and $Q_2'\,(P_2'\,p_2'\,C'\,P_1')$ have a common ray $P_1'\,Q_2'$.   Hence the intersections of the other corresponding rays are collinear, *i.e.* L′, l', and C are collinear.

for lines which pass through C, the new lines are the loci of the new points corresponding to the old points on these lines.

Thus in the transformation defined here, to every old line or old point there corresponds a new line or new point such that the old lines through the old point have their new lines through the corresponding new point, and the old points in an old line have their new points in the corresponding new line, and *vice versâ*.

It is obvious that C is its own new point and $\alpha\beta$ its own new line.

We shall next prove that the lines joining the points on any line C L through C to their new points, are concurrent in a point lying in $\alpha\beta$.

Let $L_0 P_0 Q_0$, $L_1 P_1 Q_1$, $L_2 P_2 Q_2$ be three straight lines concurrent in R, meeting C A in P_0, P_1, P_2 and C B in Q_0, Q_1, Q_2 and C L in L_0, L_1, L_2; and let P_0', P_1', P_2', Q_0', &c. be the corresponding new points.

Then $P_0' Q_0'$, $P_1' Q_1'$, $P_2' Q_2'$ will be concurrent in R′, and will meet the new line for C L in L_0', L_1', L_2', where L_0', L_1', L_2' are the new points for L_0, L_1, L_2.

We have

$$R'\{C\ L_0'\ L_1'\ L_2'\} = R'\{C\ P_0'\ P_1'\ P_2'\} = \alpha\{C\ P_0'\ P_1'\ P_2'\}$$
$$= \alpha\{C\ P_0\ P_1\ P_2\} = R\{C\ P_0\ P_1\ P_2\} = R\{C\ L_0\ L_1\ L_2\}.$$

Thus C $L_0'\ L_1'\ L_2'$ and C $L_0\ L_1\ L_2$ are equi-harmonic ranges, so that $L_0 L_0'$, $L_1 L_1'$, and $L_2 L_2'$ are concurrent in a point λ.

Further, since $\alpha\beta$ is a line which corresponds to itself, its intersections with C L and C L′ are corresponding points, whose join therefore passes through λ. Thus λ lies in $\alpha\beta$.

The point λ we shall call the *vertex* for C L, or for any point in it. It is the point of concurrence of all the lines joining old points in C L to their new points.

It is now obvious that we should get exactly the same transformation as before if we were to employ C L, C L′, and λ, and any other old line through C with its new line and its vertex, instead of C A, C A′, α, and C B, C B′, β.

It is easy to see that if C goes to infinity in a direction perpendicular to $\alpha\beta$, then that part of the system which remains at a finite distance from $\alpha\beta$ reduces to the system studied in Section I., the *old* and *new lines* becoming *in-* and *out-rays*; and the *old* and *new points* becoming *object-points* and *image-points*, while $\alpha\beta$ becomes the axis.

Reverting to the general transformation from old points to new points, we shall next prove the dualistic correlative to the vertex-property to hold good.

Let α be the vertex for any point P, and P′ the corresponding new point, and let α' be the new point for α. The line α P P′ looked at as an old line has its new line passing through α' and P′. Thus P′ is the intersection of the old line α P and its new line.

Hence C P′, the new line whose old line has α as vertex, is the locus of the intersections of all old lines through α with their corresponding new lines. C P′ we shall call the *base-line* for α.

Thus we have the Proposition :

All the old lines through an old point on the axis intersect their corresponding new lines in a fixed line through C, the base-line of α.

By means of this property, we have the following simple construction to find the new point corresponding to any old point R, having given two points α β on the double-line, their new points α' and β', and their respective *base-lines*, C A′ and C B′.

Draw R α P′, R β Q′ meeting C A′ in P′, C B′ in Q′. Then R′, the intersection of α' P′ and β' Q′, is the new point for R.

Double-Points.

In our general transformation it appears in virtue of the vertex-property that the anharmonic ratio of points in any line through C is unaltered by the transformation, and this can be shown to be true for other lines as well.

For if P Q R S be four collinear old points, and these be joined to a point O, we get a pencil cutting any line C L through C in $p\,q\,r\,s$, a range equi-harmonic with P Q R S. Now transform the whole figure, and we get two equi-anharmonic ranges $p'\,q'\,r'\,s'$ and P′ Q′ R′ S′, of which the former is equi-anharmonic with $p\,q\,r\,s$. Hence {P Q R S} = {P′ Q′ R′ S′}.

Now the line $\alpha\,\beta$ corresponds to itself. Hence it must have two double-points D_1 and D_2, real or imaginary.

Thus C D_1, C D_2, and $D_1\,D_2$ are *double-lines* (lines which transform into themselves) and C, D_1, D_2 are *double-points.*

A double-point has its vertex in coincidence with itself, so that the construction of new points has to be modified when the given object-points lie in the double-lines. We may proceed thus :—

Let P_1 be an old point on C D_1 and P_1' its new point. Through C draw any other line (C D_2 would do) cutting $D_1\,D_2$ in E, and from any arbitrary point O in $D_1\,D_2$ draw O P_1, OP_1' cutting C E in p and p'.

Let Q_1 be another old point in $C\,D_1$. Draw $Q_1\,p$ cutting $D_1\,D_2$ in R.

Draw $R\,p'$ cutting $C\,D_1$ in Q_1', the new point for Q. The proof depends on the fact that the anharmonic ratio of four collinear points is unaltered by the transformation, for by construction

$$0\{C\,D_1\,P_1\,P_1'\}=0\{C\,D_1\,p\,p'\}=R\{C\,D_1\,p\,p'\}=R\{C\,D_1\,Q_1\,Q_1'\}$$
$$\therefore \ \{CD_1\,P_1\,P_1'\}=\{CD_1\,Q_1\,Q_1'\} \ \therefore \ \{CD_1\,P_1\,Q_1\}=\{CD_1\,P_1'\,Q_1'\}.$$

We can thus find the new point for any old point in a double-line, if *one* old point in that line along with its new point are given.

Section IV.

We shall now give a brief analytical treatment of the general transformation of Section III.

Taking C′ A B A′ B′ α β P Q P′ Q′ to denote the same points as before, let us take A B C as the fundamental triangle for areal coordinates, and let the coordinates of the various points be as indicated here :—

	A.	A′.	B.	B′.	α.	β.	P.	Q.	P′.	Q′.
x	1	$\dfrac{1}{1-a}$	0	$\dfrac{1}{1-b}$	$\dfrac{1}{1-\alpha}$	$\dfrac{1}{1-\beta}$	$\dfrac{1}{1-p}$	0	x_1	x_2
y	0	$\dfrac{a}{a-1}$	1	$\dfrac{b}{b-1}$	$\dfrac{\alpha}{\alpha-1}$	$\dfrac{\beta}{\beta-1}$	0	$\dfrac{1}{1-q}$	y_1	y_2
z	0	0	0	0	0	0	$\dfrac{p}{p-1}$	$\dfrac{q}{q-1}$	z_1	z_2

The equation to Pα is

$$\begin{vmatrix} x & y & z \\ \dfrac{1}{1-p} & 0 & \dfrac{p}{p-1} \\ \dfrac{1}{1-\alpha} & \dfrac{\alpha}{\alpha-1} & 0 \end{vmatrix} = 0, \ \text{or} \ x\alpha p+yp+z\alpha=0. \quad . \quad . \quad (2)$$

That of the line CA′ is

$$x+ya=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The intersection of these is P′. Hence

$$x_1:y_1:z_1=a\alpha:\alpha:p(a\alpha-1). \quad . \quad . \quad . \quad . \quad (4)$$

The equation to Qβ is $\quad x\beta q+yq+z=0.$
That of CB′ is $\quad\quad\quad\quad x+yb=0.$
Hence we have for the coordinates of Q′

$$x_2:y_2:z_2=b\beta:\beta:q(b\beta-1). \quad . \quad . \quad . \quad (5)$$

The equation of PQ is

$$px + qy + z = 0. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The equation of P'Q' is

$$x'\{\alpha q(b\beta - 1) - \beta p(a\alpha - 1)\} + y'\{pb\beta(a\alpha - 1) - qa\alpha(b\beta - 1)\}$$
$$+ z'\alpha\beta(a - b) = 0. \quad . \quad (7)$$

Now suppose (6) to pass through a fixed point (x_0, y_0, z_0) so that

$$px_0 + qy_0 + z_0 = 0. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Eliminating p from (7) and (8) we have

$$q[x_0'\{\alpha x_0(b\beta - 1) + by_0(a\alpha - 1)\} - y_0'\{b\beta y_0(a\alpha - 1) + a\alpha x_0(b\beta - 1)\}]$$
$$+ x_0'\beta z_0(a\alpha - 1) - y_0'b\beta z_0(a\alpha - 1) + z_0'x_0\alpha\beta(a - b) = 0. \quad (9)$$

This shows that if the line P Q passes through a fixed point $x_0 y_0 z_0$ then P' Q' passes through another whose co-ordinates x_0', y_0', z_0' will be got by equating to zero separately the coefficient of q in (9) and the part independent of q. Solving these, we get

$$x_0' : y_0' : z_0' = b\beta(a\alpha - 1)y_0 + a\alpha(b\beta - 1)x_0 : \alpha(b\beta - 1)x_0$$
$$+ \beta(a\alpha - 1)y_0 : -z_0(a\alpha - 1)(b\beta - 1).$$

This we may write in the form

$$\frac{z_0'}{z_0} = \frac{x_0'}{kx_0 + ly_0} = \frac{y_0'}{mx_0 + ny_0}. \quad . \quad . \quad . \quad (10)$$

Now the condition that a line through C corresponds to itself is

$$\frac{x}{y} = \frac{x'}{y'}, \quad \text{or } mx^2 + (n - k)xy - ly^2 = 0. \quad . \quad (11)$$

This is a quadratic, giving *two* values of $x : y$ which will be *real* if $n - k^2 + 4ml$ is positive, *i. e.* if

$$\{\beta(a\alpha - 1) - a\alpha(b\beta - 1)\}^2 + 4\alpha(b\beta - 1)b\beta(a\alpha - 1) > 0.$$

Let us now take a new triangle formed by A B and the two *double-lines* as new fundamental triangle, using ξ, η, ζ as areal coordinates referred to it. In what follows we shall use A, B, C for the three double-points.

If

$$\left.\begin{array}{l} \xi = a_1 x + b_1 y + c_1 z \\ \eta = a_2 x + b_2 y + c_2 z \\ \zeta = a_3 x + b_3 y + c_3 z \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

then $\Delta x = \begin{vmatrix} \xi & b_1 & c_1 \\ \eta & b_2 & c_2 \\ \zeta & b_3 & c_3 \end{vmatrix}$, where $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Now $\zeta = 0$ becomes $z = 0$, $\therefore a_3 = b_3 = 0$. . . . . (13)

Again $\xi\eta = 0$ is equivalent to

$$(a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z) = 0, \text{ and to}$$
$$m x^2 + (n-k)xy - l y^2 = 0.$$

Hence $a_1 a_2 = m$, $b_1 b_2 = -l$, $a_1 b_2 + a_2 b_1 = -k + n$ . . (14)

Also $a_1 c_2 + a_2 c_1 = 0$, $b_1 c_2 + b_2 c_1 = 0$, $c_1 c_2 = 0$, . . (15)

$\therefore \; a c_1 c_2 + c_1^2 a_2 = 0 \quad \therefore c_1^2 a_2 = 0.$
Similarly $c_2^2 a_1 = 0$, $c_1^2 b_2 = 0$, $c_2^2 b_1 = 0$ $\Big\}$ . . . . (16)

If $c_2 \neq 0$ then $c_1 = 0$, $a_1 = 0$, $b_1 = 0$, which is impossible.

Hence $c_1 = c_2 = 0$. Thus

$$\begin{aligned}
\Delta \cdot x &= \xi b_2 c_3 - \eta b_3 c_2, \\
\Delta \cdot y &= -\xi a_2 c_3 + \eta a_1 c_3, \\
\Delta \cdot z &= \zeta(a_1 b_2 - a_2 b_1),
\end{aligned} \Bigg\}, \quad . \qquad . \qquad . \; (17)$$

(10) becomes
$$\begin{aligned}
\frac{\zeta'}{\zeta} &= \frac{b_2 \xi' - b_1 \eta'}{k(b_2 \xi - b_1 \eta) + l(-a_2 \xi + \eta a_1)} \\
&= \frac{-a_2 \xi' + a_1 \eta'}{m(b_2 \xi - b_1 \eta) + n(-a_2 \xi + a_1 \eta)}.
\end{aligned}$$

Using (14) this reduces to

$$\frac{\zeta'}{\zeta} = \frac{(a_1 b_2 - a_2 b_1)\eta'}{\eta\left(kl + mn - l^2 \dfrac{a_1}{b_1} - m^2 \dfrac{b_1}{a_1}\right)} = \frac{-(a_1 b_2 - a_2 b_1)\xi'}{\xi\left(kl \dfrac{a_1}{b_1} + mn \dfrac{b_1}{a_1} + 2lm\right)}. (18)$$

This we may write

$$\xi' : \eta' : \zeta' = f\xi : g\eta : h\zeta, \quad . \quad . \quad . \quad . \quad . \; (19)$$

which brings out the symmetrical character of the transformation.

It is easy to show that for this transformation, the locus of the intersections of a set of old lines passing through a fixed point ξ_0, η_0, ζ_0, with their new lines is

$$\left(\frac{1}{g} - \frac{1}{h}\right)\frac{\xi_0}{\xi} + \left(\frac{1}{h} - \frac{1}{f}\right)\frac{\eta_0}{\eta} + \left(\frac{1}{f} - \frac{1}{g}\right)\frac{\zeta_0}{\zeta} = 0, \quad . \; (20)$$

which in general is a conic circumscribing the triangle of double-points, but reduces in the case when $\zeta_0 = 0$ to a straight line through C.

And the envelope of the lines joining the old points of a fixed line $l_0 \xi + m_0 \eta + n_0 \zeta = 0$ to their new points, is the conic

$$l_0^2(g-h)^2 \xi^2 + m_0^2(h-f)^2 \eta^2 + n_0^2(f-g)^2 \zeta^2 - 2m_0 n_0(h-f)(f-g)\eta\zeta$$
$$- 2n_0 l_0(f-g)(g-h)\zeta\xi - 2l_0 m_0(g-h)(h-f)\xi\eta = 0, \quad . \; (21)$$

which touches the three double-lines.

In the particular case where $n_0 = 0$ it reduces to a pair of coincident lines through C, whose intersection with AB is the *vertex* of the line $l_0\xi + m_0\eta = 0$.

We note that if the coordinates of a point on AB are $x_1, y_1, 0$, those of their vertex $\xi_1, \eta_1, 0$, and those of their base-point $X_1, Y_1, 0$, then

$$\xi_1 : \eta_1 = x_1(f-h) : y_1(g-h), \quad . \quad . \quad . \quad . \quad (22)$$

and

$$X_1 : Y_1 = x_1(g-h) : y_1(f-h). \quad . \quad . \quad . \quad (23)$$

From the symmetry of the transformation with reference to the three double-points, A, B, C, it is clear that in our fundamental constructions we could interchange say B and C.

Thus for every object-point there is a B-vertex and a B-base-point lying in AC.

From the purely mathematical point of view, the simplest way to study analytically the transformation of Section III. would be to begin with the equations (19) as defining the transformation, and afterwards identify it with that defined by the vertex-property. But from our present point of view the course we have adopted seems better, especially as a pair of double-points may be imaginary.

Concluding Note.

In the foregoing study of the Axial Dioptric System and its Cardinal Points, the problem of formulizing the calculations required for determining the positions of the Cardinal Points, when the refracting surfaces and the media are given, has been avoided. Euler, Lagrange, Gauss, and others have applied continued fractions to such calculations. For practical purposes possibly the direct graphic construction (on a sufficiently large scale) of the images of two arbitrary extra-axial object-points, treating the refractions successively, and thereafter of the cardinal points (by the aid of the constructions of Section I.) might be the most satisfactory method, seeing that the data in practical cases are only approximately exact.

But however that may be, it seems to have been felt by many writers on the subject that the avoidance of the somewhat complicated treatment by continued fractions would be desirable in establishing the *theory* of these points. Clerk Maxwell in his paper in the Quarterly Journal of Mathematics, vol. ii., does this by assuming the existence of a " perfect optical instrument " fulfilling certain conditions, and a considerable resemblance will be found between some parts of the present paper and of that of Clerk Maxwell.

Möbius, again, who wrote on the subject both before and

after Gauss, begins his later paper (*Leipziger Berichte,* vol. vii. (1855)) by stating that he had recently become convinced that the chief properties of a system of lenses can be deduced by a purely geometric method much more easily than by continued fractions. But in the actual working out of his method, he uses continued fractions after all. To mention more recent writers, Herman in his lately published Treatise bases his treatment on the well-known optical theorem of Cotes; but the symbolic expressions involved in that theorem seem quite as complicated as continued fractions. And Chrystal, in his paper in the Proceedings of the Edinburgh Mathematical Society, vol. xiv. (1895), gives a very concise and complete summary of the theory, avoiding intermediate calculations, but purely algebraic in its character.

I may mention that I had worked out the foregoing method in all its main features and in its chief details when my acquaintance with the theory of the cardinal points was derived solely from the treatises of Heath and of Herman, and the paper by Chrystal. On looking up the literature bearing on the subject that was accessible to me, I found that, so far as details are concerned, very much had been anticipated by previous writers, amongst whom I might specially mention Bravais (*Ann. de Chimie et de Physique,* 3rd ser. vol. xxxiii. (1851)) who points out the existence of what I have called the *vertex* for a point; and A. Martin (*ibid.* 4th ser. vol. x. (1867)) who develops the ideas of Gauss, Bravais, Verdet, and Listing, and who points out the existence of what I have called the *base-plane* for an axial point (*plan refracteur*); as well as Möbius and Clerk Maxwell, already mentioned.

But I did not find any anticipation of my fundamental idea of method; and the very considerable simplification I have been able to introduce into the geometric treatment of the subject seems to me to justify publication.

XLII. *On the Radioactivity of Metals generally.* By J. C. McLennan *and* E. F. Burton, *University of Toronto* *.

Introduction.

IN a former paper by the authors † on the conductivity of a mass of ordinary air confined within a large metallic receiver, it was shown that about 37 per cent. of the conductivity was due to an exceedingly penetrating radiation

which, traversing the air in the room, passed through the walls of the receiver. It was also shown that the conductivity of fresh atmospheric air, after being confined in a metallic cylinder, rapidly fell to a minimum value, that a gradual rise then took place, and that a steady state was reached after some hours.

On account of the known decay of the constituent in atmospheric air responsible for excited radioactivity, the fall in the initial conductivity was attributed to its presence in the cylinder, and the subsequent rise to an emanation of a similar character given off by the walls of the containing vessel ; the steady state representing a condition of equilibrium where the rate of decay of this emanation was equal to the rate at which it was produced.

It was also pointed out that the limiting value of the conductivity thus reached varied with the metals forming the walls of the receiver. In experiments with lead, tin, and zinc the conductivity was highest with lead and lowest with zinc. In view of these results and on account of the known complex character of the radiations from such highly radioactive substances as radium and thorium, a series of experiments was made to investigate still further the radioactivity of ordinary metals.

As a result of these experiments we find that when a cylinder of any metal is inclosed within a second of the same material, insulated from it, and surrounded by air or other gases, it gradually acquires a negative charge, and after a short time reaches a state of equilibrium at a definite potential below that of the inclosing cylinder.

So far the experiments made indicate that the negative charge acquired by the cylinder results from a process in which an excess of positively charged corpuscles is expelled from its surface. This process is probably identical with the α radiation observed by Rutherford * and others in the highly radioactive substances radium, thorium, and uranium, and experiments are now being made by the writers to determine its relation to the effect observed by Guthrie †, and developed by Elster and Geitel ‡, J. J. Thomson §, Richardson ||, Strutt ¶, and others, that a metal—platinum, for example,—when heated to a dull red, will discharge a negatively charged body placed near it, but not one positively charged.

          * Phil. Mag. Feb. 1903, p. 177.
          † Phil. Mag. [4] vol. xlvi. p. 257 (1873).
          ‡ Wied. *Ann.* vol. xxxviii. p. 27 (1889).
          § Phil. Mag. vol. xliv. p. 203 (1897).
          || Proc. Camb. Phil. Soc. vol. xi. p. 286.
          ¶ Phil. Mag. [6] vol. iv. p. 98 (1902).

Experiments.—Metal Effect.

In these experiments cylinders of aluminium, zinc, lead, tin, and copper were used, the outer ones being 120 cms. in length and 24 cms. in diameter, and the inner 110 cms. in length and 19 cms. in diameter. The measurements were made with a quadrant electrometer whose sensitiveness was such as to give a deflexion of 1000 mms. on a scale situated at 1 metre from the needle for a potential of 1 volt.
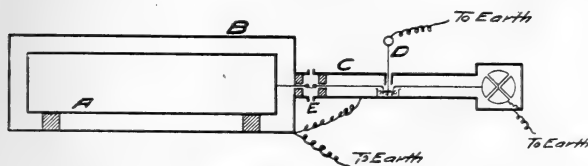
Fig. 1.

The apparatus and its connexions are shown in fig. 1, where A and B represent the cylinders of the metal examined. The wire which led from the inner cylinder to the electrometer was of brass, and was surrounded by a tube of the same metal to screen off induction effects. This tube, as well as the outer cylinder B and the metallic screen over the electrometer, was kept joined to earth during an experiment. The inner cylinder and the free quadrants were earthed by means of the brass rod D, which carried a platinum tip and made contact with a small platinum plate attached to the connecting wire as shown in the figure. The tube was provided with a small opening so arranged that the withdrawal of the rod D did not affect the capacity of the system. Plugs and supports of paraffin-wax provided insulation for the apparatus.

In making a measurement the connecting rod D was withdrawn and the apparatus left intact. The inner cylinder B and the free quadrants then slowly acquired a negative charge which finally reached a limiting value. The value varied with the metal used in the construction of the cylinders A and B, but did not vary with samples of the same metal obtained from different sources. It was also the same whether the needle of the electrometer was positively or negatively charged. With lead, tin, and copper the limiting value was reached in about one hour, but with aluminium and zinc not before four or five hours had elapsed.

Readings for pairs of cylinders of these metals at the

temperature of the room are set forth in fig. 2, where curves are drawn with the time expressed in hours for abscissæ, and the
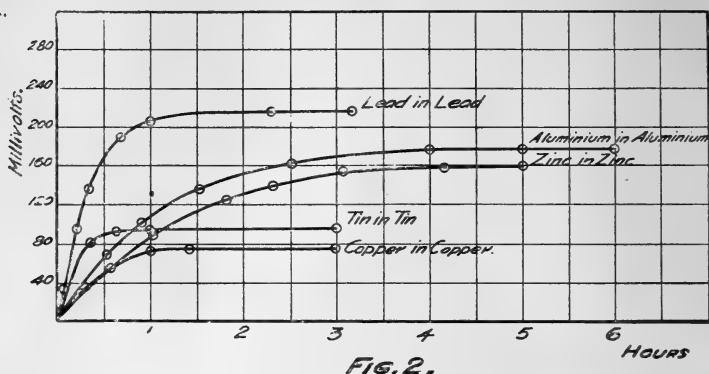
FIG. 2.

negative potential attained by the inner cylinder expressed in millivolts for ordinates. The limiting values for the different metals are as follows :—

	Millivolts.
Aluminium	178
Zinc	160
Lead	216
Tin	95
Copper	73

In experimenting with lead cylinders the pressure of the air in B was reduced from 750 mms. to 20 mms. of mercury, and a slight drop in the limiting potential of about 3 or 4 per cent. was observed.

Measurements were also made with different gases at atmospheric pressure between these cylinders, and it was found that the limiting potential slightly increased with the density of the gas used. A set of readings taken with hydrogen, air, oxygen, and carbon dioxide is as follows :—

	Millivolts.
Hydrogen	205
Air	216
Oxygen	220
Carbon dioxide	233

Experiments were also made with lead by varying the diameter of the inner cylinder. With cylinders 19, 13, and 4 cms. in diameter respectively, no difference was observed in the limiting potential; but the time required to

reach it slightly increased with the use of cylinders of smaller diameter.

The experiments were also varied by placing the lead cylinders in a tank filled with water, the surrounding layer of water having a thickness of 13 cms. Under these conditions the limiting potential was found to be unchanged.

It was also observed that this metal effect was produced when Röntgen and Becquerel rays were allowed to traverse the cylinders. With these radiations, however, the limiting negative potentials were reached in the course of a few seconds and varied but little from the values obtained for the different metals under ordinary conditions.

A small quantity of radium chloride of activity 1000 confined in a glass phial was used as the source of Becquerel rays, and was placed, in the experiment, on a support close to the outside of the cylinder B. The effect was also produced with a quantity of uranium oxide placed near the cylinder, but a much longer time was required than with the radium to reach the steady state.

The following are the values of the limiting negative potentials obtained under the different conditions.

Metal.	Negative Potential in Millivolts.		
	Natural Radiation.	Röntgen Rays.	Becquerel Rays.
Aluminium	179	179	179
Zinc	160	176	180
Lead	216	175	201
Tin...................	95	98	100
Copper	73	72	69

Volta Effect.

It is evident that with cylinders of different metals the arrangement shown in fig. 1 afforded a means of exhibiting the Volta effect, and of approximating to a measure of the contact differences of potential for a series of metals.

With the rod D in position a difference of potential would exist between the cylinders A and B. Its withdrawal would be followed by a gradual equalization of these potentials which would result in a charge being communicated to the connecting wire and the free quadrants. This again would set up a current to the surrounding tube C, and finally the free

quadrants would attain a steady potential, the current between the cylinders being equal to that between the connecting wire and its surrounding tube. The potential assumed by the free quadrants under these circumstances, while approximating to, would be less than the contact difference of potential for the metals used in the two cylinders.

The values obtained in this way for any two given metals, while differing in sign, should be numerically the same whichever metal was used for the inner or outer cylinder. But on trial with a number of pairs of metals it was invariably found that the readings differed when an interchange was made of the metals in the cylinders. This is illustrated by the curves in fig, 3, which give the results for the metals
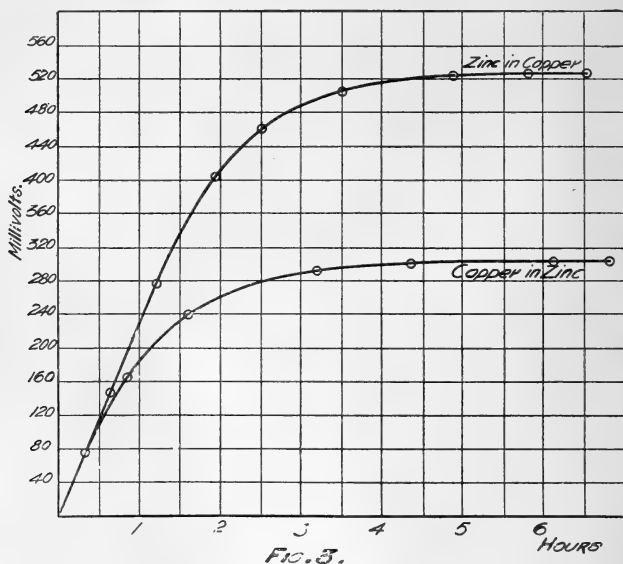
FIG. 3.

zinc and copper, the upper representing the negative potentials acquired by the quadrants with an inner cylinder of zinc, and the lower the positive potentials with one of copper. The final potentials, it will be seen, are 527 and 304 millivolts respectively.

This result finds its explanation in the metal effect described above. We have shown that with zinc cylinders the inner, in virtue of this effect, attains a potential of 160 millivolts below that of the outer, while with copper the corresponding value is 73 millivolts. If, then, we suppose the two "effects" to act concurrently, the final readings observed with the zinc and copper cylinders will represent their algebraic sum.

Denoting the metal effect for zinc by x and that for copper by y, and the Volta effect for the two metals by V, we have

Limiting Potential
(millivolts).
$$V + x = 527,$$
$$V - y = 304,$$
$$x = 160,$$
$$y = 73,$$

from which we obtain the values 367 and 377 millivolts for the contact difference of potential for the metals zinc and copper, an agreement sufficiently close to confirm our hypothesis.

Similar measurements have been made with all combinations of the metals aluminium, zinc, lead, tin, and copper, taken in pairs, with uniformly close results, but the Volta effects obtained in this way were in every case proportionately less than the generally accepted values.

As an additional illustration of these measurements, the curves representing the results obtained with an inner cylinder of lead and an outer of each of the metals are
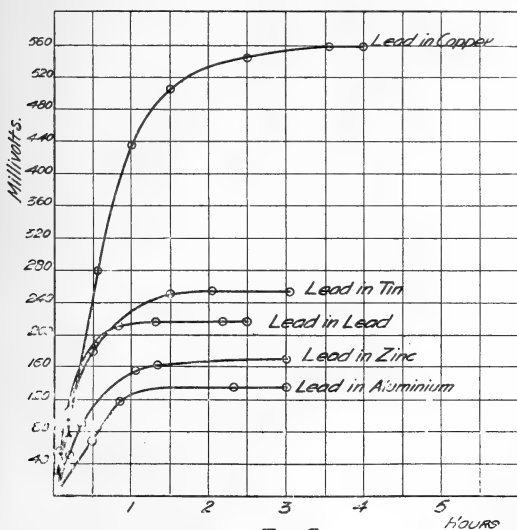
Fig. 4.

shown in fig. 4. Combining the limiting potentials of these with the metal effect for lead, 216 millivolts, the

potential-differences for the different pairs of metals are as follows :—

Difference of Potential.
(millivolts).

Aluminium⎞	
Zinc ⎞ 35	
Lead ⎞ 45	
Tin ⎞ 37	
Copper ⎞ 306	

values which are considerably below those generally adopted.

Conclusions.

The gas between the two cylinders always contains a number of ions, and, on account of the greater rate of diffusion of the negative ions, it is possible that an excess of these would impinge upon the inner cylinder in a given time, and thus leave it negatively charged. The resulting potential, however, should be the same for all metals on this hypothesis.

Again, it is possible that the very penetrating radiation which is present in ordinary air may consist of negatively charged matter, and that the negative charge taken up by the inner cylinder represents the amount of this radiation intercepted by it. But the high value obtained for aluminium, together with the results obtained with Röntgen rays, is against this conclusion.

It seems rather that a process is going on at the surface of the metal, whereby an excess of positively charged corpuscles is being continually emitted, and that the steady state attained represents a condition of equilibrium in which the current between the cylinders is equal to the rate of efflux of the positive charges.

XLIII. *A New Form of Platinum Resistance-Thermometer, specially adapted for the Continuous-flow Calorimeter.* By H. T. BARNES, *D.Sc., Assistant Professor of Physics, and* D. McINTOSH, *M.A., Demonstrator in Chemistry, McGill University* *.

ONE of the difficulties to be overcome in constructing a sensitive platinum thermometer is to restrict the size of the coil of wire forming the bulb. In order to have a sufficient length of wire it is often necessary to make the bulb inconveniently large, and thereby sacrifice quickness of register.

The form of thermometer which we have devised meets this difficulty to a considerable extent, and has enabled us to use shorter bulbs.

* Communicated by Prof. H. L. Callendar, F.R.S.

The main point about the thermometer is that a core passes through the centre of the bulb, through which a liquid or gas can be made to flow. It is therefore particularly well adapted for the continuous-flow calorimeter, where the temperature of a flowing liquid or gas is measured. It is quite as easy to construct as the mica-frame type, and it has many advantages over the latter for the particular use for which it was designed, although it would be unsuited for certain types of temperature measurement.

To construct the thermometer, a glass tube about 5 mms. in diameter and 25 cms. long is coated by beeswax for a considerable length, and a thread is cut in the wax on a machine-lathe. By means of hydrofluoric acid the glass is etched away, and a permanent thread is produced on the tube. When the wax is removed a point of glass is raised on the tube by means of a blowpipe, leaving a margin of about half a centimetre from the end. The platinum wire is wound on double in the usual way, the point of glass preventing the wire from slipping. The two ends of the coil are fastened down to the glass after winding by points of hot glass, leaving at least 1 cm. free at the ends.
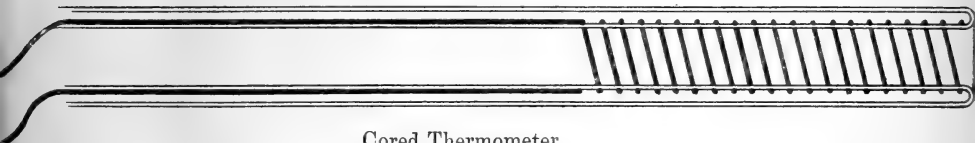
Cored Thermometer.

These free ends are fused to silk-covered copper wires of about No. 24 gauge in the blowpipe, which serve as connecting-leads. The copper wires must be longer than the glass tube, and are fastened securely to the tube in two places by binding-thread. The compensating leads of copper wire, connected by a short piece of platinum wire, are bound in the same way to the tube, and the platinum wire caught on a point of hot glass. The length of this platinum connecting wire will depend on the length of wire in the bulb. Usually it varies from 1 cm. to 3 cms. Besides being a convenient way of joining the compensating leads together at the bulb, the short bit of wire serves, as suggested by Callendar, the purpose of compensating for the conduction of heat away from the wire by the leads. As soon as the wires are fastened in place the points of glass are made as flat as possible by means of a small blowpipe-flame. A thin glass tube is then selected, which just slides over the bulb and connecting wires. The two tubes are carefully fused together around the end so as not to melt the glass near the wire. At

the other end the two tubes are of unequal length, with the inner projecting several centimetres beyond the outer. The copper leads are bent over the outer tube and bound securely. The two tubes are closed at this point by shellac or marine glue, or, where temperatures higher than that of boiling water are to be measured, the space may be closed by binding around a rubber strip.

The length and diameter of the bulb will depend on the fineness of the thread and of the wire, and on the length of wire required to give the requisite resistance. As a rule, 6-mil wire is suitable for most thermometers ; but we have used 4-mil wire with very satisfactory results. The thermometer has a small heat capacity, and as the liquid passes through the centre of the bulb as well as over the outside, it is exceedingly quick reading.

It is particularly suited for measuring steam-points. When it is placed in a hypsometer the steam passes out through the thermometer and keeps the entire length of stem at the steam temperature. It has been shown by one of us* that the length of stem exposed at the high temperature is an important consideration for very accurate measurement. The use of copper wire as leads, which is much to be recommended, has one disadvantage in having a high conducting-power for heat. With the mica-frame thermometer, unless a great length of stem is immersed in the steam-jacket, the conduction of heat from the air-space around the bulb by the leads produces a small error.

With the cored thermometer no such stem-correction is needed, since the leads are heated for their entire length by the flowing steam.

In taking an ice-point the water around the ice in which the bulb is immersed is drawn up through the thermometer and discharged again into the mixture. This can be easily arranged with a small laboratory water-pump.

The sulphur-point is more difficult to obtain ; but, as a rule, for accurate temperature measurement it is not necessary to go as high as that. If it is desired to determine the constant of the wire, where this is not known, a mica-frame thermometer should be tested which has been made from a piece of the same sample of wire. Temperature measurements by the platinum thermometer are now so universal that it is easy to obtain the pure wire with known constants.

We have tested a number of these cored thermometers of different resistances, and find them exceedingly quick and accurate.

McGill University, June, 1903.

* Phil. Trans. vol. cxcix. p. 195 (1902).

XLIV. *On the Practical Attainment of the Thermodynamic Scale of Temperature.*—Part II. *By* J. Rose-Innes, *MA.*, *B.Sc.**

IN the first part of this paper, which was published in the Philosophical Magazine for July 1901, I remarked that the question of finding any temperature on the thermo-dynamic scale resolves itself into two minor problems. These two minor problems are :—(i.) To find the absolute value of the freezing-point of water ; and (ii.) to find the value of the interval between the freezing-point of water and the temperature under discussion (*loc. cit.* pp. 135–136). It was also remarked that these two problems are fairly distinct though not entirely so ; and I hope in this second part of my paper to trace out the connexion between them in greater detail than was done in the first part.

One of the most important physical constants is the absolute temperature of the freezing-point, but its value is not known with any great accuracy at the present time. In the first part of this paper I gave two tables containing estimates of the freezing-point. The first table was based upon M. Chappuis's experiments with the constant-volume thermometer, M. Amagat's experiments on isothermal compressibility, and the well-known Joule-Thomson experiments. The following results were obtained :—

$$\text{Hydrogen} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 273{\cdot}153$$
$$\text{Nitrogen} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 273{\cdot}361$$

(*loc. cit.* p. 137). The second table was based upon Regnault's measurements of the expansion of gases under constant pressure, and the Joule-Thomson experiments. The numbers obtained were as follows :—

$$\text{Hydrogen} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 273{\cdot}00$$
$$\text{Air} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 273{\cdot}16$$

The difference between the two estimates in either table is surprisingly large when we consider the skill of the experimenters furnishing the numerical data. An interesting question arises as to the cause of this difference : and an easy way of disposing of the whole matter is to assume that the difficulty is caused entirely by errors in the observed values of the Joule-Thomson effect. There is certainly a good deal to be said in favour of this assumption, as the Joule-Thomson experiments were so difficult to carry out that they were doubtless subject to a considerable experimental error. But this assumption, though plausible, ought not to be

* Communicated by the Author.

accepted as the true explanation until it has been tested in some way or other, since there is always the possibility that a portion of the data not due to Joule and Kelvin is likewise erroneous.

The most hopeful method of attacking the question appears to be to devise some way of arriving at an estimate of the freezing-point using different gases, but without employing the Joule-Thomson numerical data. One way of securing this result is given in the following investigation.

It was shown by Lord Kelvin that when a gas is forced through a porous plug, we have

$$t\left(\frac{dv}{dt}\right)_p - v = \mathrm{JK}\frac{\delta t}{\delta p}$$

(Reprinted Papers, vol. iii. p. 179).
Divide by t^2

$$\frac{1}{t}\left(\frac{dv}{dt}\right)_p - \frac{v}{t^2} = \frac{\mathrm{JK}}{t^2}\frac{\delta t}{\delta p}.$$

Integrate this equation with respect to t along an isopiestic between the limits t_0 and t_1; we thus obtain

$$\frac{v_1}{t_1} - \frac{v_0}{t_0} = \int_{t_0}^{t_1}\frac{\mathrm{JK}}{t^2}\frac{\delta t}{\delta p}\,dt.$$

The symbols v_1 and v_0 denote the volumes at t_1 and t_0 for the same pressure p; if we integrate along a second isopiestic p' between the same limits of temperature t_0 and t_1 we shall obtain

$$\frac{v_1'}{t_1} - \frac{v_0'}{t_0} = \int_{t_0}^{t_1}\frac{\mathrm{JK}}{t^2}\frac{\delta t}{\delta p}\,dt.$$

It was shown by Joule and Lord Kelvin that $\frac{\delta t}{\delta p}$ is independent of the pressure, hence by subtraction

$$\frac{v_1 - v_1'}{t_1} - \frac{v_0 - v_0'}{t_0} = 0,$$

which leads to

$$t_0 = (t_1 - t_0) \div \left\{ \frac{v_1 - v_1'}{v_0 - v_0'} - 1 \right\}$$

$$= (t_1 - t_0) \div \left\{ \frac{v_1}{v_0} - \left(\frac{v_1'}{v_0'} - \frac{v_1}{v_0}\right)\frac{v_0'}{v_0 - v_0'} - 1 \right\}.$$

Let the suffix 1 refer to the boiling-point, and the suffix 0 to the freezing-point; then the above equation enables us to calculate out the absolute value of the freezing-point.

Employing Regnault's data as quoted by Lord Kelvin (*loc. cit.* p. 181), we obtain the following numbers for the value of t_0

Hydrogen .... 273·13
Air .... 273·21

If we prefer to work with the numbers quoted by M. Chappuis in his paper " L'échelle thermométrique normale et les échelles pratiques " we shall have

Hydrogen .... 273·04
Nitrogen .... 273·13

(*loc. cit.* pp. 3 & 8).

There is another way of reaching an estimate of t_0 without making use of the Joule-Thomson numerical results. I have shown in my paper already quoted that for such gases as air and hydrogen we may write

$$pv = Rt - \frac{1}{v} \Sigma \frac{Ra_n}{(n+1)t^{n-1}}$$

(Phil. Mag. ii. p. 133). Let the suffix 0 applied to p and t refer to the freezing-point, and let the suffix 1 refer to the boiling-point. Then keeping the volume constant and equal to v we have

$$p_1 v^2 = Rt_1 v - \Sigma \frac{Ra_n}{(n+1)t_1^{n-1}},$$

$$p_0 v^2 = Rt_0 v - \Sigma \frac{Ra_n}{(n+1)t_0^{n-1}}.$$

Next keeping the volume constant and equal to v' we obtain

$$p_1' v'^2 = Rt_1 v' - \Sigma \frac{Ra_n}{(n+1)t_1^{n-1}},$$

$$p_0' v'^2 = Rt_0 v' - \Sigma \frac{Ra_n}{(n+1)t_0^{n-1}}.$$

By subtraction we get

$$p_1 v^2 - p_1' v'^2 = Rt_1(v - v'),$$

$$p_0 v^2 - p_0' v'^2 = Rt_0(v - v').$$

Hence

$$\frac{p_1 v^2 - p_1' v'^2}{p_0 v^2 - p_0' v'^2} = \frac{t_1}{t_0},$$

$$t_0 = (t_1 - t_0) \div \left\{ \frac{p_1 v^2 - p_1' v'^2}{p_0 v^2 - p_0' v'^2} - 1 \right\}.$$

This last equation, after some manipulation, may be written

$$t_0 = (t_1 - t_0) \div \left\{ \frac{p_1}{p_0} - \left(\frac{p_1'}{p_0'} - \frac{p_1}{p_0} \right) \frac{p_0' v'^2}{p_0 v^2 - p_0' v'^2} - 1 \right\}.$$

In this equation we may use the numerical data furnished by M. Chappuis in his paper, and we thus obtain the following numbers :—

<div style="text-align:center">

Hydrogen .... 273·04
Nitrogen........................ 273·13

</div>

The last three pairs of estimates have been obtained without employing the Joule-Thomson measurements directly. We are obliged to rely on their experimental results in order to arrive at the formulæ used, but none of their numerical data enter into the final calculations. As the result the difference between the estimates formed from hydrogen and from nitrogen is in all cases considerably diminished. We may therefore infer that errors in the accepted numbers for the Joule-Thomson effect were really responsible for a large portion of the original difference.

But though the difference between the estimates derived from hydrogen and from nitrogen has been diminished in the later calculations, it has been by no means completely removed. We note also that the difference, though varying in amount, remains persistently of one sign. If it were a matter of one pair of estimates only the difference between them might very well be attributed to some accidental disturbing agent interfering with that particular set of experiments ; but it seems hardly likely that a difference persistently in one direction can be the result of mere accident. At any rate it seems worth while to take into consideration an alternative hypothesis, viz. :—that there is some permanent physical cause at work, independently of the Joule-Thomson experiments, which serves either to raise slightly the estimate of the freezing-point derived from nitrogen, or to lower slightly the estimate derived from hydrogen. The amount of error introduced into the uncorrected estimate of the freezing-point in either case is supposed to be comparable with $0°\cdot08$ C.

We are not able at present to assert that any known physical cause is certainly the source of the error here spoken of ; but there are indications which lead us to suppose that its introduction may be due to the influence of the walls of the containing vessel. The ordinary mathematical investigation concerning the Joule-Thomson effect leads to the conclusion that the thermodynamic scale agrees very closely with the

hydrogen or nitrogen scale ; but this investigation assumes that the gas in the thermometric vessel possesses the same state as in the Joule-Thomson experiments. Thus in obtaining the fundamental differential equation we have to consider the work done by the streaming gas on both sides of the porous plug ; now such an expression as pv, the work done by the issuing gas against the atmosphere, clearly refers to the pressure and volume of the gas actually in use, and not to the pressure and volume of the same gas contained in a separate vessel, should these last two chance to be different from the former. In a palladium vessel the volume of unit mass of hydrogen might be—probably would be—something very different from the volume of unit mass of hydrogen at the same temperature and pressure in the stream of the Joule-Thomson experiments. We should therefore be travelling entirely beyond our data if we were to assume on the strength of the Joule-Thomson experiments that the readings of a constant-pressure hydrogen thermometer were approximately those of the thermodynamic scale when the containing vessel was made of palladium.

For thermometric purposes hydrogen is usually inclosed in a glass or platinum vessel ; we may feel sure that in such a vessel the effect of the walls on the density of the gas would be nothing like as big as in the case of a palladium vessel, but we cannot be certain at present that there is no effect at all. The most trustworthy experiments bearing on this point are some measurements made by M. Chappuis. That physicist examined the increase of pressure at constant volume of hydrogen in a vessel of iridio-platinum, and also in a vessel of hard glass. He found that when the pressure at the freezing-point is 100 cms. of mercury, the increase with the former vessel was 36·6254 cms. of mercury, and with the latter vessel the increase was 36·6217. (*L'échelle thermométrique normale et les échelles pratiques*, p. 3.) The difference between these two numbers is small, but it is worth taking into account. If we assume that the walls of the containing vessel exert an influence which has the same sign for both vessels, but which is greater for one vessel than for the other, then the true value of the increase of the pressure will be outside the interval between the observed values, differing from either of them by an amount comparable with the said interval. Thus if we imagine that the influence of the platinum vessel upon the gas is the more marked, we might readily suppose the true value of the increase of the pressure to be as low as 36·6150 without straining the evidence. The reciprocal of 0·00366150 is 273·11, while the reciprocal

of 0·00366254 is 273·03 ; thus we see that we might reasonably suppose the value of the freezing-point obtained from hydrogen to be in error by as much as 0°·08 C.

It might perhaps be urged as an objection to the above reasoning that the difference in the increase of pressure observed by M. Chappuis has not been proved to be due to the influence of the walls of the vessel ; and that such a difference might very well be caused by those unavoidable errors of experiment which accompany even the most skilful observations.   Such an objection should not be disregarded ; but it remains of importance to notice that a method, carried out by a competent observer, which might have indicated that the fifth significant figure remains unaffected by the nature of the containing vessel, entirely fails to furnish such a result.

If there is really some physical cause at work which introduces an error comparable with 0°·08 C. into the uncorrected estimate of the freezing-point, then such a cause might affect the second place of decimals very perceptibly in measurements of differences of temperature between 0° C. and 100° C. This would happen if the disturbing agency were introduced somewhat irregularly between 0° C. and 100° C., as I have already shown in the first part of this paper (*loc. cit.* pp. 139–141). It would thus appear that the errors introduced by the walls of the thermometer may be more serious than the thermodynamic correction to either gas-scale ; and as we do not know for which of the two gases this error is more likely to arise, we cannot really tell which of the two constant volume thermometers most correctly measures differences of temperature on the thermodynamic scale.

XLV. *On Ionization produced by the Motion of Positive and Negative Ions.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,

IN the July number of the ' Philosophical Magazine ' Dr. Stark has published some remarks on the theory of ionization by collision which refer to my work on that subject.   It might be gathered from his remarks that the theories which I have proposed and worked out represent the views previously held by other physicists, and that in the reference I made to his work I have greatly misrepresented the theories which he has put forward.

I may therefore be permitted to make some remarks on the points which Dr. Stark has brought forward, as some of his statements are apt to be misleading.

In the first paper which I published on ionization by collision, in ' Nature,' 9th August, 1900, I called attention in the first paragraph to the cathode rays and Becquerel rays which have been known to have the property of ionizing the gases through which they pass. The particles constituting these rays travel with a very high velocity, and I pointed out that the chief result of my experiments was the definite proof of the fact that negative ions produce others by collision with neutral molecules when they are travelling with comparatively small velocities. I also quoted experiments which showed that negative ions produce others by collision with neutral molecules much more readily than positive ions.

As far as I am aware, the only other experiments from which a clear proof can be deduced of the genesis of ions by collision when small electromotive forces are acting, are those of Stoletow * on Ultra-Violet light.

I called attention to these experiments in my first paper as there is no publication previous to August 9th, 1900, in which it is suggested that Stoletow's experiments can be explained by a collision theory.

My first experiments with Röntgen rays were made at the Cavendish Laboratory, and at that time Prof. Thomson maintained that the best explanation of these experiments was to be found in the theory of surface layers which he had given in explanation of Stoletow's experiments†, as he considered that the potentials which I used were too small to give rise to ionization by collision. In support of this view Prof. Thomson quoted a paper by Rutherford and McClung (which he communicated to the Royal Society on the 15th June, 1900), from which it could be deduced that it would require a fall of potential of at least 175 volts in order to produce ions by collision.

In dealing with the subject of ionization by collision I have from the first considered the possibility of the action of the positive ions. I have recently made some experiments in order to obtain some definite information on this subject. An outline of the method which I have used is given in the ' Electrician ' (3rd April, 1903). The investigations establish all the essential points which the theory demands, and give

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* Stoletow, *Journal de Physique*, Series 2, Tome ix. 1890.

† J. J. Thomson, Phil. Mag. Dec. 1899.

results from which the ionizing action of the positive ions may be determined. With regard to these experiments Dr. Stark states :—" Further, J. A. McClelland has already made an observation which is similar to the phenomenon studied by Townsend, but more simple." Professor McClelland's experiments *, to which this statement refers, were made on the conductivity that takes place between a cylinder and a concentric wire heated to incandescence. The phenomena connected with the passage of a current through a gas surrounding a hot wire are so complicated that Professor McClelland was unable to come to a definite conclusion as to the action of the positive ions. Professor McClelland states that the particular experiments which seem to indicate that positive ions produce others by collision may be explained by the action of the negative ions. Even if we accept the first explanation and attribute the results of his experiments to positive ions, we cannot be sure that they are the same as the positive ions generated in a gas at ordinary temperature as in my experiments. On this point Professor McClelland says :—" Possibly the positive ions which are active in producing secondary ionization also come from the wire, which may account for the apparent difference between them and the positive ions investigated in other cases of ionization."

Dr. Stark is therefore under a misapprehension as to the work in which I have been engaged.

The first paper which Dr. Stark has published on the genesis of ions by collision seems to be a communication to the *Physikalische Zeitschrift*, which bears the date 13th October, 1900. Since then he has published numerous articles on the subject, in which he has added to the theory the " catalytic action " of the metal on the gas. It appears that I have not understood the full import of the catalytic action. My experiments with Röntgen rays were made between two parallel plates, one of aluminium, the other of brass. One set of experiments were made with the plates one centimetre apart, and the other set with the plates two centimetres apart. The results showed that the conditions under which ionization by collision is brought about are precisely the same in the two cases. Dr. Stark interprets these experiments as illustrating the catalytic action of brass on air. Presumably therefore the catalytic action is the same at two centimetres from the metal as it is at one centimetre. As later experiments with shorter distances between

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* J. A. McClelland, Proc. Camb. Phil. Soc. xi. p. 296 (1901).

the plates have shown that the properties of gases are the same at one millimetre from a plate as they are at two centimetres, we see that the catalytic action of a metal extends undiminished from a millimetre to two centimetres at least. Dr. Stark's view therefore leads to a most improbable conclusion. It is difficult to believe in the peculiar influence which a metal is thus supposed to exercise, and as far as I can see the experimental evidence is altogether opposed to it.

The one passage in my paper which refers to Dr. Stark's work is the following: "According to Stark, a negative ion must travel between two points differing in potential by 50 volts in order to acquire sufficient velocity to produce new ions on collision, because there is a fall of potential of 50 volts near the anode when a continuous discharge takes place in air." To what extent I have thus misrepresented his work may be gathered from the following two passages which occur in his paper, *Annalen der Physik* [4] vol. viii. (1902). On p. 816 the following statement occurs: "die Ionisirungsspannung des positiven Ions gegen Platin in Luft beträgt 340 Volt, diejenige des negativen im Luftinnern etwa 50 Volt." And on p. 819: "In der erwähnten Abhandlung wurde dargelegt, dass die auf einer positiven Schichtlänge liegende Spannungsdifferenz eine obere Grenze der Ionisirungsspannung der negativen Ionen im Gasinnern (in Luft 50, in Stickstoff 45, in Wasserstoff 33 Volt) ist, dass ferner der normale Kathodenfall die Ionisirungsspannung der positiven Ionen gegen das Kathodenmetall darstellt."

I maintain that the voltages 340 and 50 are both too large, and that the views which Dr. Stark holds with regard to falls of potential either at the electrodes or in layers near the electrodes are inadequate proofs of his suppositions.

With regard to the potential-difference required to produce ions by collision, I have been led to my conclusions from experiments which do not admit of interpretations in favour of any particular value. The calculations which I have given show that ionization by collision takes place in a small percentage of cases when negative ions collide with molecules with a velocity acquired by a fall of potential of 5 volts. It may be seen from the paper in the 'Electrician' to which I have referred that the ionization produced by positive ions can easily be detected with a potential-difference of 280 volts between the electrodes.

<div style="text-align: right">
Yours very truly,
JOHN S. TOWNSEND.
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XLVI. *The Fluorescence and Absorption Spectra of Sodium Vapour. By* Prof. R. W. WOOD *and* Mr. J. H. MOORE*.

[Plates XIII. & XIV.]

The Fluorescence of Sodium Vapour.

THE green fluorescence which sodium vapour exhibits when illuminated with an intense beam of white light was first observed and studied by Wiedemann and Schmidt (Wied. *Ann.* lvii. p. 447, 1896). The method which they employed was to heat the metal in an exhausted glass bulb, concentrating a beam of sunlight on the vapour by means of a lens. A cone of green light is seen where the intense beam enters the mass of metallic vapour. Examined with the spectroscope this light was found to consist of a band in the red, a narrower band nearly in the position of the D lines, and a broad green band, which appeared to be broken up into channels or bands. The wave-lengths of some of these bands were roughly determined, but the authors do not appear to have ascertained exactly the relation which they bore to the absorption-bands which appear in the same part of the spectrum.

Inasmuch as we have at the present time no very satisfactory theory of fluorescence, and as practically all quantitative work has been done with solutions, it seemed worth while to make a careful study of the relation between the fluorescent light emitted by sodium vapour and the light absorbed by the vapour under the same conditions. The chief points of interest which have been brought out by these investigations are the establishment of the fact that the fluorescent spectrum coincides exactly with the absorption spectrum, band for band and line for line, and a determination of the relation existing between the wave-length of the light which provokes the fluorescence and the nature of the fluorescent spectrum. The somewhat remarkable fact has been ascertained that the D-line absorption is in no way connected with the production of the fluorescence, though the absorption at this point of the spectrum is much more intense than at any other.

By means of improved apparatus we have not only been able to photograph the fluorescent spectrum by means of a concave grating, but have been able to observe by means of a grating the fluorescent spectrum when provoked by approximately monochromatic light obtained with the Fuess monochromatic illuminator. The results of the work throw

* Communicated by the Authors.

a great deal of light on the mechanics of the sodium mole-
cule, and will doubtless prove of considerable value in the
development of the theory of fluorescence.

Apparatus Employed.

The fluorescence as observed in exhausted glass bulbs is
never very intense, and the experiments can be continued
only for a few moments owing to the speedy corrosion of the
glass surface. Moreover, it is not possible to make use of
very dense vapour, the generation of which requires a high
temperature, owing to its action on glass. The form of tube
which was employed by one of us in a previous investigation
on the subject (Wood, Proceedings of the Royal Soc. vol. lxix.
p. 157 ; also Phil. Mag. Jan. 1902) enables vapour of great
density to be obtained, but owing to the rapid distillation to
colder parts of the tube the experiment cannot be continued
long enough for satisfactory photographic records. It is
important not only to have a very dense vapour but also to
have the vapour confined within a small region and sharply
bounded, in order that the light may not be weakened by
absorption before it reaches the denser portions. To meet
these requirements a new form of tube was designed and
constructed which gave admirable results. With it a fluores-
cence ten or fifteen times as bright as anything that can be
obtained with glass bulbs was obtained and maintained con-
tinuously for five or six hours without recharging the tube.
This tube we have had in action for fully forty hours, and it
is only just beginning to show signs of leakage around the
brazed joints, due to the action of the sodium at a red heat
on the silver with which the joints were brazed. The tube
can be very easily constructed, and when once charged can
be used over and over again without any preparation what-
ever for the exhibition of this most beautiful example of
fluorescence. Its general form is shown in Plate XIII. fig. 1.
The large tube was a piece of three-inch (16 gauge) Shelby
seamless steel tubing two feet in length. A ten-inch piece
of thin $\frac{3}{4}$-inch steel tubing was brazed with silver into the
large tube at the centre, making a right angle with it.

Immediately below this tube and at a right angle to both
a small iron crucible was brazed into the larger tube, pro-
jecting into its interior to such a distance that its rim was
visible through the side tube. This crucible was made by
boring out a $\frac{3}{4}$-inch iron cylinder. A small piece of brass
tubing was also brazed into the large tube near one end
through which the whole could be exhausted. The crucible
was filled with lumps of sodium, and the ends of the large

tube closed with plate-glass carefully cemented on with sealing-wax. The side tube was then closed in the same manner and the whole exhausted to a pressure of about a millimetre by means of a mercurial pump. The tube was then sealed off from the pump, a small piece of glass tubing having been cemented into the brass tube and drawn down to a capillary.

The tube was now mounted in a horizontal position and a large Bunsen burner placed beneath the crucible, which was soon raised to a red heat. The dense sodium vapour poured out of the mouth of the crucible and gradually condensed on the cooler portions of the tube. Sunlight from a heliostat was sent down the large tube, a lens placed close to the glass window forming an image of the sun immediately above the neck of the crucible. On looking down the side tube an intensely brilliant green cone of light was seen many times brighter than anything that has ever been obtained with glass bulbs. The floating specks of oxide which appear when the tube is first heated, and which shine with a dazzling white light, soon disappear and leave the fluorescence entirely uncontaminated. It is apparent that with this arrangement light enters at once into the densest vapour without suffering previous loss by absorption in vapour of less density. Moreover, the fluorescent light passes down the observation-tube without having to traverse more than a very thin layer of the vapour, a matter of considerable importance, as we wish to examine the fluorescent light unmodified by subsequent absorption. The large amount of sodium which can be stored in the crucible enables us to deliver a dense stream of vapour in front of the observation-tube for a very long time, which is absolutely essential if photographic records are to be obtained.

Spectrum of the Fluorescent Light.

The spectrum of the fluorescent light was first examined with a two-prism Steinheil spectroscope. The spectrum consisted of a red band and a green band, the latter appearing distinctly fluted. No trace of any bright line or band at or near the position of the D lines has ever been seen in any of our experiments. Its presence in the spectrum described by Wiedemann and Schmidt, and also by one of us, may possibly have been due to the fact that in both of these cases the vapour was contained in a glass bulb heated by a Bunsen burner. This flame coloured by the sodium of the glass may have been responsible for the appearance of a bright line in the place mentioned, a matter which can be very easily settled by repeating the experiments with the bulbs.

The marked resemblance which the fluted spectrum bears to the absorption at once suggests that it may be due to the fact that the fluorescent light has to pass through a certain amount of vapour before reaching the spectroscope, in other words that it does not belong to the fluorescent spectrum at all, but is the result of absorption. To determine whether or not this was the case an absorption comparison spectrum was formed by throwing some of the light which had passed through the tube into the instrument by means of a pair of mirrors and a small right-angle prism. It was at once apparent that the bright lines and bands of the fluorescent spectrum were exactly out-of-step with those of the absorption spectrum. To secure a fixed record of this fact a camera was attached to the spectroscope and the two spectra photographed. The spectrogram confirmed the visual observations in every respect, but the dispersion was too small to make the pictures very satisfactory.

A Rowland concave grating with 15,000 lines to the inch of one metre radius was then used in place of the spectroscope, and some excellent photographs obtained with an exposure of less than an hour. The fluorescent spectrum was found to extend from wave-length 5340 to wave-length 4600 in the green and blue region. All of the photographs show in addition a faint solar spectrum extending from the end of the fluorescent spectrum down to the H and K lines. This is due to a small amount of white light which is scattered by occasional specks of oxide, or perhaps reflected from the wall of the tube. So far as we have been able to determine the fluorescent spectrum is not contaminated with solar lines since it is located in a less actinic region, and the scattered light is not of sufficient intensity to leave any appreciable record in this region.

These photographs brought out the remarkable fact that the fluorescent spectrum is the exact complement of the absorption spectrum. The two spectra were photographed in contact on the same film, and either one might have been a contact print taken from the other. In the upper spectrum, for example, there were two broad light bands with a fine dark line down the centre, while in the lower spectrum occurred two broad dark bands with bright lines down the centre. The same thing was true for all of the irregularities of shading in the very complicated fluted spectrum. A number of these photographs are reproduced in Plate XIV. figs. 1 & 2. As most of the fine detail will doubtless be lost in the process of reproduction a very careful drawing of the two spectra has been prepared from the original negative, which is reproduced with the direct records (Plate XIV. fig. 3,

absorption spectrum above, fluorescent below). The fluorescent spectrum in the green-blue region may be divided into three groups of bands:—(1) Those from 5340 to 5080, consisting of six broad hazy bands which correspond in position to the fluted bands of the absorption spectrum; higher dispersion would doubtless resolve them into lines. (2) Those from 5080 to 4865, a group of irregularly fluted bands, the heads of which point towards the red, *i. e.* they are strongest on the less refrangible side, and shade off on the side of the shorter wave-lengths. (3) Those from 4865 to 4600, which appear under the dispersion used as uniform narrow bands. The spectrum is the exact complement of the absorption spectrum taken with the same instrument, and further work with the concave grating of 14 feet radius will show whether all of the very minute and narrow lines in the flutings of the latter spectrum are present in the former. We feel sure that the spectrum can be photographed with the large grating without difficulty.

Visual observations have shown that the red fluorescence is also fluted, though the flutings are only to be seen when the vapour is very dense and the illumination very intense. It has not yet been definitely proven whether this fluting belongs properly to the fluorescent spectrum or is the result of absorption. A new form of tube has been designed which it is believed will give a much denser vapour and make the settlement of this question possible.

The complementary nature of the photographs of the fluorescent and absorption spectra might lead one to suppose that the wave-length absorbed was re-emitted without change of wave-length. To test whether or not this was the case it was necessary to illuminate the vapour with monochromatic light.

The Fuess monochromatic illuminator, which enables one to cut out a region of any width from a very intense solar spectrum and focus this light at a definite point in space, was arranged so as to send a cone of violet light down the tube, the rays meeting just above the mouth of the crucible. No fluorescence was detected. On gradually increasing the wave-length of the light by turning the graduated screw which rotated the prisms of the instrument it was found that the fluorescence appeared when wave-length 4600 was reached. On looking into the end of the large tube a very beautiful phenomenon was seen. The cone of deep blue monochromatic light was distinctly visible owing to traces of oxide floating about, while at the point where the rays met in the dense sodium vapour there appeared a brilliant spot of *yellowish-green* fluorescent light. As the wave-length was further

increased this spot increased in brilliancy, still remaining green, passed through a maximum, and then gradually faded away, disappearing entirely when the illuminating light became yellow. The vapour remained dark until the wave-length of the light exceeded that of the D lines, when the red fluorescence gradually developed, passing through a maximum in the same manner and then fading away.

The fluorescence of the vapour produced in this way was much less brilliant than in the former experiment, since the total radiant energy thrown into it was very much less than when direct sunlight was employed, still the light sent down the lateral tube was intense enough to give a spectrum when the Steinheil spectroscope was directed down the tube. This spectrum was very feeble however, and it was only with great difficulty that the changes which took place when the wave-length of the illuminating beam was changed could be followed. When blue light was employed the complete green fluorescent spectrum seemed to be present with a maximum intensity at the yellow end. As the wave-length of the light was increased the point of maximum intensity in the fluorescent spectrum moved towards the blue. The important bearing which the phenomenon has on the theory of fluorescence made a more careful study of the relation between the wave-lengths of the absorbed and emitted light very desirable. To accomplish this the arrangement of the apparatus was changed in the following manner. The monochromatic illuminator was rotated through a right angle so that its emitting slit was horizontal instead of vertical. The dense sodium vapour above the mouth of the crucible was now illuminated with a thin horizontal sheet of monochromatic light (the image of the horizontal slit of the illuminator projected in space by a lens). This arrangement of course limits the fluorescence to a thin horizontal layer of vapour, which when viewed through the lateral tube appears as a narrow line of bright green light. By viewing the sheet of vapour edgewise we get a much more intense light, and since its width is small we can dispense with a spectroscope entirely, simply viewing it through a prism or grating.

A small transmission grating of 14480 lines to the inch, which gives a first-order spectrum nearly as bright as a 60° flint prism, was mounted in front of the lateral tube, and the line of fluorescent light viewed through it, the head being covered with a black cloth to cut out all extraneous light. It was found that by slightly inclining the large tube to the illuminating beam it was possible to illuminate a minute projection on the inside edge of the lateral tube with the monochromatic light which caused the fluorescence; in other

words, the near end of the horizontal image of the slit was brought onto this projection. The line of fluorescent light was thus tipped at one end with a small point of light similar to the light which produced the fluorescence. The spectrum of this small illuminated spot, which was of course a narrow band (practically an illuminated point), fell alongside of the fluorescent spectrum, enabling a comparison to be made at once. Any exception to Stokes's law would make itself evident as an extension of the fluorescent spectrum on the more refrangible side of the position occupied by the small spot of illuminating light.

Some very remarkable and significant results were obtained with this disposition of the apparatus. Starting with violet illumination there appeared through the grating only the small comparison-spot of light which moved along as the wave-length was increased. As soon as wave-length 4600 was reached the fluorescent spectrum appeared. Its appearance is indicated in fig. 3, Plate XIII. (*a*). A strip of blue fluorescent light continuous with the reference-spot showed that light of the same wave-length as the absorbed light was being emitted by the vapour. A region of darkness occurred on the less refrangible side (usual sense of the term) and then came a broad green band with a maximum of intensity on the end towards the yellow. Then came another gap extending considerably above the D lines, followed by a *very faint* trace of the red fluorescent band (not shown). On gradually increasing the wave-length of the light the following changes were noted :—The spot of reference light, accompanied by its fluorescent prolongation, moved up the spectrum, pushing the dark region before it, while the point of maximum fluorescence in the wide green band moved down the spectrum to meet the advancing spot. These changes are represented in fig. 3 (*b*) (*c*) (*d*), Plate XIII., the point of maximum fluorescence nearly coinciding in position with the reference-spot in (*d*). On still further increasing the wave-length the fluorescence became very faint, and appeared to extend a little further down the spectrum than the position occupied by the small spot of light. This means an exception to Stokes's law when the wave-length of the illuminating light (green) has the maximum value which still yields fluorescence.

In fig. 4, Plate XIII., an attempt has been made to represent these changes in the conventional way (Müller Pouillet, *Lehrbuch der Physik*, ii. 1, p. 368). Everything shown in this diagram is, however, due to fluorescence, the deviated continuous spectrum of the light exciting fluorescence having been omitted. It will be seen that there is an emission of fluorescent light of the same wave-length as that of the exciting light, which gives the diagram an appearance not unlike

the diagrams where the deviated continuous spectrum is shown. To make this diagram exhibit the changes just described, one has only to move a narrow vertical slit across it from right to left.

It was impossible to tell whether flutings were present in any portion of the fluorescent spectrum or not, owing to the comparatively wide source, and small dispersion employed. There is no reason for believing that they are not, but it does not seem safe to assume that the spectra are identical when the fluorescence is produced by white light and monochromatic blue light. The extent of the spectrum, and the distribution of the intensity in it, has been shown to be different in the two cases, and the flutings, say in the green, which are present when white light is used, may be absent when the fluorescence is produced by light of much shorter wave-length. To test this the spectrum will have to be photographed when the tube is illuminated with monochromatic blue light. To accomplish this with the present apparatus would probably require an exposure of six or eight hours. More favourable conditions are expected with a new form of tube now under construction, and it seems probable that we shall eventually learn a good deal about the dynamics of the sodium molecule. The results already obtained apparently prove that the light of longer wave-length emitted when the vapour is fluorescing, is not the result of damping of the vibration, as assumed in Lommel's theory, but an emission resulting from the fact that the ions of longer free period are set in vibration either by the waves of shorter period, or by the ions of short period which are thrown into vibration by these waves. It is not worth while to speculate about this, however, until the fluorescent spectrum produced by monochromatic blue light has been photographed.

In general the fluorescence of a substance has its maximum intensity when the wave-length of the light is that of the light most strongly absorbed. Sodium vapour is an exception to this rule, for the D-line absorption, which is far more intense than the fluted absorption, has nothing to do with the fluorescence. On Lommel's theory of fluorescence the absence of any lateral emission of light by an absorbing medium is explained in one of two ways. Either the absorption is of waves of different period (an octave below for example) from the free period of the ions, or else the damping factor is so large that the emitted light lies in the infra-red region. In the case of sodium vapour neither explanation is sufficient to account for the absence of fluorescence when the wave-length of the exciting light is that of the D-lines, for since Kirchhoff's law is obeyed in this case, the absorbed waves and emitted waves have the same period, and the absorption

is a resonance phenomenon ; moreover the damping factor must be very small since we have interference with large path difference in the case of sodium light. The non-luminous vapour employed in these experiments may of course be in a different state from the vapour in a sodium flame, but even if this be the case it does not seem justifiable to assume a large amount of damping, for this should cause a displacement of the absorption-lines with reference to the position which they occupy in the case of absorption by a sodium flame.

It is worthy of note that lines corresponding in position to the position of the fluted bands are absent in the emission-spectrum of sodium vapour, except perhaps in the case of the temperature-emission studied by Evershed, which does not appear to have been studied under high dispersion.

There seems to be no way of explaining the emission of the green light when the vapour is illuminated with blue light. It cannot be the result of the damping of the ions whose free period is that of the blue light, for it is in all probability a discontinuous spectrum complementary to the fluted absorption-spectrum.

It will be interesting to see whether the absorption of the vapour is directly affected by the circumstance that it is fluorescing at the same time. This was found to be the case in some experiments made by Burke upon uranium glass. It is also important to determine in what way the absorption and fluorescence are influenced by pressure.

These matters will be investigated in the near future.

The statement that when the wave-length of the exciting light is that of the D-lines no fluorescence is produced requires modification. Strictly speaking this is not true, though it is almost certain that the D-line absorption is in no way responsible. This is due to the fact that the beam from the monochromatic illuminator is not strictly monochromatic, being in fact a band varying from ten to twenty Ångström units in width. As we shall show in the part of this paper dealing with the fluted absorption of the vapour, the fine lines can be traced up to the very edges of the broad band produced by the widening of the D-lines when the vapour is very dense. It is unquestionably the absorption at these lines which gives rise to the very feeble reddish fluorescence which can be seen when the light furnished by the monochromatic illuminator is symmetrical about the D-lines. Strictly monochromatic light of the wave-lengths of D_1 and D_2, no matter how intense, we feel sure would produce no fluorescence, unless the flutings actually cross this region, which is very probably the case. The only light which produces no fluorescence is green light in the vicinity of $\lambda = 5530$ and the violet

below 4600, which as we have shown is all that is transmitted by the vapour when it is very dense.

The Absorption Spectrum of Sodium Vapour.

The fluted absorption-spectrum of sodium vapour was first observed and studied by Roscoe and Schuster in 1874. Subsequent investigations were made by Liveing and Dewar in connexion with their work on the reversal of the lines of metallic vapours, and also by one of the writers.

Previous experiments by one of us having shown that it was impossible to secure photographs of the fluted spectrum with the concave grating, using the arc as a source of light, that were not contaminated by bright lines from the vapour of the lamp, it was necessary to find a source of light of great intensity and having a continuous spectrum. After experimenting with various sources of light we finally adopted the Nernst lamp, which was found to fulfill the conditions specified.

In the first series of experiments the sodium was vapourized in an atmosphere of hydrogen, generated by electrolysis, and dried by passage over calcium chloride and phosphorus pentoxide. In order to remove traces of oxygen which caused the tube to smoke, the gas was finally passed over red-hot copper gauze. The metal was heated in tubes of thin steel, the ends of which were closed with plate-glass. In the later experiments the tubes were exhausted with a mercury-pump, and the metal volatilized in a vacuum. The latter method was found to be most satisfactory. The spectra were found to be identical in the two cases.

The tubes were either heated by means of Bunsen burners, or by a coil of No. 20 iron wire, insulated from the tube by a thin layer of asbestos board. The vapour is more uniform when the tube is heated electrically, for if the upper side of the tube is colder than the lower, a non-homogeneous medium results, the density being greatest along the floor of the tube. A Bunsen burner is better, however, for some experiments, where a very dense vapour is required. With it the metal does not distil so rapidly to the colder parts of the tube, and with careful regulation of the flow of gas to the burner, it was used very successfully in many of the experiments.

The absorption-spectrum was photographed with a 14-foot concave grating in the first order, the time of exposure varying from twenty minutes to an hour, according to the density of the vapour. The second-order spectrum was observed from time to time during the exposure in order to keep the density of the vapour properly regulated.

It was found that the best results were obtained with a

slit-width of 0·065 mm. The width of the slit for the iron comparison-spectrum was 0·030 mm. In order to eliminate any errors due to changing the width of the slit, photographs were taken of both spectra with a slit-width of 0·030 mm., and the measurements made with the two sets of plates compared. The plates were measured on the dividing-engine of the laboratory, and the measurements can be considered accurate to within 0·05 of an Ångström unit. The ultra-violet region was explored in the same manner, using a tube closed with quartz plates. No trace of any flutings was found, but the lines of the principal series were strongly reversed.

The fluted absorption-spectrum makes its first appearance when the D-lines are two or three Ångström units in width. It begins as nine bands, the heads of which point towards the violet end of the spectrum, *i. e.* in each band the absorption decreases from the head towards the longer wave-lengths. The heads of the bands as they first appear are :—

$$
\begin{array}{rl}
\text{I.} & \ldots\ldots \ 4783\cdot35, \\
\text{II.} & \ldots\ldots \ 4809\cdot92, \\
\text{III.} & \ldots\ldots \ 4837\cdot72, \\
\text{IV.} & \ldots\ldots \ 4865\cdot60, \\
\text{V.} & \ldots\ldots \ 4894\cdot94, \\
\text{VI.} & \ldots\ldots \ 4932\cdot97, \\
\text{VII.} & \ldots\ldots \ 4962\cdot96, \\
\text{VIII.} & \ldots\ldots \ 5001\cdot94, \\
\text{IX.} & \ldots\ldots \ 5040\cdot71.
\end{array}
$$

There is also evidence of absorption-lines in the first seven bands ; these lines, however, do not come out strong at this stage, but appear as a slight shading on the bright background. In band I. can be seen the line 4793·10, in band II. line 4820·72, in band IX. 5053·50, and in band VII. 4979·28, which appear as fainter heads of small bands into which the others break up. Bands V. and VI. are at about the centre of the region of absorption. (Fig. 3, Plate XIV., iron comparison-spectrum.)

A very slight increase in the density of the vapour is sufficient to bring out the heads of the bands as strong dark lines, as well as the fainter heads in the bands themselves. The bands are now seen to be made up of an immense number of lines, some of which are broad and some exceedingly fine, and in some of the bands there is an appearance as if two series were superposed. (Fig. 4, Plate XIV.)

At this time the region of absorption extends from about 4600 to 5200, a series of three bands, the more refrangible

edges of which are broken up into smaller bands appearing
between 5079 and 5200, and a series of twelve bands, each
of which is divided into smaller bands coming into view
between 4780 and 4600. The wave-lengths of the principal
lines in the fluted spectrum have been carefully measured by
comparing them with standard lines in the iron spectrum.

A fluted absorption-spectrum makes its appearance in the
red and orange portion, when the density of the vapour is
such as to give the green-blue flutings at their best. This
spectrum has not been photographed and measured at the
present time owing to the difficulty of getting plates suffi-
ciently sensitive to the red. The Erythro plates, which were
used by one of us in securing photographs of this region with a
14-foot grating some years ago, are no longer on the market,
and the few experiments which we have made in sensitizing
our own plates have been only partially successful. This
region will be studied as soon as suitable plates can be
obtained.

As the density of the vapour is further increased, the
greenish-blue region disappears entirely, and the fine lines
can be traced nearly up to wave-length 54 from the blue end
of the spectrum, and from the red end they are seen to fill
up the spectrum quite up to the broad D-line absorption,
which is now 40 Ångström units in width. On crossing this
broad black band the fine lines make their appearance again,
which makes it seem probable that the red fluted spectrum
crosses the region occupied by the D-lines. A further
increase in the density results in blotting out the red, orange,
and yellow completely, leaving only a rather narrow green
region, the centre of which is at wave-length 5530 and the
violet below 4600. The colour of the transmitted light is
now a very deep violet. The fine lines can be pushed into the
green band from its opposite edges, and traces of them have
been found all through it. Finally a very black broad band
appears at the centre of the green strip, which in a spectro-
scope of small dispersion appears as a rather narrow line.
The centre of this band is at wave-length 5530 approxi-
mately. We have not been able to photograph it with the
grating, owing to the feebleness of the light, and the difficulty
of keeping the vapour at this great density for a sufficient
length of time, but its position was determined visually with
a grating of about 8 feet radius. This band has been seen
by previous observers (Liveing and Dewar), and is described
as persisting until the last traces of the fluted spectrum dis-
appeared, as the vapour cooled off. We cannot understand
this, for we have only found it when the vapour had the
maximum density attainable, namely, when a blast-lamp

played directly against the bottom of the tube, raising it to a bright red heat.

In order to see whether an increase in the length of the absorbing column of the vapour produced the same effect as increasing the density of a short column, a steel tube five feet in length was used.

In this were placed eight lumps of sodium of the same size at intervals of about 6 inches. Eight Bunsen burners were regulated so that when one of them was placed under a lump of sodium the D-lines were strongly reversed, but no trace of the fluted spectrum appeared. On adding burner after burner exactly the same sequence of events was observed as in the previous experiments, where the density of the vapour was increased by raising the temperature of the tube. Though this was to be expected it seemed worth while to try the experiment.

We have measured the wave-lengths of about 460 of the strongest lines between wave-lengths 4616 and 5738.

The tables of these wave-lengths appear in the Astrophysical Journal for September of this year, together with a complete set of the photographs of the fluted absorption-spectrum.

Johns Hopkins University,
    Baltimore, June 1903.

XLVII. *The Viscosity and Composition of some Colloidal Solutions.* By HENRY GARRETT, *B.Sc., Lond., Ph.D., Heidelberg*[*].

THE object of this research was to study the changes in the viscosity (η) of typical colloidal solutions, gelatine, silicic acid, and albumen, when the temperature and concentration are varied, and thereby to arrive at a clearer understanding of the constitution of such substances.

Two methods were used :—(a) O. E. Meyer's method, in which the viscosity is deduced from the damping of the oscillations of a disk vibrating in the liquid ; (b) the Hagen-Poiseuille method of flow through a capillary tube.

The results obtained were quite inexplicable on the older theories of colloids, but agreed well with Quincke's " foam-cell " theory, which can be shortly outlined. A colloidal solution, on this theory, is not a homogeneous fluid like water, but consists of a mixture of two solutions, having surface-tension at the surface of contact, one rich in colloid, the other

poor in colloid. The result is that the richly colloidal solution forms itself into cells which may be either full of the poorer solution or of uniform composition. Further, the cells may be isolated or may hang together in threads or masses. Granting this hypothesis it will be readily understood that the damping of a disk set swinging in such a pseudo-solution will depend upon several factors, (*a*) the internal friction of the richly colloidal solution ; (*b*) the internal friction of the matrix of poor colloid ; (*c*) the external friction of the viscous colloidal cell-wall against the less viscous matrix ; and (*d*) the surface-tension at the surface common to the two solutions. With *gelatine* the principal results found were these :—

(1) Although, as is well known, the logarithmic decrement of a disk oscillating in water, or other homogeneous fluid, is constant, yet in a gelatine solution the decrement varied considerably, even when the temperature and concentration remained unchanged. In general the decrement increases with the time during which the disk has been immersed in the solution. In the case of solutions slowly cooled to the temperature under observation the logarithmic decrement was a linear function of this time. There did not appear to be any definite maximum of decrement, but a fixed minimum was shown to exist, viz., the decrement—as found by interpolation—at the moment when the plate was introduced into the solution. This value was the only one which remained the same from day to day and with various solutions of the same strength. When the plate was taken out of the solution, well washed with hot water, cooled, and reintroduced, the same "anfangsdekrement" was obtained.

(2) Occasional disturbances, such as the passing of a vehicle in the street (50 yards away) produced with homogeneous liquids no noticeable effect, but created immense irregularities in the decrement given by gelatine solutions.

(3) Below a certain temperature t_u (depending on the strength of the jelly) the decrement increased continuously whether the plate was washed or not. Moreover, the viscosity got by Poiseuille's method also increased steadily with the time. Under these circumstances the introduction of an already solidified portion of the jelly caused the viscosity to increase more rapidly than before—an observation comparable with the crystallization of a supersaturated solution of a crystalline substance on the injection of a crystal.

(4) The logarithmic decrement of a disk swinging in a dilute (1 to 3 per cent.) solution observed after a few large vibrations was smaller than when observed with only small swings.

(5) The oscillating disk method gave much higher values for η than the transpiration method, although König found close agreement, with the two methods, for homogeneous liquids so long as the viscosity did not exceed 0·018 gr./cm. sec.

(6) β gelatine, *i. e.* gelatine which has been boiled for some time, does not show these results.

With *silicic acid* the results obtained were these :—

(1) No perceptible increase of logarithmic decrement within a few hours but a marked increase of viscosity (by both methods) within longer periods. Thus a 3·67 per cent. solution showed an increase of about 200 per cent. in the value of η in four months. Shaking a solution, whose viscosity has thus increased, with air reduces the viscosity ; but shaking it in a vacuum increases η.

(2) A solution which has been raised to the boiling-point and reduced to the original temperature shows increased viscosity ; if cooled and warmed to the original temperature decreased viscosity.

(3) In the case of silicic acid the log. dec. was not much influenced by occasional disturbances.

(4) The decrement measured after large swings was—unlike gelatine—greater than without such preliminary oscillations.

(5) As the amplitude of swing became smaller the log. dec. grew larger and attained a maximum after which it again became smaller.

(6) The oscillating disk method again gave considerably higher values for the viscosity than the transpiration method, even when the viscosity was not much greater than that of water.

With *albumen* solutions the following results were obtained :—

(1) Gradual increase of log. dec. similar to gelatine, the decrement being again a linear function of the time during which the disk was immersed in the liquid.

(2) The decrements taken after one or two large oscillations were—unlike gelatine but like silicic acid—larger than without such swings.

(3) Like silicic acid also the first decrement measured during a series of oscillations is smaller than the next and here also a maximum decrement is observed.

(4) The disk method gave much higher values for η than the transpiration method, but what was most noticeable was that boiling reduced η as got from the disk method but

increased the transpiration value so that the two became almost identical, *e. g.* :—

	η before boiling.	η after boiling.
By disk . . .	0·2278	0·2032
By transpiration	0·1250	0·1851

To explain these results, consider the effect of setting a disk in slight oscillation within a liquid of the constitution assumed by Quincke. The colloidal cells which have fixed themselves upon the disk are carried by it through the liquid and come in contact with new cells, so that the mass of cells hanging on to the plate increases and with it the resistance to vibration, or, in other words, the logarithmic decrement. If the oscillation is too large the cell-walls will be drawn out and become thinner ; the thickness of the wall may become less than twice the distance of the action of molecular force (2*l*) in which case the surface-tension becomes less. The thickness of cell-wall may even become zero and the cells then tear themselves entirely away. Hence after a large oscillation the damping will be less than after a small one.

Below the temperature t_u the richly colloidal solution is separating slowly out of the liquid as a liquid precipitate. Its water content gets gradually less, and it finally goes solid. The solidification is hastened by the addition of solid jelly as described.

Continued boiling destroys the cells and a gelatine solution then behaves as a homogeneous fluid.

In the case of silicic acid and albumen the cell-walls appear to be much less easily torn. Stretching them, so long as they remain thicker than 2*l*, does not increase the surface-tension.

On long standing the two constituents of silicic acid become by liquid precipitation more and more unlike, and the surface-tension—involving the viscosity—proportionately greater. During a large oscillation of the disk in these solutions some of the cells are brought into contact with the disk and with other cells hanging upon it. They run together and increase the area of the surface of colloidal liquid in contact with the disk and thereby the surface-tension. Hence the decrement after a large swing is greater than after a small one.

Boiling the silicic acid has the effect of breaking up the cells, thus increasing the surface-area and surface-tension so that η appears greater after boiling. Mechanical shaking (*in vacuo*) has the same effect. Moreover, boiling drives out some of the air dissolved in the cell-walls and, as is well

known, small quantities of dissolved substances lower the surface-tension. On shaking with air we have the two opposite actions—increase of surface-tension due to breaking up of the cells and decrease of surface-tension due to solution of air, the latter being the greater.

Hence the three colloids experimented upon all behave as non-homogeneous liquids, showing only differences of degree. In each case the viscosity at any temperature is not a constant quantity but depends on the size of the contact surface of the two constituents. The logarithmic decrement of a disk oscillating in the solution in each case depends upon the size of the amplitude. The observed differences point to the gelatine cell-walls being thinner than $2l$, while those of silicic acid and albumen are thicker than $2l$.

XLVIII. *Notices respecting New Books.*

La Telegrafia Senza Filo. Di AUGUSTO RIGHI *e* BERNARDO DESSAU. Con 259 figure intercalate nel testo. Bologna : Ditta Nicola Zanichelli. 1903. Pp. viii + 518.

THE authors of this work inform us in their preface that their book is not intended to be a scientific treatise on wireless telegraphy. This statement must, however, be accepted as true in a limited sense only—the work is strictly non-mathematical, and from beginning to end we have not come across a single mathematical symbol or equation. But in another sense the work is, though popular, yet truly scientific—the results of the recent remarkable developments of electrical theory being carefully and lucidly explained. There is none of that looseness of language and misleading vagueness which are only too often associated with so-called popular books on scientific subjects. Any person possessing a knowledge of elementary physics should find the volume delightful reading ; while the more advanced student will find numerous references to original sources of information.

The work is divided into four parts. Part I. deals with electrical phenomena generally, including electrostatics, electric currents, the electromagnetic field, and what is practically a verbal statement of the equations of the electromagnetic field. Part II. is devoted to electromagnetic waves : beginning with an account of the oscillatory discharge of a condenser and Feddersen's experiments, the authors lead up to Hertz's brilliant investigations, giving illustrated descriptions of the more important forms of oscillators and resonators used by different experimenters, and a very complete account of detectors of electromagnetic radiation. Part III. is the most important section of the work, and contains a brief historical sketch of the development of telegraphy, and a full account of the

systems devised by Popoff, Marconi, Lodge and Muirhead, Braun, and Slaby and Arco. Part IV. deals with systems based on photo-electric effects.

The book is one which should prove of interest to the general reader as well as the serious worker in science.

Mechanics, Molecular Physics and Heat : A Twelve Weeks' College Course. By R. A. MILLIKAN, *Ph. D., Assistant Professor of Physics in the University of Chicago.* Chicago : Scott, Foresman and Co. 1902. Pp. vii + 242.

ALTHOUGH there are numerous laboratory manuals of physics in existence, we welcome this addition to the literature of the subject, both because it will render English teachers familiar with the methods employed at the University of Chicago, and also on account of its being unlike the ordinary laboratory manual. This latter it is, indeed, not wholly intended to be, as the author informs us in his preface. One of the chief characteristics of the book is the large amount of space devoted to the elucidation of the *principles* involved in the experiments selected. The number of experiments dealt with is not large, but they are all typical of important classes of *accurate* measurements, and the apparatus employed, most of which is of modern design, is such as to admit of a high degree of accuracy. The illustrations and descriptions of the apparatus should prove of great interest to teachers in England, and the book as a whole reaches the highest standard of excellence.

Der Stickstoff und seine Wichtigsten Verbindungen. Von DR. LEOPOLD SPIEGEL, *Privatdocent an der Universität Berlin.* Mit Eingedruckten Abbildungen. Braunschweig : F. Vieweg und Sohn. 1903. Pp. xii + 912.

THIS elaborate and comprehensive treatise on nitrogen and its compounds will form an extremely useful book of reference for chemists, chemical physicists, and physiologists. Important as is the part played by carbon in the chemistry of living tissues, nitrogen is hardly less important. The author has compiled a treatise in which all but the most recent researches relating to nitrogen compounds are dealt with, and has supplied numerous references to original papers. A mere glance at the long list of references occurring at the foot of almost every page gives some idea of the enormous amount of labour which must have been bestowed on the preparation of the book. It is a most thorough and exhaustive account of the subject.

As regards the general plan of the book, the author seems to have adopted the principle of proceeding from the less to the more complicated portions of the subject. After a full account of the occurrence, properties, and preparation of nitrogen, he deals with the halogen compounds, the oxides, the sulphur and hydrogen compounds, the nitrides of the metals, phosphorus and arsenic

compounds, carbon, silicon, titanium, zirconium, and boron compounds. Then comes a section dealing with the part played by nitrogen in closed ring systems, alkaloids and proteids, and lastly a section devoted to the detection and estimation of nitrogen in analytical operations. An index is provided at the end of the book.

Hauptsätze der Differential und Integral Rechnung als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt von Dr. ROBERT FRICKE. Braunschweig : F. Vieweg und Sohn. 1902.

WE commend this book to the consideration of those among us who would like to see the technical student burdened with no more mathematics than what seems to be necessary for his needs. The teachers of pure mathematics and even of applied mathematics in our colleges have been accused of not merely giving the embryo engineer more mathematics than he needed, but of neglecting to teach him some of the most necessary parts. Now here we have a book on the calculus, not in the author's opinion a treatise on the subject, but simply a compilation of the principal theorems in a form useful for the student in the Technical ' Hochschule ' of Brunswick, and intended to be an aid to him as he attends lectures on the higher mathematics. It begins with a discussion of functions, treated to a large extent graphically, and then passes on to the theory of limits, differentiation, and the usual discussion of maxima and minima, properties of plane curves, infinite series, very much as we find in similar mathematical courses throughout our country. Then follows integration with the usual applications in Geometry and Algebra, then an introduction to differential equations, finishing with an appendix on the complex variable. The German technical student evidently gets strong meat, and there is no evidence that his special practical wants are considered. Some of the demonstrations are new, at all events to readers of English books, and these demonstrations are essentially analytical. If we leave the preface out of account, there is nothing in any of the chapters which would lead a reader to infer that the author had the technical student particularly in mind. The book is, in fact, an admirable epitome of a first course on the infinitesimal calculus.

Mathematical Papers by the late George Green. Paris : Librairie Scientifique. A. Hermann : Libraire de S.M. le Roi de Suède et de Norwege. 1903.

THIS is a fac-simile reprint of Ferrers' edition of Green's famous papers. It has evidently been reproduced by a process of photo-lithography, the only page printed from type being the title-page, which differs of necessity from the title-page in the original edition. The reproduction has been remarkably well done, and can be recommended to those who do not possess the original book.

It is unnecessary to refer in these days to the great importance of Green's work in the theory of the potential. Green's theorem is the great touchstone of transformation in mathematical physics ; and the papers on the Reflexion and Refraction of Light and on Double Refraction are of the highest interest.

Dynamics of Rotation, an elementary introduction to rigid dynamics. By A. M. WORTHINGTON. Fourth Edition. Longmans, Green & Co.

THIS is an admirable little book and should be in the hands of all teachers of dynamics. Elementary dynamics should certainly be extended to take in some of the simpler theory of the motion of real bodies. When a boy sees how much can be done in applying simple dynamical reasoning to the familiar motion of a top he will begin to have some regard for the subject. The one essential is for the student to get a true idea as to the meaning of moment of momentum, and to see clearly that its rate of change measures the acting couple. The quaternion aspect of the subject is specially to be commended, and might, we think, have been introduced more explicitly into the discussion. Once this great theorem is grasped, many of the problems of spinning tops and bicycle balancing are solved by inspection, at least in a qualitative sense. As an introduction to the study of the dynamics of extended rigid bodies, there is, we venture to think, no better book in our language, if indeed in any language.

Lehrbuch der Physik. Von O. D. CHWOLSON. Vol. I. Translated from the Russian into German by H. Pflaum. Braunschweig : F. Vieweg und Sohn. 1902.

THIS book by the well-known St Petersburg physicist is issued with a short commendatory note by Professor Wiedemann, with whom the idea originated to bring the book into wider circulation by means of a German translation. This first volume is divided into six parts, a brief introduction being followed by nine chapters devoted to Mechanik (practically our dynamics). Then we have nine chapters on measuring-instruments and the way to use them. The fourth part (of six chapters) considers the properties of gases, the fifth part, with ten chapters, the physical properties of liquids, and finally the sixth part the doctrine of rigid bodies, including crystallization, elasticity, friction, and impact. In all these sections the effects of change of temperature are freely introduced, a sufficient exposition of the principles of thermometry in the introductory chapter making this possible. The mode of treatment is essentially experimental, although the elementary processes of differentiation and integration are used when necessary. An unusual feature is the introduction into the general chapter on wave-motion of a discussion of the principle of

diffraction, in which no direct mention is made of the optical phenomena involved. These will be taken up in the second volume. It is not easy, nor is it desirable, for an author of a book on general physics to bring forward many novelties or new modes of treatment. What we desire in a book of this kind is well digested material, well arranged, and clearly expounded. In these respects Professor Chwolson's *Lehrbuch* could scarcely be excelled. If the later volumes on electricity and light are as well planned as the present first volume, an admirable text-book on physics, free from the faults which too often mar a mere compendium of scientific knowledge, will have been given to the world of students.

XLIX. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. v. p. 175.]

January 21st, 1903.—Prof. Charles Lapworth, LL.D., F.R.S., President, in the Chair.

THE following communications were read :—
1. 'The Figure of the Earth.' By William Johnson Sollas, M.A., D.Sc., LL.D., F.R.S., F.G.S., Professor of Geology in the University of Oxford.

The almost precise correspondence of great terrestrial features with a circular form seems to be frequently overlooked. The Aleutian curve has its centre in lat. 6° N., long. 177° W., that of the East Indies about 15° N. and 118° E., and round the latter centre are several concentric curves. The northern part of South America, the Alpine-Himalayan chain, the western shore of North America, and a portion of Australia, may be similarly reduced to geometric form. A great circle swept through the centres of the East Indian and Aleutian arcs runs symmetrically through the bordering seas of Asia as far as Alaska, borders the inland lakes of America, passes the Californian centre, extends through the middle of the Caribbean Sea, runs parallel with the coast of the Antarctic Continent, and returns to the East Indian centre without touching Australia. This course is in remarkable correspondence with the general trend of the great zone of Pacific weakness. If the pole of this circle in the Libyan Desert is placed towards an observer in a globe, the African Continent appears as a great dome surrounded by seas and separated from the Pacific by an irregular belt of land. A second great circle defined by Lake Baikal, and with its centre at 'the morphological centre of Asia' of Suess, and passing through the East Indian centre, may be regarded as the direction-circle for the Eurasian folding. These two centres intersect at an angle of 39°, and, on bisecting this angle, a mean directive circle is found, with its pole near the sources of the White Nile, 6° north of the

Equator. The axis of terrestrial symmetry through this pole passes through the middle of Africa and of the Pacific Ocean. The smallest circle which will circumscribe Africa has its centre near this pole, and within it the symmetry of the fractured African dome is observable. Outside this comes a belt of seas, and outside that again the Pacific belt of continents, the Antarctic, South America, North America, Asia, and Australia. Mr. Jeans has concluded on mathematical grounds that the ' pear-like shape of the earth ' might have been possessed by it at the time of its consolidation ; and he has suggested that Australia may represent the 'stalked end' of the ' pear.' The author's observations would lead him to place it in Africa, and to regard the Pacific as covering the 'broad end.'

2. ' The Sedimentary Deposits of Southern Rhodesia.' By A. J. C. Molyneux, Esq., F.G.S.

The greater portion of the area of Southern Rhodesia lies on granite and gneiss, and on the schists and slates that contain the auriferous veins worked in ancient times, and now being again opened up on an extensive scale. The remaining area is on sandstone and other sedimentary beds, with coal-deposits, and regions of volcanic rocks. To explain the deposition and order of these sediments several sections are given ; one being along a line extending from the Zambesi River on the north, through Bulawayo and the central plateau, to the Limpopo River on the south, a distance of over 400 miles. Another section, with remarks thereon, is copied, by permission, from a report by Mr. C. J. Alford, F.G.S., on the coal-bearing rocks of the Mafungibusi District.

From Bulawayo fine sandstones continue for about 170 miles to the north, when there is a sudden drop in the surface of the country, caused by a long line of cliffs of red sandstone, which extends from the Zambesi Falls Road right across this portion of Rhodesia, and finally merges into the Mafungibusi Hills far away to the north-east. This is the great escarpment, formed by the erosion of 400 feet of coarse grit with angular pebbles. To the north-west of this escarpment, and running parallel with it, is a long and narrow valley formed of soft shales which are known as the Matobola Flats. Here the beds dip at 5° south-eastward. Thus, in proceeding farther to the north-west, underlying beds are revealed, with a lower series of Coal-Measures containing seams of workable coal. Below the Coal-Measures are quartzites and current-bedded grits, which rise up and form the Sijarira Range, a flat plateau 15 miles across. Its north-western side, however, is almost precipitous, and is capped by folds of quartzites. There is a drop of 1100 feet in a few miles, and the rest of the country is almost flat as far as the Zambesi River. To the west is the gorge of the Lubu River, and it is there seen that the sediments rest upon pegmatites and gneiss.

Another section shows the contact of fine sediments and metamorphic rocks down the railway-line to the south-west past Sisi

siding, where certain plant-remains were found. By these sections the boundary or line of unconformity is traced from the Mafungibusi district, round the promontory of granites and shales which form the backbone of Matabeleland, to the Tuli district and Sabi River on the south. Except in the Tuli district, where an unconformity between the veined sandstones and the Coal-Measures is noticed, there are no definite breaks in the order of stratification; and it is by the general arrangement of superposition and characteristic features that the strata fall into certain groups. No attempt is made to correlate the strata with the Cape and Karoo systems; and for the present the author gives the following provisional classification:—

		Thickness in feet
Taba 'Sinduna Series	Sandstones and volcanic rocks ...	200
Forest Sandstones	....	1000
Escarpment Grits	............⁚...............	400
Upper Matobola Beds (fossiliferous).	Coal-Measures	300
Busse Series .... (fossiliferous).	Sandstones and grits	300
Lower Matobola Beds......	Coal-Measures	200
Sijarira Series	Quartzites and current-bedded sandstones	2000

Great unconformity.

Basement-rocks:—Gneiss, schists, and pegmatites of Mafungibusi and Lubu.

Fossils have been found in the Coal-Measures, comprising mollusca, plant- and fish-remains, which are described in appendices. These indicate the age of these beds to be Permo-Carboniferous.

The Coal-Measures yield coal of excellent quality, and the areas in which seams outcrop, or have been developed, are described under the names of the Mafungibusi, Sesami, Sengwe, Lubu, Sebungu, and Wankies Coalfields in the north, and the Tuli and Sabi Coalfields in the south.

Reference is made to the numerous mineral springs, of varying temperature, that are dotted along the Zambesi Valley, and to mounds of travertine, containing recent freshwater- and land-shells, that have been accumulated by extinct springs.

Volcanic rocks are well displayed in a long area extending from Macloutsie to the Bubi River, 200 miles; and the extinct craters are still recognizable at Fort Tuli, which gives the name to this tract of Tuli Lavas. Sheets of basalt are interbedded with the Forest Sandstones at the Bubi and Gwampa Rivers; and at a portion of the escarpment above the Sesami Coalfield, basalt forms a capping and extends back about 24 miles.

Three Appendices are added: one, on a New Species of *Acrolepis* from the Sengwe Coalfield, by A. Smith Woodward, LL.D., F.R.S., F.G.S.; a second, on some Lamellibranch Mollusca, by Wheelton Hind, M.D., F.R.C.S., F.G.S.; and a third, on some Fossil Plants from Rhodesia, by Mr. E. A. Newell Arber, M.A., F.G.S.
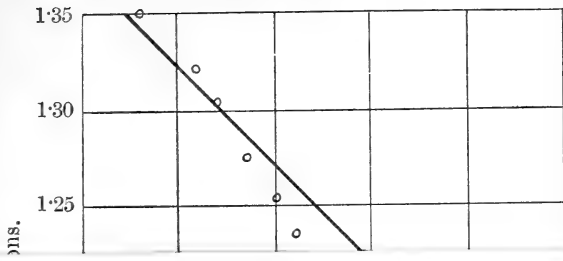
DIAGRAM 4.

DIAGRAM 6.
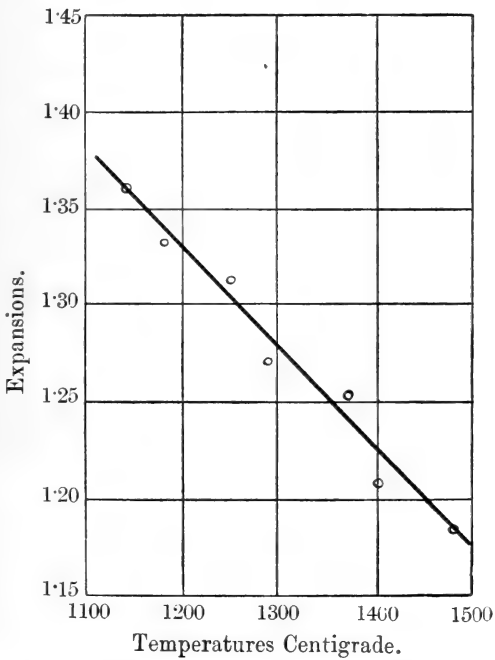
Fig. 2.

Fig. 1.
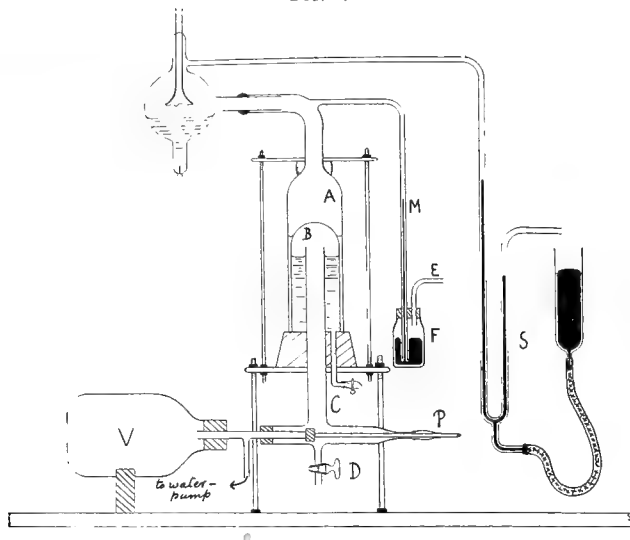
Diagram 1.
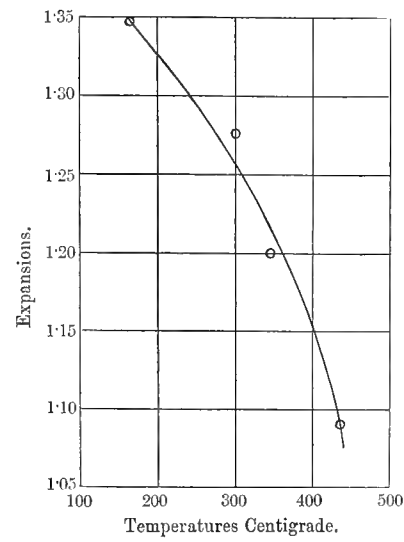
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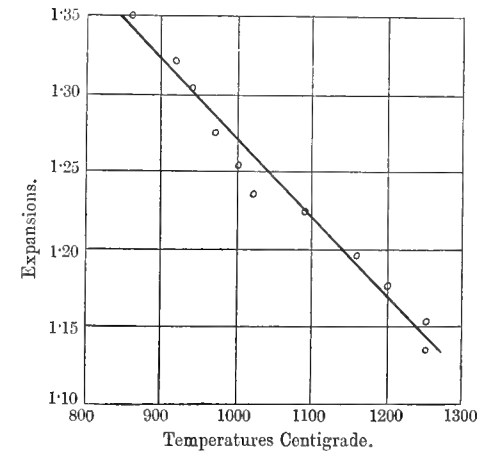
Fig. 2.
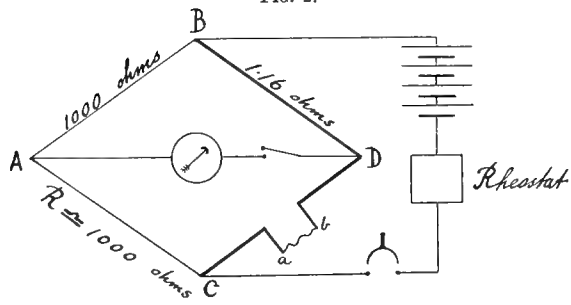
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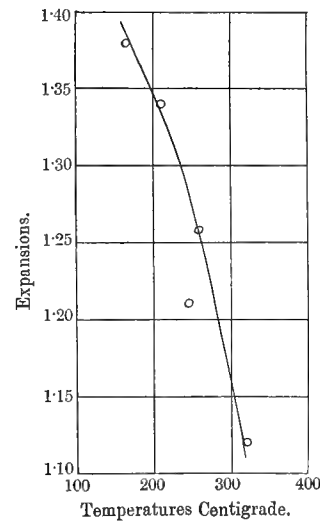
Diagram 5.

Fig. 3.

Diagram 3.

Diagram 6.
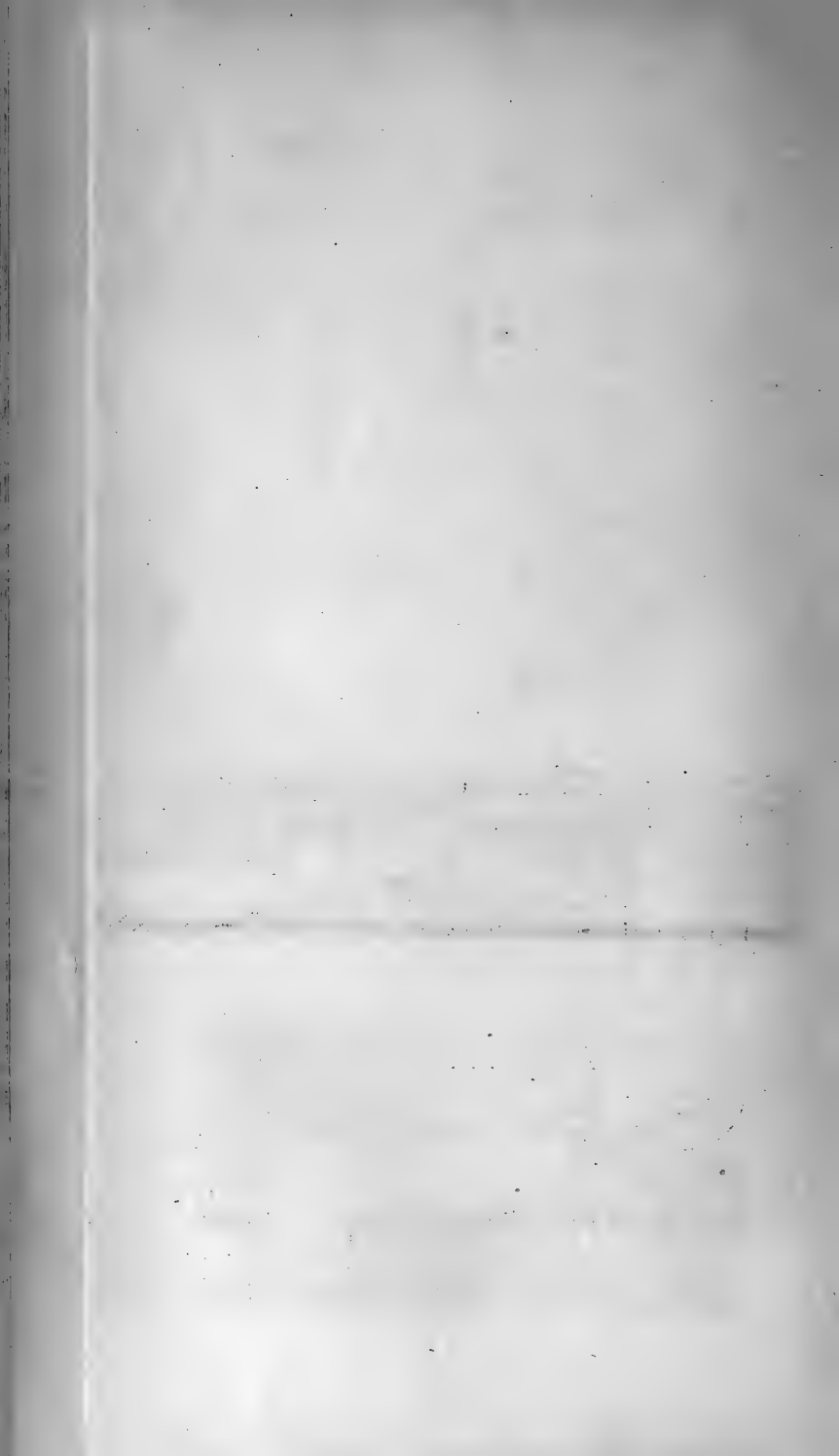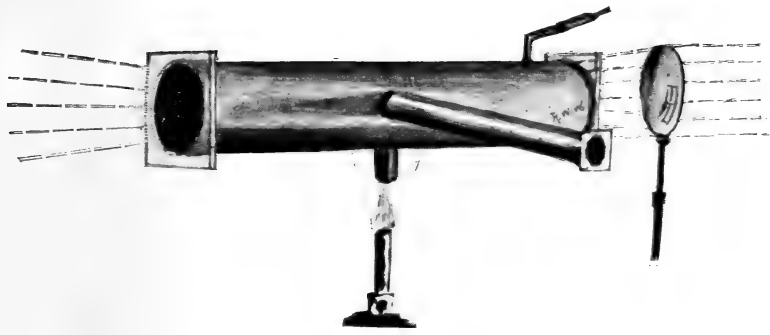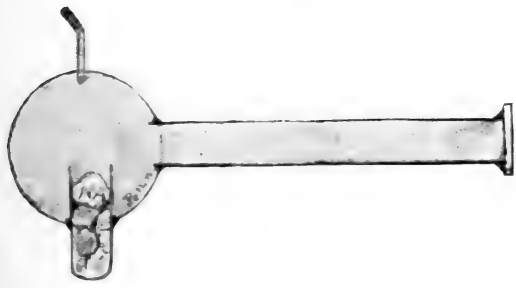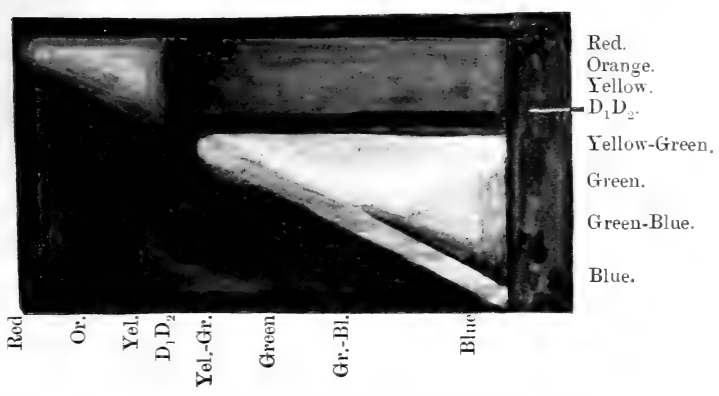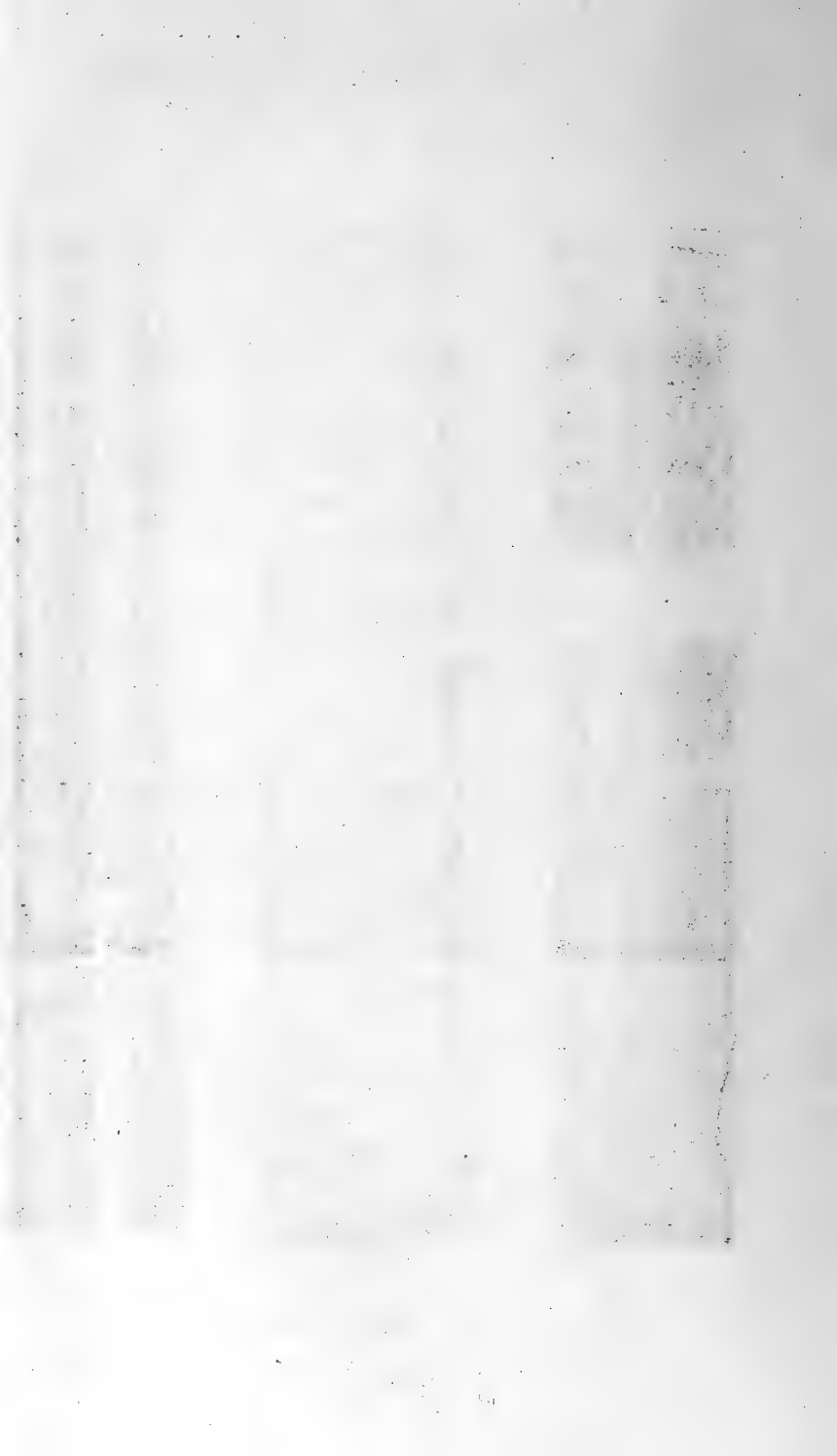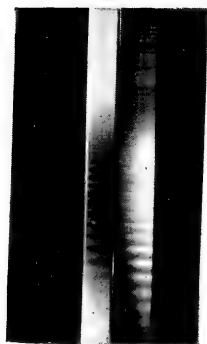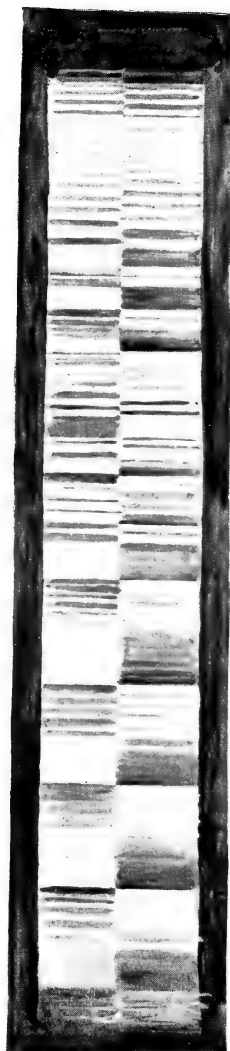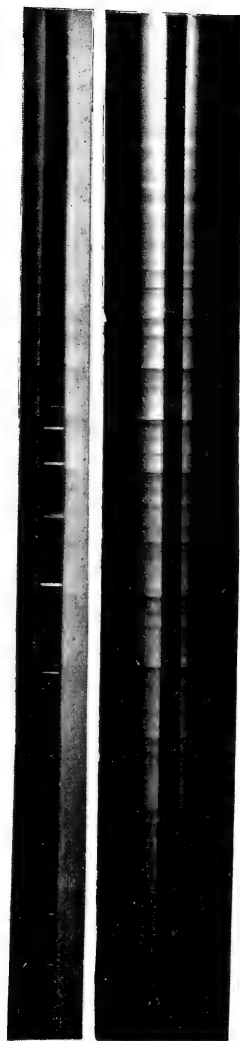
To P_2O_5 tube
and Vacuum tube.

Fig. 2.

M

P_2O_5

E

G

E'

Z

Y

q

p

r

X'

D

X

B

d

e

To P_2O_5 tube
and Vacuum tube.

f

F

d'

e'

b'

c

f'

To Water Pump.

$CaCl_2$

W

Q

Br.

R

T

u

v

z

l

l'

h'

h

i

i

j

j

K'

K

U

V

A

b

P

Fig. 1.

E

e

X

D

d

d'

e'

B

Fig. 2.

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Red.
Orange.
Yellow.
$D_1 D_2$.

Yellow-Green.

Green.

Green-Blue.

Blue.

Red.   Or.   Yel. $D_1 D_2$.   Yel.-Gr.   Green   Gr.-Bl.   Blue.

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

L. *On the Work done by Forces Operative at one or more Points of an Elastic Solid.* By Lord RAYLEIGH, O.M., F.R.S.*

A N investigation of the waves generated in an isotropic elastic solid by a periodic impressed force, localized in the neighbourhood of a point, was first given by Stokes†. A simpler treatment of the problem will be found in my paper "On the Light from the Sky, &c." ‡, and more fully in 'Theory of Sound,' § 378. It will be desirable to recapitulate the principal steps.

If α, β, γ be the displacements at any point of the solid, X′, Y′, Z′ the impressed forces reckoned per unit of *mass*, we have equations such as

$$\frac{d^2\alpha}{dt^2} = (a^2 - b^2)\frac{d\delta}{dx} + b^2\nabla^2\alpha + X', \quad . \quad . \quad . \quad (1)$$

in which a and b are the velocities of dilatational and transverse waves respectively, and δ represents the dilatation expressed by

$$\delta = \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}. \quad . \quad . \quad . \quad . \quad (2)$$

If, as throughout the present paper, α, β, γ, &c. be proportional to e^{ipt}, $d^2\alpha/dt^2 = -p^2\alpha$, and (1) &c. become

$$(a^2-b^2)d\delta/dx + b^2\nabla^2\alpha + p^2\alpha + X' = 0, \quad . \quad . \quad (3)$$

$$(a^2-b^2)d\delta/dy + b^2\nabla^2\beta + p^2\beta + Y' = 0, \quad . \quad . \quad (4)$$

$$(a^2-b^2)d\delta/dz + b^2\nabla^2\gamma + p^2\gamma + Z' = 0. \quad . \quad . \quad (5)$$

These are the fundamental equations. For our purpose we may suppose that X', Y' vanish throughout, and that Z' is finite only in the neighbourhood of the origin. It will be convenient to write

$$k = p/b, \quad h = p/a. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The *dilatation* δ is readily found. Differentiating (3), (4), (5) with respect to x, y, z and adding, we get

$$\nabla^2\delta + h^2\delta + a^{-2}dZ'/dz = 0. \quad . \quad . \quad . \quad (7)$$

The solution of (7) is

$$\delta = \frac{1}{4\pi a^2}\iiint \frac{dZ'}{dz}\frac{e^{-ihr}}{r}\,dx\,dy\,dz,$$

r denoting the distance between the element at x, y, z near the origin (O) and the point (P) under consideration. If we integrate partially with respect to z, we find

$$\delta = -\frac{1}{4\pi a^2}\iiint Z'\frac{d}{dz}\left(\frac{e^{-ihr}}{r}\right)dx\,dy\,dz, \quad . \quad . \quad (8)$$

the integrated term vanishing in virtue of the condition that Z' is finite only within a certain space T. Moreover, since the dimensions of T are supposed to be very small in comparison with the wave-length, $d(r^{-1}e^{-ihr})/dz$ may be removed from under the integral sign.

It will be convenient also to change the meaning of x, y, z, so that they shall represent as usual the coordinates of P relatively to O. Thus, if $Z_1 e^{ipt}$ denote the whole force applied at the origin, so that

$$Z_1 = \rho\iiint Z'\,dx\,dy\,dz,$$

in which ρ is the density,

$$\delta = \frac{Z_1}{4\pi a^2\rho}\frac{d}{dz}\left(\frac{e^{-ihr}}{r}\right), \quad . \quad . \quad . \quad . \quad (9)$$

giving the dilatation at the point P.

In like manner we may find the *rotations* ϖ', ϖ'', ϖ''', defined by

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 2\varpi', \quad \frac{d\alpha}{dz} - \frac{d\gamma}{dx} = 2\varpi'', \quad \frac{d\beta}{dx} - \frac{d\alpha}{dy} = 2\varpi'''. \quad (10)$$

For from (3), (4), (5) we have

$$\nabla^2\varpi' + k^2\varpi' + \tfrac{1}{2}b^{-2}dZ'/dy = 0, \quad . \quad . \quad (11)$$
$$\nabla^2\varpi'' + k^2\varpi'' - \tfrac{1}{2}b^{-2}dZ'/dx = 0, \quad . \quad . \quad (12)$$
$$\nabla^2\varpi''' + k^2\varpi''' \qquad\qquad = 0, \quad . \quad . \quad (13)$$

whence $\varpi''' = 0$, and

$$\varpi' = \frac{Z_1}{8\pi b^2 \rho} \frac{d}{dy}\left(\frac{e^{-ikr}}{r}\right), \qquad \varpi'' = -\frac{Z_1}{8\pi b^2 \rho} \frac{d}{dx}\left(\frac{e^{-ikr}}{r}\right). \quad (14)$$

These are the results given in my paper of 1871.

The values of δ, ϖ', ϖ'', ϖ''' determine those of α, β, γ. If we take

$$\alpha = \frac{d^2\chi}{dx\,dz}, \qquad \beta = \frac{d^2\chi}{dy\,dz}, \qquad \gamma = \frac{d^2\chi}{dz^2} + w, \quad . \quad (15)$$

where

$$w = k^2 A \frac{e^{-ikr}}{r}, \qquad \chi = A\left(\frac{e^{-ikr}}{r} - \frac{e^{-ihr}}{r}\right), \quad . \quad . \quad (16)$$

and

$$A = \frac{Z_1}{4\pi k^2 b^2 \rho}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

it is easy to verify that these forms give the correct values to δ, ϖ', ϖ'', ϖ'''. As regards the dilatation,

$$\delta = \frac{d}{dz}(\nabla^2\chi + w),$$

in which

$$\nabla^2\chi = A\left(-k^2\frac{e^{-ikr}}{r} + h^2\frac{e^{-ihr}}{r}\right).$$

This reproduces (9).

As regards the rotations, we see that χ does not influence them. In fact

$$\varpi''' = 0, \qquad \varpi' = \tfrac{1}{2}\frac{dw}{dy}, \qquad \varpi'' = -\frac{dw}{dx};$$

and these agree with (14). The solution expressed by (15), (16), (17) is thus verified, and it applies whether the solid be compressible or not.

In the case of incompressibility, $h = 0$. If we restore the time-factor e^{ipt} and throw away the imaginary part of the solution, we get

$$\alpha = \frac{\mathrm{A}k^2 xz}{r^3}\left[\left(-1+\frac{3}{k^2 r^2}\right)\cos(pt-kr)\right.$$

$$\left. -\frac{3}{kr}\sin(pt-kr)-\frac{3}{k^2 r^2}\cos pt\right], \quad . \quad . \quad (18)$$

$$\gamma = \frac{\mathrm{A}k^2}{r}\left[\left(1-\frac{z^2}{r^2}+\frac{3z^2}{k^2 r^4}-\frac{1}{k^2 r^2}\right)\cos(pt-kr)\right.$$

$$\left.+\left(\frac{1}{kr}-\frac{3z^2}{kr^3}\right)\sin(pt-kr)-\left(\frac{3z^2}{k^2 r^4}-\frac{1}{k^2 r^2}\right)\cos pt\right], (19)$$

the value of β differing from that of α merely by the substitution of y for x. The value of A is given by (17), and $Z_1 \cos pt$ is the whole force operative at the origin at time t.

At a great distance from the origin (18), (19) reduce to

$$\alpha = -\frac{Z_1}{4\pi b^2 \rho}\frac{xz}{r^2}\frac{\cos(pt-kr)}{r}, \quad . \quad . \quad . \quad (20)$$

$$\gamma = +\frac{Z_1}{4\pi b^2 \rho}\left(1-\frac{z^2}{r^2}\right)\frac{\cos(pt-kr)}{r} \quad . \quad . \quad (21)$$

Upon this ('Theory of Sound,' § 378) I commented:—
"W. König (Wied. *Ann.* xxxvii. p. 651, 1889) has remarked upon the non-agreement of (18), (19), first given in a different form by Stokes, with the results of a somewhat similar investigation by Hertz (Wied. *Ann.* xxxvi. p. 1, 1889), in which the terms involving $\cos pt$, $\sin pt$ do not occur, and he seems disposed to regard Stokes's results as affected by error. But the fact is that the problems treated are essentially different, that of Hertz having no relation to elastic solids. The source of the discrepancy is in the first terms of (3) &c., which are omitted by Hertz in his theory of the æther. But assuredly in a theory of elastic solids these terms must be retained. Even when the material is supposed to be incompressible, so that δ vanishes, the retention is still necessary, because, as was fully explained by Stokes in the memoir referred to, the factor $(a^2 - b^2)$ is infinite at the same time."

Although the substance of the above comment appears to be justified, I went too far in saying that Hertz's solution has no relation to elastic solids. It is indeed not permissible to omit the first terms of (3) &c. merely because the solid is incompressible; but if, though the solid is compressible, it be in fact not compressed, these terms disappear. Now

Hertz's solution, corresponding to the omission of the second part of χ in (16), makes

$$\alpha = \frac{d^2}{dx\,dz}\left(\frac{e^{-ikr}}{r}\right), \quad \beta = \frac{d^2}{dy\,dz}\left(\frac{e^{-ikr}}{r}\right), \quad \gamma = \left(\frac{d^2}{dz^2} + k^2\right)\left(\frac{e^{-ikr}}{r}\right),$$
$$\cdots \quad (22)$$

and accordingly

$$\delta = \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0 :$$

values which satisfy

$$(\nabla^2 + k^2)\,(\alpha,\,\beta,\,\gamma) = 0. \quad \cdots \quad (23)$$

Thus (3), (4), (5) are satisfied, and the solution applies to an elastic solid upon which no forces act except at the origin. The only question remaining open is as to the character of the forces which must be supposed to act at that place. This is rather a delicate matter ; but it is evident at any rate that the forces are not of the simple character contemplated in the preceding investigation. It would appear that they must be double or multiple, and have components parallel to x and y as well as z. By a *double* force is meant the limit of a *couple* of given moment when the components increase and their mutual distance decreases, analogous to the double source of acoustics.

I now propose to calculate the work done by the force Z at the origin as it generates the waves represented by (18), (19). For this purpose we require the part of γ in the neighbourhood of the origin which is in quadrature with the force, *i.e.* is proportional to $\sin pt$. From (19) we get

$$\frac{Ak^2\sin pt}{r}\left[\left(1 - \frac{z^2}{r^2} + \frac{3z^2}{k^2r^4} - \frac{1}{k^2r^2}\right)\sin kr + \left(\frac{1}{kr} - \frac{3z^2}{kr^3}\right)\cos kr\right],$$
$$\cdots \quad (24)$$

the last term (in $\cos pt$) not contributing. Expanding $\sin kr$, $\cos kr$ and retaining the terms of order kr, we get for the square bracket

$$\tfrac{2}{3}kr + \frac{z^2}{r^2}(0).$$

Thus

$$(24) = \tfrac{2}{3}k^3A \sin pt,$$

when r is small, so that the part proportional to $\sin pt$ is in

the limit finite and independent of z/r. If W be the work done in time dt,

$$\frac{dW}{dt} = Z_1 \cos pt \cdot \tfrac{2}{3} k^3 p A \cos pt \,;$$

and by (6), (17)

$$\text{mean } \frac{dW}{dt} = \frac{k^2 Z_1^2}{12\pi b\rho}. \quad \ldots \ldots (25)$$

The right-hand member of (25) is thus the work done (on the average) in unit of time.

This result may be confirmed by a calculation of the energy radiated away in unit time, for which purpose we may employ the formulæ (20), (21) applicable when r is great. The energy in question is the double of the kinetic energy to be found in a spherical shell whose thickness $(r_2 - r_1)$ is the distance travelled by transverse waves in the unit of time, viz. b. In the expression for the kinetic energy the resultant (velocity)^2 at any point x, y, z is by (20), (21) proportional to

$$\frac{x^2 z^2}{r^4} + \frac{y^2 z^2}{r^4} + \frac{(r^2 - z^2)^2}{r^4} = 1 - \frac{z^2}{r^2} = 1 - \mu^2,$$

a quantity symmetrical with respect to the axis. Also $\sin^2(pt - kr)$ is to be replaced by its mean value, viz. $\tfrac{1}{2}$. Thus the kinetic energy is

$$\tfrac{1}{2}\rho \left(\frac{Z_1}{4\pi b^2 \rho} \right)^2 \int_{-1}^{+1} \int_{r_1}^{r_2} \tfrac{1}{2} p^2 (1 - \mu^2) 2\pi \, d\mu \, dr,$$

the double of which is identical with (25).

We will now form the expression for the resolved displacement at P due to $Z_1 \cos pt$ acting at O (parallel to OZ), the displacement being resolved in a direction PT in the plane

Fig. 1.
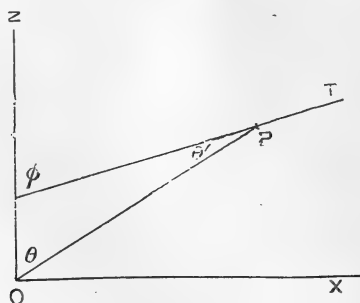
ZOP making an angle θ' with OP (fig. 1). The angle between PT and OZ is denoted by ϕ, so that $\phi = \theta + \theta'$.

The resolved displacement is

$$\gamma \cos \phi + \alpha \sin \phi, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

α and γ being given by (18), (19), in which we write

$$z/r = \cos \theta, \quad x/r = \sin \theta.$$

We find

$$\frac{\gamma \cos \phi + \alpha \sin \phi}{Ak^2/r}$$

$$= \cos (pt - kr) \left[-\sin \theta \sin \theta' - \frac{\cos \phi}{k^2 r^2} + \frac{3}{k^2 r^2} \cos \theta \cos \theta' \right]$$

$$+ \frac{\sin (pt - kr)}{kr} \left[\cos \phi - 3 \cos \theta \cos \theta' \right]$$

$$+ \frac{\cos pt}{k^2 r^2} \left[\cos \phi - 3 \cos \theta \cos \theta' \right],$$

a symmetrical function of θ and θ' as required by the general principle of reciprocity ('Theory of Sound,' § 108). The value of A is given by (17) in which, however, we will now write $\mathfrak{F}$ for Z_1, so that

$$A = \frac{\mathfrak{F}}{4\pi k^2 b^2 \rho} . \quad . \quad . \quad . \quad . \quad (27)$$

The above equation gives the resolved displacement at P in a direction making an angle θ' with r due to a force $\mathfrak{F} \cos pt$ at O acting in a direction inclined to r at angle θ. If we suppose that a force $\mathfrak{F}' \cos pt$ acts at P in direction PT and inquire as to the work done by this force upon the motion due to $\mathfrak{F}$, we have to retain that part of the resolved displacement due to $\mathfrak{F}$ which is in quadrature with $\mathfrak{F}' \cos pt$, viz. the part proportional to $\sin pt$. The mean work is given by the symmetrical expression

$$\frac{k^2 \mathfrak{F} \mathfrak{F}'}{8\pi b \rho} \left\{ \frac{\cos (\theta + \theta') - 3 \cos \theta \cos \theta'}{k^2 r^2} \left(\cos kr - \frac{\sin kr}{kr} \right) \right.$$

$$\left. - \frac{\sin kr}{kr} \sin \theta \sin \theta' \right\} . \quad . \quad . \quad . \quad (28)$$

If the forces are parallel, $\phi = 0$, $\theta' = -\theta$, and (28) becomes

$$\frac{k^2 \mathfrak{F} \mathfrak{F}'}{8\pi b \rho} \left[\sin^2 \theta \frac{\sin kr}{kr} + \frac{1 - 3 \cos^2 \theta}{k^2 r^2} \left(\cos kr - \frac{\sin kr}{kr} \right) \right]. \quad (29)$$

If we further suppose that kr is very small, the square bracket reduces to the value $\frac{2}{3}$, and we get

$$\frac{k^2 \mathfrak{F} \mathfrak{F}'}{12\pi b \rho} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Comparing with (25) we see that the work done by $\mathfrak{F}'$ on the motion due to an equal $\mathfrak{F}$ is the same as that done by $\mathfrak{F}$ itself, as should evidently be.

If in (29) $\theta = 90°$, so that the forces are perpendicular to the line joining the two points of application, we get

$$\frac{k^2 \mathfrak{F}\mathfrak{F}'}{8\pi b\rho} \left[\frac{\sin kr}{kr} + \frac{1}{k^2 r^2}\left(\cos kr - \frac{\sin kr}{kr}\right)\right]. \quad . \quad . \quad (31)$$

As we have seen, when kr is small (31) is finite and positive. It vanishes when

$$\tan kr = \frac{kr}{1 - k^2 r^2}, \quad . \quad . \quad . \quad . \quad (32)$$

and this occurs first in the second quadrant.

In general, when there are a number of forces acting at detached points, the whole work done must be obtained by a double summation of (28). If the forces are continuously distributed, the sum becomes a double integral. A particular case, of interest in connexion with the problem of electrical vibrations along a circular wire[*], occurs when the forces act tangentially at the various points of a circular arc. Here $\theta' = \theta$, and (28) becomes

$$\frac{k^2 \mathfrak{F}\mathfrak{F}'}{8\pi b\rho} \left\{ \frac{1 + \cos^2\theta}{k^2 r^2} \left(\frac{\sin kr}{kr} - \cos kr\right) - \frac{\sin kr}{kr}\sin^2\theta \right\}. \quad (33)$$

Aug. 2, 1903.

LI. *Change of Length of Ferromagnetic Substances under High and Low Temperatures by Magnetization.* *By* K. Honda, *Rigakuhakushi, and* S. Shimizu, *Rigakushi*[†].

[Plate XV.]

THE interesting experiments of Hopkinson on the magnetization at high temperatures promise us some important results in the magnetic change of length at such temperatures. On account, however, of the experimental difficulties, this interesting subject has as yet scarcely been investigated. Barrett[‡] was the first to touch the subject: with a rise of temperature of about $50°$ C. he observed no effect on the magnetic change of length in iron and cobalt,

* Compare Pocklington, Proc. Camb. Phil. Soc. ix. p. 324 (1897); 'Nature,' lxii. p. 486 (1903). It would seem that (33) must lead to a more complicated expression for the energy radiated than that in Dr. Pocklington's investigation.

† Communicated by the Authors.

‡ Barrett, Phil. Mag. [4] xlvii. p. 51 (1874); 'Nature,' xxvi. pp. 515 586 (1882); *Beibl.* vii. p. 201.

but in nickel the contraction was reduced to about two-thirds of its ordinary value. A few years ago one of us* studied the same effect in iron, tungsten-steel, and nickel at temperatures ranging from 18° C. to 100° C. In weak fields the magnetic elongation of iron was slightly diminished by heating ; but in strong fields it was increased. In tungsten-steel the elongation was always diminished ; in nickel the contraction was considerably reduced, except in weak fields, in which a minute increase of contraction was sometimes observed.

In these experiments the range of temperatures was very limited, so that the remarkable effect of high temperatures was not observed. In the present case the experiments were pushed beyond the critical temperature of iron ; in addition to this, the change of length in liquid air was also examined.

§ 1. *Arrangement.*

The apparatus for measuring the change of length by magnetization at high and low temperatures was substantially the same as that used for the study of the effect of tension on the magnetic change of length † (see figs. 1 & 2, Pl. XV.). Repeated experiments showed that this arrangement gave very consistent results, but that it was rather preferable to flatten the surface of the vertically suspended wire in contact with the rotating cylinder.

To diminish the effect of tension on the magnetic change of length, as well as the yielding at high temperatures due to tension, thick rods about 1 cm. in diameter and 21 cms. in length were employed.

In the experiments at high temperatures the suspended weight was generally 1 or 2 kilograms ; its effects were consequently almost insensible. To the upper end of the ferromagnetic rod a copper rod about 1 cm. thick and 25 cms. long was jointed by means of a copper screw and then brazed. The lower end of the rod was likewise attached to a similar copper rod about 20 cms. long. The screwed part in each end of the ferromagnetic rod was about 1·5 mm. This connected system hung vertically from a stout support by means of a brass stand with three levelling-screws. The support on which the tripod-stand rested was provided with a brass plate with a hole-slot-plane arrangement. The free end of the bar was connected with a copper wire about 1·5 mm.

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thick which was stretched by a weight dipping in a vessel of water.

Heating was effected by means of an electric current. The heating coil was wound on a copper tube 40 cms. long and 2·5 cms. in diameter well insulated with asbestos-paper. The coil was wound anti-inductively two turns per centimetre by a wire about 1 mm. thick. Two heating coils of the same dimensions were prepared, the one wound with a german-silver and the other with a nickel wire. The former coil was used in the experiments below 700° C., and the latter for higher temperatures. The melting-point of nickel is about 1500° C., while its magnetic property is lost at a temperature below 400° C.; hence above this temperature the presence of the metal does not at all disturb the magnetic field. Thus by using a nickel wire for experiments at high temperatures we may dispense with a costly platinum wire or foil.

The heating coil was fixed by means of screws to the upper copper rod, while in its lower end was a hole through which the lower copper rod passed without being in contact. The air current which might enter or escape through this narrow opening was diminished on one side by another partition placed 3 cms. above the lower end of the coil, and on the other side by a bundle of fibrous asbestos attached to the copper rod just below the same end.

The exposed parts of the copper rods and wire were well covered with asbestos-paper, except the part where the wire came in contact with the rotating cylinder. This precaution was necessary to diminish the loss of heat, and also to avoid the oscillatory displacement of the image in the field of the telescope due to thermal expansion and contraction caused by the air current.

The temperature of the samples to be tested was measured by means of a platinum platinum-rhodium junction which was loosely placed in contact with the sample in its middle part, while the rest was insulated with asbestos-paper. The other junction well insulated was inserted into a copper tube dipped in a water bath of the temperature of the room. The thermoelectric current in the circuit was measured by a d'Arsonval galvanometer, and the constant of the pyrometer was determined by means of a mercury-thermometer below 300° C. and by the melting-point of zinc and sodium chloride above that temperature.

A magnetizing-coil with a waterjacketed arrangement was placed coaxially with the suspended rod; it was 40 cms. long and gave a field of 39·44 C.G.S. units at the centre due to a current of one ampere. When adjustments were finished

the exposed parts were well covered with fibrous asbestos and cotton in order to diminish the loss of the generated heat, care being taken to produce no sensible resistance to the elongation or contraction of our specimens.

For the measurement of the change of length in liquid air the above arrangement was modified in the following way:— The specimen was stretched upwards by means of a spring instead of stretching it by a suspended weight. The shaded portions of the protruding supporter were made of brass, while the unshaded portion was made of wood to lessen the conduction of heat. Care was specially taken to stretch the copper wire in the direction of the axis of the rod.

The magnetizing coil with a Dewar's tube inside it was placed coaxially with the rod. The contraction or elongation of the specimen was measured by a rotating cylinder in contact with a vertical copper wire. The temperature of the liquid air was assumed to be $-186°$ C.

§ 2. *Method of Observation.*

The experiments at low temperatures were conducted in the following manner:—The change of length at the temperature of the room was first determined and compared with the corresponding result obtained by the heating arrangement. The comparison showed that these two results nearly coincided with each other. The liquid air was then poured into the Dewar's tube containing the specimen, and the exposed parts round the magnetizing coil were carefully protected with cotton. Owing to the violent evaporation of the liquid a small oscillation of the image in the telescope was first observed, but after about 15 minutes the image became steady. The magnetic change of length was then measured in the usual way.

The experiments at high temperatures were undertaken in the following order:—The change of length at the temperature of the room was first determined. Then an electric current from a dynamo was passed through the heating-coil for one or two hours till the temperature of the core became very constant; then the current from the dynamo was replaced by one from an accumulator in order to get rid of the fluctuation of temperature due to irregularities in the dynamo-current. Twenty or thirty minutes passed before the observations could be taken, when the temperature had become very steady. The change of temperature caused a deflexion of more than 6 cms. per degree in the field of the observing-telescope, whereas the deflexion due to magnetization was almost 5 cms. in nickel

and 1·5 cm. in soft iron; hence it can be easily judged how great difficulty is experienced in obtaining a constant temperature.   Since the demagnetization was carefully effected before each observation and then an instantaneous deflexion was noted, a slow displacement of the image, such as 1 mm. in several minutes, could not cause any sensible error in the results.   In our experiments, during the whole set of observations which usually required ten or fifteen minutes, the displacement of the zero point was only two or three millimetres, except in a few cases.

After one set of observations had been taken, a current of increased strength was again supplied by the dynamo for about an hour or more till the temperature became nearly constant. The current-supply from the dynamo was then changed for that from the accumulator to repeat the same subsequent processes.   In this way experiments at successively increasing temperatures from the ordinary to the highest were performed. Our heating-coil gave a rise of temperature of about 1000° C. in the core by passing a current of 11 amperes.

When the specimen was once heated to a high temperature it underwent a permanent change with regard to the magnetic change of length; thus it was necessary to try first the experiment at low temperatures before our specimens had been heated to high temperatures.

The sensibility of our apparatus was such that an elongation or contraction of $4·8 \times 10^{-8}$ of our specimen could easily be observed.

We tested six different samples shown in the following table :—

Metal.	Length.	Diameter.	Demagnetizing factor.
Soft iron	22·05 cms.	1·025 cm.	0·0683
Tungsten steel (rod)...	22·93	0·940	0·0562
,, ,, (prism)	22·00	$0·948 \times 0·953$	0·0751
Cast cobalt	22·10	1·044	0·0722
Ann. cobalt	22·03	1·083	0·0758
Nickel.....................	22·03	1·121	0·0801

§ 3. *Results of Experiments.*

The change of length of our specimens at the temperature of the room (8° C. — 17° C.) is shown in figure 3 (Pl. XV.).

The full lines represent the curves of the change of length plotted against the internal field $(H' = H - LN)$; the dotted lines refer to the external field H. The comparison of the corresponding curves shows us the influence of the demagnetizing force.

In the experiments at high temperatures the internal field for a given magnetizing current varies with the temperature, as the intensity of magnetization changes with it. A full knowledge of the field in which experiments were carried on requires the determination of the intensity of magnetization at each temperature and field. Since our experiments did not extend so far the curves of the length-change at different temperatures were drawn for the external field. But if we refer to fig. 3, it is easy to see how the forms of these curves are to be changed if the internal field be used instead of the external one.

Nickel.—The change of length in nickel under high and low temperatures is graphically shown in fig. 4. The temperature markedly reduces the magnetic contraction of the metal. At a temperature of $240°$ C. the contraction in $H = 800$ is already reduced to half its ordinary value, and at $400°$ C. it almost vanishes. With the ovoid of the same specimen Prof. H. Nagaoka and Mr. S. Kusakabe found the critical temperature to be $400°$ C. In liquid air the contraction is reduced in weak fields, but is increased in strong fields. The relation between the change of length and the temperature for given external fields is given in fig. 5. Each curve has a minimum point, the temperature of which decreases as the field is increased. We also notice that the contraction vanishes asymptotically as the temperature approaches to $400°$ C. It is to be remembered that on account of the demagnetizing force each curve does not represent the contraction in a constant effective field, but shows the general feature of contraction with regard to temperature. The above results are consistent with the former experiments.

Soft iron.—The change of length in soft iron is given in fig. 6.

As the temperature is raised the contraction in high fields gradually disappears, and at $312°$ C. the change of length is similar to that of tungsten-steel at ordinary temperature. With further increase of temperature the elongation, after passing a maximum, gradually decreases. We could trace the elongation up to $970°$ C., which is far higher than its critical temperature. The effect of cooling by liquid air is

decidedly large in strong fields, producing an increase of contraction.

The relation between the change of length and the temperature is given in fig. 7. It is remarkable to observe that the maximum elongation in weak fields, which is characteristic for iron, remains almost constant for temperatures ranging from −186° C. to 200° C. Above this temperature the elongation increases till it reaches a maximum, and then rapidly decreases.

Tungsten steel.—The results of experiments in tungsten steel are given in figs. 8, 9, & 10. The course of the curves and its variation with temperature are similar to those of soft iron at temperatures higher than 500° C. The change of length seems to disappear nearly at its critical temperature, namely 900° C., a value obtained by Prof. H. Nagaoka and Mr. S. Kusakabe. The former result obtained by one of us approximately agrees with the corresponding result in the present experiment.

With tungsten steel we first studied the effect of temperature; when the specimen was cooled down to its initial temperature it underwent a considerable permanent change with regard to the change of length. So the experiment in liquid air was performed with another rod of square section cut from the same specimen as the cylinder. The curves for 10° C. and −186° C. are given in figure 10, which shows a slight effect of cooling on the change of length. Cooling decreases the elongation of the alloy in weak fields, but increases it in strong fields.

Cast cobalt.—The results of observations on cast cobalt are shown in figs. 11 & 12. As the temperature is raised the magnetic contraction in weak fields gradually lessens, and the elongation in strong fields increases till it reaches a maximum. At temperatures higher than 800° C. the initial contraction altogether disappears, and the course of the curves resembles that of iron and steel at high temperatures. If the temperature be further increased the elongation diminishes steadily but at a diminishing rate, and even at such a high temperature as 1020° C. we still observe a considerable elongation of the metal. From the course of the curves in fig. 12 it is easy to see that for $H = 800$ the elongation does not vanish up to a temperature of 1200° C., which is higher than its critical temperature by 100° C.

With our arrangement it was not possible to push the experiments still further, as the melting-point of copper was not far from that temperature. It is also to be observed that

the field of maximum contraction gradually decreases as the temperature is raised, and that the temperature of maximum elongation in a given field diminishes as the field is increased.

With the same specimen the effect of high temperature was also first studied, and when the specimen was cooled down to its initial temperature, it totally changed its character with regard to the magnetic change of length. It was therefore not possible to examine the effect of cooling in the metal in its cast state.

Annealed cobalt.—The effect of high temperatures on annealed cobalt presents an extraordinary feature, as may be seen from fig. 13. The specimen was annealed in a charcoal fire for about four hours after being carefully wrapped in asbestos-paper. The change of length at ordinary temperature is abnormal. As the temperature is raised, beginning with that of liquid air, the contraction increases at first slowly and then rapidly, till it reaches a maximum. It then decreases, and after passing the state of no contraction it is changed to an elongation which again increases with the temperature up to a maximum, and then gradually diminishes. At such a high temperature as 1034° C. we could still observe a considerable elongation of the metal. To judge from the course of the curves, the temperature at which the elongation at last vanishes is a little lower than in cast cobalt. It is interesting to observe that the curve of the length-change at a temperature near 450° C. is similar to that of iron at ordinary temperature. The cobalt slightly elongates in weak fields, but it contracts in strong fields. At temperatures higher than 500° C. the cast and annealed cobalts resemble each other in their behaviour in respect of the change of length.

The curves (fig. 14) showing the relation between the change of length and the temperature present a peculiar feature, having generally one maximum and one minimum. In low fields, however, two small maxima and minima are observed. Since each curve in the figure almost passes through a point (464° C.) on the axis of temperature it follows that there is a certain temperature at which the change of length in annealed cobalt nearly disappears for all magnetizing fields, and that the change occurs in an opposite sense in every field, according as the specimen is heated above or below that temperature. It appears, then, that annealed cobalt undergoes some molecular change at that temperature.

General remarks.—On comparing the above results in soft

iron, tungsten-steel, cast and annealed cobalts, we notice the remarkable fact that the changes of length of these metals at ordinary temperature, so very different from each other, assume, at sufficiently high temperatures, an extraordinarily simple character; they tend to become proportional to magnetic force, a fact which has no doubt an important bearing on the theory of molecular magnetism.

It is also to be observed that the change of length of the ferromagnetic substances at their critical temperature nearly disappears, and even in cases for which we actually observe it the amount of the change is only a small fraction of the change at ordinary temperature.

In conclusion, let us give a short account of the permanent change with regard to the magnetic change of length. Cooling the specimen in liquid air has almost no permanent effect on the change of length at the ordinary temperature; but heating it to a very high temperature generally produces a considerable permanent change. As seen from fig. 15 (Pl. XV.) the heating of soft iron up to 746° C. does not sensibly affect the length-change at ordinary temperature. Here the crosses (×) denote the points obtained after heating to 746° C. In tungsten-steel the effect is very large, tending to reduce the elongation in high fields.

In cast cobalt the effect is still greater, changing totally the course of the curve, as seen from fig. 16. If the specimen is once annealed at a high temperature subsequent heating and cooling between the same limits of temperature produce a slight effect on the change of length at ordinary temperature. But if the upper range of temperature be further increased the change of length is slightly affected. This will be seen from the example of annealed cobalt.

In the experiments at temperatures higher than 700° C. the suspended weight was reduced to 1 or 0·4 kilogram, according to the circumstances. This was found necessary to avoid the gradual elongation of our specimens caused by the yielding at high temperatures.

It is hoped that the present investigation may be completed by studying the change of length in every stage of rising and falling temperatures, and also by measuring the magnetization at the corresponding temperatures. The further extension to other ferromagnetics, such as nickel-steels of different percentages, will also be undertaken in the near future.

LII. *On the Interpretation of Milne Seismograms.* By C. COLERIDGE FARR, *D.Sc.**

[Plate XVI.]

THE question whether a horizontal pendulum Seismograph acts as a clinograph or whether its records must in part be ascribed to horizontal movements of the Earth's surface has received discussion by Milne†, Omori‡, and by others also whose arguments I have not been able to peruse. Both Prof. Milne and Dr. Omori conclude that the tilts represented by the maximum displacement of the boom would indicate earth movements too large to be admissible.

Without attempting to give a complete theory§ of the movement of the boom, or denying that horizontal movements may occur when the inertia weight will act as a steady point, yet from the ordinary elementary theory of forced vibrations‖ the equation giving the movement of a vibrating body, whose "free" frequency is $n/2\pi$, under the action of a periodic force $E \cos pt$, is

$$u = \frac{E}{\sqrt{(n^2-p^2)^2 + k^2\rho^2}} \cos (pt - \epsilon). \quad . \quad . \quad (1)$$

If k, the resistance due to friction, be small compared with the difference of the squares of the frequencies, the resulting vibration has an amplitude

$$u = \frac{E}{n^2 - p^2}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whilst, as is well known, the period of the vibration takes the period of the disturbing cause.

Though an earthquake record is probably due to several terms of the form $E \cos pt$ with different coefficients of decay, yet in many instances I have seen a considerable portion of the record appear to consist of waves of one wave-length, in which case the above simple formula will give at least a general idea of the earth movement corresponding to any diagram; but to obtain it a knowledge of "p" as well as "n" is necessary, for which purpose the tape must be driven sufficiently fast

* Communicated by the Physical Society: read March 13, 1903.
† 'Nature,' vol. lxv. p. 202, and B. A. Report Seismological Committee.
‡ Publications of the Earthquakes Investigation Committee, No. 5, Tokyo, 1901, p. 45, *et seq.*
§ The theory of the horizontal pendulum is discussed at length in *Comptes Rendus des Séances de la Commission Seismique Permanent* 1902, which reached me after this paper was written.
‖ Rayleigh, 'Sound,' 1st ed. vol. i. p. 38.

for the period of the forced vibration to be determined, which is not the case with Milne seismographs as at present constructed.

A possibility in these diagrams which appears to have been overlooked is that of interference effects between the forced and free vibrations of the boom. In order to ascertain the effect of periodically loading the pillar, I attached wooden boxes on its east and west sides. These were filled with sawdust. Two chains, fastened together by a rope passing over two pulleys fixed to the ceiling, were hung one over the centre of each box, and the rope was of such a length that the chains just touched the sawdust in the boxes together. By working this arrangement up and down at definite speed, I was able to imitate in a rough way a periodic tilting of the pillar. The total movement of the boom when one chain was removed from its box and the other placed in its proper box was 1·6 mm., *i.e.* the tilt was ·35″ whilst the chain was in its box. With this apparatus I imitated in succession waves of periods of 12, 13 ... to 20 seconds, whilst the boom period was 16·5 seconds throughout. I had previously increased the tape speed sufficiently to be able to count the number of vibrations of the boom. Some of the diagrams thus obtained are reproduced (figs. 1 to 4, Plate XVI.). In every case the number of vibrations performed by the boom was exactly the same as the number of motions of the chain. In every case also the amplitude due to periodic displacement was greater than that due to steady loading, though in no case did the simple formula (2) give the observed extent of the swing. The latter is always less than it should be. The discrepancy is probably due to three causes: (1) The imperfect representation of a sine curve with the apparatus. (2) It does not follow (as indeed it is one of the objects of this paper to point out) that because the pillar was tilted ·35″ when statically loaded, that it was also tilted that amount when periodically loaded by the same weight—it depends on whether $n^2 - p^2$ as applied to the pillar was $> =$ or < 1. (3) A certain amount of viscosity exists in the pillar, which prevents it responding promptly to its load.

The diagrams show interference effects well, especially that representing 15 sec. waves. That some of the throbbings so common in earthquake diagrams from the Milne instrument, and called by Professor Milne "Echoes"[*], are in reality interference effects I have little doubt. The 15 sec. diagram shows a series of lens-shaped throbbings remarkably like a very common feature of a seismogram. The interval

given by Milne of usually from 2 to 6 minutes* between the shock and its echo is exactly that between the beats. On the 15 sec. diagram the interval is about three minutes for a boom period 16·5. With a nearer approach to isochronism the interval would be longer until, if the periods are sufficiently near, the free vibration will have been damped out before opposition in phase can occur. This is evident in the 16 sec., 17 sec., and 18 sec. diagrams.

The object of this paper is to point out that on ordinary elementary theory it is erroneous to derive information regarding the movement of the earth from the measurement of the boom of a Milne seismograph as at present constructed, as we have no knowledge to what extent synchronism may affect the result. To obtain the necessary knowledge the tape must be driven at a higher speed, or it might be obtained in some cases from the interference effects. In cases where there is a near approach to isochronism between the boom period and the wave period, the amplitude of swing depends largely on the damping effect of friction (equation 1), which is entirely ignored at present in this connexion.

LIII. *A Penetrating Radiation from the Earth's Surface. By* H. LESTER COOKE, *M.A., Demonstrator of Physics, McGill University, Montreal* †.

C. T. R. WILSON‡ has examined the rate of discharge of an insulated charged conductor placed inside a closed vessel, the inclosed gas not being exposed to any known ionizing agent. The effect of different gases and varying pressures was studied. He found that the rate of discharge was approximately proportional to the pressure and also to the density of the gas employed. This discharge takes place through the gas, and must be ascribed to a production of ions proportional to the pressure and density of the gas; in other words, to the amount of gaseous matter present.

To account for this there are several explanations possible. The ionization may be due to :—

(1) a radiation or active emanation from the walls of the inclosing vessel ;

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* British Association Reports, p. 72 (1900).

† Communicated by Prof. Rutherford, F.R.S. A preliminary account of these experiments was given before the American Physical Society, Washington, Dec. 1902. An abstract of the paper was published in the 'Physical Review.' See Rutherford and Cooke, Phys. Rev. lxxxiv. p. 183.

‡ C. T. R. Wilson, Proc. Roy. Soc. lxviii. p. 151 lxix. p. 277.

(2) an external radiation capable of penetrating the walls of the inclosing vessel ;

(3) a spontaneous ionization of the gas itself ; or

(4) a combination of any two, or of all three, of the foregoing.

In order to explain the facts by (3) it must be assumed that the spontaneous ionization is due to the action of the particles of gas on themselves, and not due to any form of radiation or collision between adjacent particles. If this latter were the case the amount of ionization, instead of being simply proportional to the number of particles present, would vary as the square of this number. It would therefore seem advisable, before accepting this "Spontaneous" theory, to examine the other possible explanations of the phenomena. It was on this account that the following research was undertaken.

It was decided to employ an electroscope in these experiments. The observations could be made either with an electrometer and large testing-vessel (or small vessel in which the gas is under high pressure), or by means of an electroscope, which could be made small and portable. As the effects are somewhat small to come within the range of an ordinary electrometer, and as the method of experimentation required the use of heavy metal screens, and necessitated observations being taken at different places and under varying conditions, it was decided that the electroscope most nearly fulfilled the experimental requirements.

The form of instrument used, which is very similar to that employed by Wilson, is shown diagrammatically in the accompanying figure (fig. 1). The case of this electroscope is of brass, about 2 millimetres thick, and the volume of gas inclosed measures 1100 c.c. A metal rod, A, passes through an ebonite cork in the centre of the top plate, and is terminated by a sulphur bead B. In the lower end of this bead a strip of rolled copper wire is secured, and to this the gold leaf, which acts as a potential indicator, is fastened. Another wire, C, bent as shown in the diagram, passes through the ebonite cork, and is capable of being turned so as to be brought into electrical contact with the gold leaf system, and thus serves as a charging device. In the bottom of the vessel is placed a small glass dish, covered with perforated tinfoil, and containing calcium chloride ; this serves the purpose of removing the moisture from the inclosed gas without causing any electrostatic disturbance. The divergence of the leaves was read with a microscope having a 100 division scale in the eyepiece.

In beginning these experiments it was found that the effect of care in the preparation of the instrument was to greatly decrease the ordinary rate of collapse of the leaves.

Fig. 1.
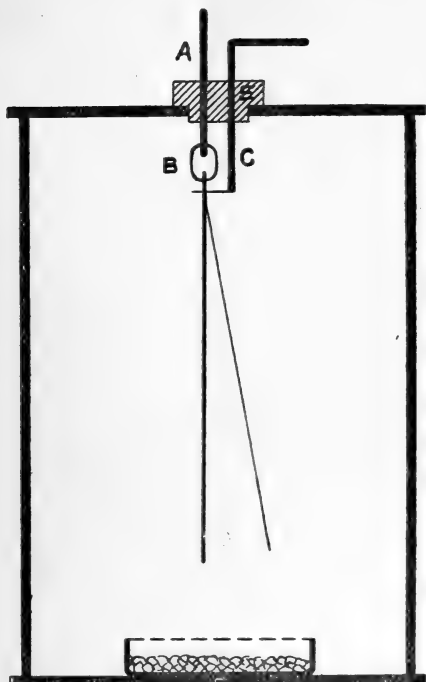
After repeated trials the following method was adopted as giving the most satisfactory and consistent results. The three brass sections were first carefully polished with metal-polish, then washed with soap and hot water, and thoroughly dried with a clean towel. The glass dish containing the drying agent was similarly washed and dried, fresh calcium chloride placed in it, and then covered with perforated tin-foil, previously cleaned with ammonia. The sulphur bead was made larger than required, and then pared down with a sharp clean knife, the utmost care being taken during this operation not to touch the bead with the fingers. All the joints and the edges of the mica windows were then carefully waxed over, and the instrument set aside for a day to give the calcium chloride time to dry the inclosed air, and render the insulation of the sulphur bead as perfect as possible.

The leakage of electricity across the sulphur bead could

be determined by observing the rate of collapse of the leaves when the rod which supported the bead was earthed, and then repeating the observation with this rod at a potential of 300 volts. The difference gave the leakage across the bead for a P.D. of 300 volts. This was generally found to be about ·02 division per hour, and was negligible in comparison with the leakage due to the ionized air. After charging the leaves the rod C was always earthed. This had the effect of slightly reducing the potential of the insulated system. This lowering of the potential, which could be accurately measured, did not in any way interfere with the results, as it was constant throughout the experiments.

The voltages employed ensured the removal of practically all the ions, the potential gradient across the gas, according to the measurements of the instrument, never falling below 20 volts per cm.

The primary object of this research was to ascertain whether there was any penetrating radiation coming from some external source, and responsible for the ionization of the air inclosed in the electroscope. For this purpose slabs of lead were cast, an inch in thickness, which could be built in the form of a cage for the insertion of the instrument. Lead was chosen as the material for these screens on account of its high absorptive power for ionizing radiations, as compared with its density ; this ratio in the case of lead being nearly double the value of the same ratio in the case of most other materials.

The instrument was charged, then placed in a definite position on an earthed support, which was carefully levelled, and the deflexion of the leaf observed. The instrument was then placed inside the lead cage and allowed to remain for about twelve hours, then removed, set up as before, and the deflexion again read. Great care was taken during these operations not to jar the instrument in any way. It was found that moving the instrument in this way did not introduce an error of more than ·2 of a division ; and as the observed collapse of the leaves was usually between 20 and 40 divisions this did not introduce any serious source of inaccuracy.

The effect of surrounding the electroscope with lead screens was always to reduce the rate of discharge of the leaves. The inch slabs of lead used in the first experiments diminished the ionization as much as 25 per cent. This result was arrived at by a continued series of observations extending over more than a month, the readings being taken and the instrument recharged night and morning.

Measurements were then made of the relation between the thickness of the lead screen and the decrease in the ionization, screens ranging in thickness from 1 mm. to about 40 cms. being employed. The results of repeated experiments indicated that this radiation was subject to the ordinary law of absorption, and that it was cut down to half value by a thickness of lead somewhere between ·25 and ·5 of an inch, being almost completely absorbed by 2 inches of lead. The greatest quantity of lead employed in the construction of a cage was about 10,500 lbs., the value thus obtained being practically the same as with a cage 2 inches thick. The individual observations were not as consistent as one could wish, sometimes showing a variation of as much as 10 per cent., but the averages obtained from a very extended series of experiments clearly indicated the above results. As will be shown further on, the current measured in these experiments was almost inconceivably minute, and thus any slight disturbing influence, which would be incapable of being detected by ordinary methods, was apt to become a serious factor of disturbance, and had to be guarded against with the utmost care. Altering the distribution of the lead slabs revealed the fact that the radiation was of practically uniform intensity in all directions, coming equally from the floor, walls, and ceiling of the laboratory.

It was thought possible that this radiation might be due in some way to the fact that radium and other radioactive substances had been used in the laboratories, and might in some way have affected the walls of the room. To test this point a series of experiments was carried out in the basement of the University Library, where no radioactive materials had been used. From observations extending over a period of more than two weeks it was definitely shown that this effect was present in the library as well as in the Physics building, and therefore could not in any way be due to contamination with radioactive substances.

The next thing to be ascertained was whether this radiation would be reduced on passing through other metals besides lead. For this purpose small slabs of polished iron were obtained which could be built up in the form of a cage as in the case of the lead. In this way iron was found to also cut down the radiation. The relation between the thickness of iron and the absorption was not determined, the object being merely to show that this screening effect was not a property peculiar to lead.

The effect of water was also tried. A brass cylinder was obtained with a top which could be bolted down and rendered

water-tight by means of a rubber washer. The electroscope was placed inside this vessel and the rate of discharge obtained with the apparatus resting exposed on the slate in the laboratory. The run was then repeated with the cylinder containing the electroscope immersed in a large tank, so as to be surrounded on all sides by a thickness of water of at least $2\frac{1}{2}$ ft. Under these conditions it was found that the rate of discharge was reduced as in the experiments where lead and iron were used. The tank used in these experiments was of iron, about an inch in thickness; it is thus doubtful how much of the radiation was cut off by the iron, and how much by the water. The only two conclusions which can be drawn with certainty from these experiments are that the iron and water combined reduced the amount of penetrating radiation, and that the water did not supply any such radiation to an appreciable extent.

Radioactivity of Ordinary Materials.

The walls of the laboratory in which the experiments were carried on were of brick. It was therefore thought advisable to ascertain what would be the effect of surrounding the electroscope with a brick cage. New bricks were obtained and two cages built, the sides of one being one brick in thickness and the sides of the other four bricks thick. Placing the electroscope inside these cages caused the rate of collapse of the leaves to *increase* 40 and 50 per cent. for the small and large cage respectively. This increase, however, was not due to a penetrating form of radiation, for when a casing of lead a millimetre thick was interposed between the electroscope and the bricks the leakage was immediately reduced to its normal value. This also precludes the idea of the leakage being due to an emanation from the bricks, which was suggested as an explanation of the effect. The rays from the bricks were evidently of a comparatively low penetrating power, probably being very similar in character to the β rays given off by radium. Experiments were tried to detect the presence of very easily absorbed rays from bricks, the base of the electroscope being removed, and the instrument placed on a brick support. Bricks covered with gold leaf, bare bricks, and powdered bricks, were successively tried in this way. The removal of the base of the electroscope slightly increased the ionization of the inclosed air, but not to an extent sufficient to indicate the presence of an appreciable amount of easily absorbed rays. Since the ionization was the same when the electroscope surrounded by lead one millimetre in thickness was placed inside the brick cage, as

when it was left exposed in the laboratory, it follows that the bricks must have supplied an amount of penetrating radiation equal to the amount they cut off. This result might have been anticipated, as the walls of the laboratory were of brick.

The activity of lead slabs which had been exposed for over two weeks on the roof of the building was tested, and also the activity of the copper roof itself. Both exhibited greater activity than materials which had been kept in the laboratory, but the radiation was not of a penetrating character. Experiment showed that the amount of penetrating radiation was approximately the same in the laboratories, on the roof, and on the bare ground outside the building; in fact this radiation seems to be present everywhere with practically uniform intensity.

The effect of placing metal sheaths as linings inside the electroscope was observed in an attempt to reduce the ionization as much as possible, and also the instrument with its base removed was placed on other non-metallic substances. In this way dried earth, polished tin, writing-paper, a stone window-ledge, pine wood, brick, lead, zinc, and iron were tried ; the results showing that they were all, with the exception of iron, more active than polished brass. Iron exhibited about the same effect as brass, but it is probably more active, for the iron used was a slab about ·75 of an inch in thickness, which must necessarily have absorbed a considerable amount of the penetrating rays. Great care was not taken in this part of the work to secure numerical values for the activity of the various substances, as the object was merely to determine if any material showed a lower ionizing effect than brass. The results would, however, indicate that the above list represents the materials arranged according to their activity, beginning with the most active. The quantitative aspect of this branch of the subject has been carefully investigated by McLennan and Burton *, and by R. J. Strutt †.

In order to calculate the number of ions produced per c.c. per sec. it was necessary to reduce the readings on the micrometer scale to volts, to estimate the electrical capacity of the insulated gold-leaf system, and to calculate the volume of the inclosed air.

The deflexions of the gold leaf in micrometer divisions could be reduced to readings in volts by observing the deflexions when the leaves were charged to different known

* McLennan and Burton, Phys. Rev. lxxxiv. p. 184.
† R. J. Strutt, 'Nature,' Feb. 19, 1903, p. 369 ; see also 'Nature,' 1903, pp. 391, 414, 439.

voltages, the curve thus obtained being practically linear between 100 and 340 volts, these being the extreme limits of potential for the observations. For different gold-leaf systems this calibration gave between 2·5 and 3 volts per division.

To determine the capacity of the gold leaves the electroscope was set on an insulating block of paraffin, and the outside cylinder connected to the quadrants of a very delicate Dolezalek electrometer, which was connected in parallel with a standard capacity of ·002 microfarad. The leaves of the electroscope were charged, and the reading observed. The leaves were then discharged, and the charge which had been attracted to the case of the instrument thus released and allowed to charge up the electrometer and standard capacity. The deflexion of the electrometer was then read, and was a measure of the charge which had been on the leaves of the electroscope. The observations were taken five successive times, using 200 and 300 volts, and showed an agreement of better than four parts in a hundred. The sensitiveness of the electrometer was 543 divisions per volt. From the data thus obtained the capacity of the system was calculated, and came to ·85 and ·91 cms., for 200 and 300 volts respectively. As the observations generally ranged between 325 and 175 volts the average capacity of the system was taken as ·88 cm.

The volume of the air inclosed in the electroscope was 1100 c.c.

With a gold-leaf system giving 2·6 divisions deflexion for a volt, the rates of collapse of the leaves for the electroscope exposed in the laboratory, and then surrounded with a very thick lead screen, were 2·4 and 1·7 divisions per hour. This gives as the number of ions 7·5 and 5 respectively. The number given by Wilson, 14, is for conditions similar to those under which the value 7·5, above, was obtained. The lower number here obtained may be accounted for by the fact that Wilson used silvered glass in the construction of his electroscope, whereas an instrument of brass 2 mms. thick was used in these experiments.

To sum up, the results of this investigation are :—

(1) The proof of the existence of a very penetrating radiation, present everywhere under ordinary conditions. This radiation is similar in properties to the radiation from radium, and is comparable to it in penetrating power. This radiation is accountable for between 30 and 33 per cent. of the natural ionization observed in ordinary testing-vessels, 33 per cent. being the greatest reduction obtained by the use of massive

lead screens. This penetrating radiation may have its origin in the radioactive matter which is distributed throughout the earth and atmosphere. It was not found possible to obtain sufficient excited activity on a wire charged negatively in the open air to show the presence of a very penetrating radiation due to it. The effect observed is too large to be accounted for by the excited activity distributed on the walls of the laboratory.

(2) That all the substances examined give forth a radiation of a not very penetrating character; that this is probably the cause of all the residual ionization in the electroscope when surrounded by heavy metal screens; and that this activity varies with different substances, being very low in the case of brass.

(3) The reduction by the experimental arrangements of the number of ions produced per c.c. per second in air under atmospheric pressure from 14 to 5.

In conclusion, I wish to thank Prof. Rutherford, at whose suggestion and under whose very kind supervision this research was undertaken and carried out.

McGill University, June 23, 1903.

LIV. *On the Law of the Condenser in the Induction-Coil.* By JAMES E. IVES, *Ph.D., Instructor in Physics in the University of Cincinnati*[*].

[Plate XVII.]

THE optimum capacity, in an induction-coil, is defined as that capacity of condenser which, when placed around the break, will give the longest spark in the secondary. It has also been found by experiment to be the least capacity that causes the sparking at the break to disappear, or if not entirely to disappear to become very small.

In the July number of the Physical Review I proposed a formula for it, based upon theoretical considerations, viz. :—

$$K = \frac{I_0}{a v_0}$$

where K is the optimum capacity; I_0 the steady current in the primary at the instant of breaking the circuit; v_0 the velocity with which the circuit is broken; and a some constant which depends upon the medium in which the break is made. Unfortunately for this theory, experiments made

* Communicated by the Author.

later, and recorded in the same paper, make it untenable at least for moderate speeds, since the speed of breaking, 70 cms. per second or less, was found to have very little effect upon the electromotive force induced in the secondary.

An attempt has therefore been made to find by experiment the relation between the optimum capacity and the other constants of the primary circuit, viz., the current, the resistance, and the inductance.

For this purpose a primary coil was used without a secondary, as it is well known that the action of the primary is unaffected by the presence of an open secondary. The break consisted of an amalgamated copper wire, one millimetre in diameter, dipping into a cup of mercury. The surface of the mercury was covered by two or three centimetres of distilled water. The breaking was done by hand. In making an experiment the capacity of the condenser, around the break, was gradually increased, and the value noted at which the sparking suddenly disappeared or became very small. The batteries used were large lead chloride cells.

A primary circuit may be regarded as made up of the three branches shown in the accompanying diagram.

The first branch, containing the primary coil and battery, and having a resistance R_p, an inductance L, and an electromotive force E; the second containing the break, and having a resistance R_b, divided between the two sides; and the third containing the condenser, and having a capacity K, and a resistance R_c. By suitable manipulation any one of these quantities can be varied while the others are kept constant.

This was done, and although I have not yet been able to derive a general law connecting the optimum capacity with these different variables, certain well-marked and somewhat remarkable results have been obtained.
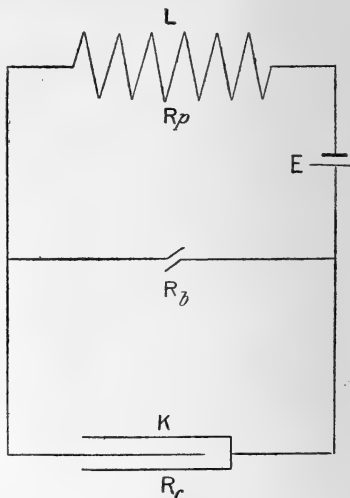
1. It was found that for the same current the optimum capacity is much greater when the breaking pole is negative than when it is positive. See Table I. and Pl. XVII. fig. 1. The change of sign was obtained by reversing the cell connexions.

TABLE I.

R_b and R_c small.

Breaking-Pole Negative.			Breaking-Pole Positive.		
Current in Amperes.	Optimum Capacity in Microfarads.	$\dfrac{K}{I^2}$.	Current in Amperes.	Optimum Capacity in Microfarads.	$\dfrac{K}{I^2}$.
1·20	·057	·0396	1·15	·033	·0250
1·80	·126	·0390	1·70	·062	·0215
2·15	·182	·0394	2·10	·097	·0221
2·60	·261	·0386	2·60	·126	·0187
3·25	·377	·0357	3·60	·283	·0219
4·00	·566	·0354	4·65	·383	·0177
4·60	·680	·0321	5·40	·480	·0165

2. If R_p or E is varied so as to vary the current, the optimum capacity varies ; in general, as a power of the current greater than the square and less than the cube. This is shown in Tables I. to IV., and in figs. 1 to 4 (Pl. XVII.). In fig. 1 the current is increased by decreasing the resistance R_p, and in figs. 2 to 4 by increasing the electromotive force of the battery E. It will be observed that the same curves are obtained in the two cases. In some of the tables is given the ratio of the optimum capacity to the square of the current. If the optimum capacity varied as the square of the current this should, of course, be a constant. It will be seen that when R_b and R_c are very small, that is, when the resistances of the break and condenser connexions are small, this is approximately true.

TABLE II.

$R_c = 1\cdot0$ ohm.

No. of Cells.	$R_b = \cdot2$ ohm.			$R_b = \cdot5$ ohm.			$R_b = 1\cdot9$ ohm.		
	Current in Amperes.	Optimum Capacity in Microfarads.	$\dfrac{K}{I^2}$.	Current in Amperes.	Optimum Capacity in Microfarads.	$\dfrac{K}{I^2}$.	Current in Amperes.	Optimum Capacity in Microfarads.	$\dfrac{K}{I^2}$.
1 ...	·45	·008	·0395	·40	·010	·0625	·35	·030	·245
2 ...	·90	·033	·0408	·85	·035	·0485	·70	·139	·284
3 ...	1·30	·084	·0497	1·25	·095	·0608	1·00	·440	·440
4 ...	1·80	·171	·0528	1·65	·179	·0656	1·35	1·201	·660
5 ...	2·30	·318	·0600	2·15	·293	·0635			
6 ...	2·75	·536	·0710	2·55	·557	·0854			
7 ...	3·25	·870	·0824	3·00	·946	·1050			
8 ...	...	...	...	3·40	2·091	·1800			

TABLE III.

$R_c = 0.$

No. of Cells.	$R_b = \cdot 5$ ohm.		$R_b = 1\cdot 0$ ohm.		$R_b = 1\cdot 9$ ohm.	
	Current in Amperes.	Optimum Capacity in Microfarads.	Current in Amperes.	Optimum Capacity in Microfarads.	Current in Amperes.	Optimum Capacity in Microfarads.
1 ...	·75	·048	·65	·090	·50	·049
2 ...	1·50	·231	1·30	·464	·95	·291
3 ...	2·20	·587	1·90	1·359	1·40	1·359
4 ...	2·90	1·156				
5 ...	3·60	1·750				

TABLE IV.

$R_b = \cdot 25$ ohm.

No. of Cells.	$R_c = 0$ ohm.			$R_c = \cdot 5$ ohm.			$R_c = 1\cdot 0$ ohm.		
	Current in Amperes.	Optimum Capacity in Microfarads.	$\dfrac{K}{I^2}$.	Current in Amperes.	Optimum Capacity in Microfarads.	$\dfrac{K}{I^2}$.	Current in Amperes.	Optimum Capacity in Microfarads.	$\dfrac{K}{I^2}$.
1 ...	·80	·033	·0516	·85	·038	·053	·90	·035	·043
2 ...	1·60	·139	·0543	1·60	·204	·080	1·70	·213	·074
3 ...	2·45	·335	·0559	2·40	·470	·082	2·40	·638	·111
4 ...	3·25	·725	·0686	3·10	1·166	·116	3·20	1·578	·154
5 ...	3·90	·980	·0644	3·90	1·750	·115			
6 ...	4·60	1·359	·0643						
7 ...	5·20	1·720	·0637						

3. The optimum capacity is a function of R_b and R_c. The results are given in Tables V. and VI. and in figs. 5 & 6. They are very remarkable, and show that the optimum capacity depends not only upon the primary current, but upon the resistances of the connexions leading to the break and to the condenser. The same primary current was, of course, used in each experiment. The form of these curves for small values of R_b and R_c was carefully examined by supplementary experiments, and found to be that shown in the figures. The curves therefore have a point of inflexion, and the optimum capacity is apparently a cubic function of these resistances.

TABLE V.

Four cells. Current = 2·1 amperes.

R_b in ohms.	Optimum capacity in Microfarads.			
	$R_c = 0.$	$R_c = 1$ ohm.	$R_c = 2$ ohms.	$R_c = 3$ ohms.
·00	·030	·184	·430	1·051
·35	·132	·350	·703	1·750
·71	·172	·470	1·086	
1·06	·237	·589	1·586	
1·41	·313	·859		
1·77	·388	1·163		
2·12	·636	1·578		
2·47	·926			
2·82	1·163			

TABLE VI.

Four cells. Current = 2·1 amperes.

R_c in ohms.	Optimum capacity in Microfarads.			
	$R_b = ·5$ ohm.	$R_b = 1$ ohm.	$R_b = 1·5$ ohm.	$R_b = 2$ ohms.
·00	·122	·213	·377	·646
·35	·236	·375	·616	1·163
·71	·340	·536	·850	1·750
1·06	·418	·726	1·163	
1·41	·480	1·049	1·700	
1·77	·589	1·450		
2·12	·859			
2·47	1·162			
2·32	1·720			

4. The two parts of R_b play exactly the same part in the primary circuit, and it is indifferent which of the two is changed. The same is true for the two parts of R_c. This was proved by a number of experiments, although it might have been presupposed.

5. The optimum capacity is a function of the *same form* both of R_b and R_c. This is shown by the results already given in figs. 5 & 6.

6. The optimum capacity is a function of the inductance of the primary. It varies less rapidly than a linear function, and soon becomes practically independent of it. This is shown by the results given in Table VII. and fig. 7 (Pl. XVII.). They were obtained by replacing the solenoid by a variable inductance, keeping the current constant by putting in or taking out

non-inductive resistance. The variable inductance was free from iron. The curves in fig. 7 suggest that the optimum capacity may be related to the inductance in the following manner :—

$$K = \frac{a R_b L}{\sqrt{b^2 L + c}} + d,$$

a, b, c, and d being constants. Curves of this form, but arising in an entirely different way, are discussed in my paper on the Induction-coil, already referred to [*].

TABLE VII.

Current $= 2 \cdot 1$ amperes. $R_c = 1$ ohm. $E = 4 \cdot 2$ volts.

Inductance in Millihenries.	Optimum capacity in Microfarads.			
	$R_b = 0$ ohm.	$R_b = \cdot 2$ ohm.	$R_b = \cdot 5$ ohm.	$R_b = \cdot 7$ ohm.
$\cdot 00$	$\cdot 182$	$\cdot 226$	$\cdot 263$	$\cdot 280$
$\cdot 03$	...	$\cdot 226$	$\cdot 300$	$\cdot 303$
$\cdot 06$	$\cdot 182$	$\cdot 244$	$\cdot 313$	$\cdot 338$
$\cdot 16$	$\cdot 182$	$\cdot 236$	$\cdot 350$	$\cdot 397$
$\cdot 43$	$\cdot 182$	$\cdot 280$	$\cdot 423$	$\cdot 414$
$1 \cdot 60$	$\cdot 192$	$\cdot 293$	$\cdot 420$	$\cdot 506$
$4 \cdot 44$	$\cdot 213$	$\cdot 303$	$\cdot 470$	
$6 \cdot 04$	$\cdot 189$	$\cdot 318$		
$7 \cdot 64$	$\cdot 283$			

In all the experiments recorded above, unless otherwise stated, the breaking-pole was negative. The solenoid used in the experiments had an air-core and an inductance of $\cdot 28$ of a millihenry.

The optimum capacity was always taken as that capacity at which the break-spark was reduced from a large spark to a very small one. For small resistances and small currents this point is usually very well marked, but as the current or resistance increases it becomes more difficult to determine.

A phenomenon which appears well worthy of notice and which frequently attracted my attention in these experiments, especially in the case of a large primary inductance, is that as the capacity around the break is gradually increased the volume and "snappiness" of the spark increases up to a certain point, and then suddenly decreases, this value of the capacity being that taken as the optimum capacity. As the capacity is increased the potential appears to increase up

* Physical Review, vol. xiv. pp. 300–304 (1902).

to a certain point, and then to diminish. This behaviour of the spark suggests a resonance phenomenon.

From these experiments it will be evident that for this kind of a mercury-break any equation representing the optimum capacity as a function of E, R_p, R_b, R_c, and L will be quite complicated.

Summarizing these results we may make the following statements with regard to the optimum capacity for the particular kind of mercury-break used.

It depends upon the sign of the breaking-pole.

It is, in general, proportional to a power of the primary current, greater than the square and less than the cube.

It depends very markedly upon the resistances of the connexions leading both to the break and to the condenser.

It is a function of the inductance of the primary, but for large inductances becomes practically independent of it.

In conclusion, I wish to express my indebtedness to Mr. Gordon Farnham, of the University of Cincinnati, for the great assistance which he has rendered me in this research.

University of Cincinnati,
June 1902.

LV. *On an Instrument for Measuring the Lateral Contraction of Tie-bars, and on the Determination of Poisson's Ratio. By* J. MORROW, *M.Sc. (Vict.), Lecturer in Engineering, University College, Bristol* [*].

THE ratio of the lateral to the longitudinal strain in a bar under the action of a simple pull or push in the direction of its axis has been the subject of much controversy and a considerable amount of research. Some French mathematicians, including Poisson and Navier, held that this ratio was $\frac{1}{4}$ for all isotropic substances, their arguments were based on a theory of the constancy of the ratio of the elastic coefficients for all materials.

Wertheim endeavoured to show that this theory was false. He filled hollow prisms with liquid and subjected them to a tensile force. The interior of the prism communicated with a capillary tube, and the change of volume of the interior was measured by the distance the fluid advanced or receded in the tube. If Poisson's ratio be denoted by σ, change of unit volume = tensile strain $(1-2\sigma)$ very approximately. He maintained that the ratio σ should really be

$\frac{1}{3}$ for all materials; but it appears doubtful whether the specimens dealt with were sufficiently isotropic to justify any definite conclusion either for or against the uniconstant form.

Practical methods for the determination of Poisson's ratio may be divided into three classes. First, those by which two coefficients of elasticity are obtained, and the required ratio inferred by calculation. Second, those which depend on the deformation of the section of a beam when bent. And, lastly, methods by which the tensile and lateral strains are actually measured in specimens of the material under direct tensile or compressive stress.

Kirchhoff experimented in 1859 upon cylinders, which by means of a weight attached to a lever were put simultaneously under the action of bending and twisting forces and the strains produced were measured accurately. The values of Poisson's ratio were calculated from the ratio of the observed displacements, and were

Mean for three steel rods . . . . ·294.
    ,,   hard drawn brass rod . . ·387.

Many others have worked on similar lines.

Cornu (and later Straubel) explored the anticlastic surface of a rectangular beam by means of the interference-fringes produced between it and a plate of glass laid on it, thus obtaining data for the calculation of σ.

Mallock (1879) also examined the anticlastic curvature. By means of a microscope he measured the movement of four fine steel wires fastened to opposite diameters of a circle on the surface, obtaining data for the radii.

Another method of investigation would be to find the displacements in the sides of a beam, either by measuring the lateral strains at any section, or by attaching a mirror to the side and finding the angle through which it turns when the beam is loaded.

If $f =$ the maximum tensile stress, the lateral contraction is $\sigma \frac{f}{E} b$ (b being the breadth of the beam and E Young's modulus) and the angle turned through by the side of the beam is $\tan^{-1} \sigma \frac{f}{E} \frac{b}{d}$.

In all the above methods the practical accuracy of the theory of elastic materials is assumed.

Coming now to actual measurements of the lateral contraction of a tie-bar, it appears that Bauschinger (*Der Civilingenieur*, vol. xxv. 1879, pp. 81-124) was the first to construct an instrument which would measure the alteration

of cross dimensions of a metal test-piece. His apparatus is complicated. It is essentially a pair of levers, the small arms of which are the diameter of small caoutchouc cylinders, and the greater the double distance of the scale from a mirror. To read the lateral contraction requires two separate observations by means of two telescopes and mirrors on opposite sides of the specimen.

Stromeyer (Proc. Roy. Soc. vol. lv. no. 334, p. 373) measured directly the lateral contraction of a bar under tension by interference methods; a dark glass attached to the specimen advanced or receded from the surface of a prism as the diameter of the specimen altered, thus producing change in the positions of the interference-bands.

Elastic Constants.

If α is the longitudinal strain produced by unit stress, and β the linear lateral strain in the material, then

$$\sigma = \frac{\beta}{\alpha}, \quad E = \frac{1}{\alpha}, \quad n = \frac{1}{2(\alpha + \beta)}, \quad \text{and} \quad k = \frac{1}{3(\alpha - 2\beta)};$$

where E, n, and k are Young's modulus and the moduli of rigidity and of bulk respectively.

If any two of the elastic coefficients be known Poisson's ratio can be calculated.

Also we have that:—

if $\sigma = \frac{1}{2}$ the material is incompressible, and the extension and lateral contraction are such that the volume remains constant.

if $\sigma > \frac{1}{2}$ k would be opposite in sign to E and n, the material not being homogeneous and isotropic.

if $\sigma = 0$ to $\frac{1}{2}$ we have an ordinary elastic body.

if $\sigma = 0$ either there is no lateral strain or the material is perfectly plastic.

$\sigma = \frac{1}{4}$ is the value of the uniconstant theory which requires $n = \frac{3}{5} k = \frac{2}{5} E.$

The New Form of Apparatus.

With the object of measuring, in the most direct way possible, the lateral contraction of a tensile specimen or the dilatation of a compressive piece when loaded in an ordinary testing-machine, the author devised the instrument shown in the figure.

Two hardened set-screws A and B, on opposite sides of the specimen S, are pressed inwards by stiff springs C, C. The relative motion of the points of these set-screws is transmitted

by the bars E, F, and the pivot P, to the extremities H, G, of the pieces K, L.

PLAN

ELEVATION
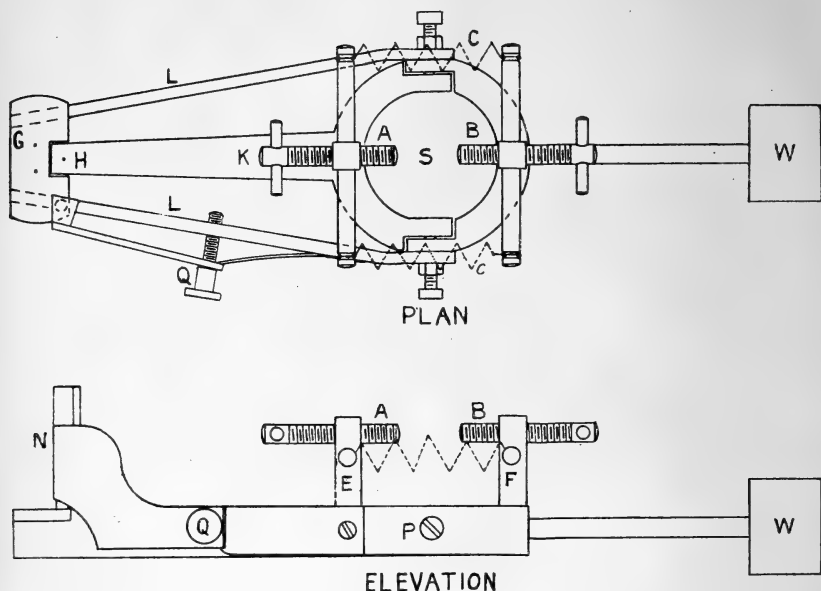
SCALE HALF SIZE

It will be noticed that the instrument touches the specimen only at the points between which the alteration of dimension is to be measured. When the specimen extends A and B approach one another, and consequently G falls with reference to H. The relative displacement of G and H is measured optically. A mirror M rests by three steel points (two of which are on G and the other on H, as shown in the plan of the instrument) on glass plates fixed to G and H. A second mirror N is attached to G in a vertical position, it is capable of a small amount of rotation about a vertical axis by the fine adjustment screw Q. The two mirrors are close together but not touching. A scale is placed some distance away, and by means of a telescope two images of the scale are seen, one in each mirror. These images are brought together in the telescope and the one is read against the other. Thus a certain convenient reading on the N image is taken as the index, and as the load is applied to the specimen the observer reads the values on the M image coinciding with this index.

Every precaution has been taken to ensure the greatest

possible accuracy. The screws A and B have spherical points in order that they may not cut into the specimen. The instrument is balanced by the weight W. The mirrors are of specially worked optically plane glass. The joint at P is made by centring the screws in the inner piece, and is constructed with the greatest care.

In order to test the instrument a specially selected bar of mild Bessemer steel was used. It was turned to one inch diameter as given by a Whitworth gauge. After placing in the testing-machine it was loaded several times up to 5 tons. The instrument was attached and allowed to stand for a short time before readings were taken.

The load was then applied by successive tons between the limits 1 and 5 tons and the scale-readings observed. The results for three series of increasing and decreasing loads are given in the following table:—

TABLE I.

Load.	Scale-Readings.			Differences.		
	1st Loading.	2nd Loading.	3rd Loading.	1st Loading.	2nd Loading.	3rd Loading.
1	195·0	201·2	201·8	2·9	2·8	2·9
2	197·9	204·0	204·7	2·8	2·9	2·9
3	200·7	206·9	207·6	2·9	2·9	2·8
4	203·6	209·8	210·4	2·5	2·8	2·8
5	206·1	212·6	213·2	2·7	2·6	2·5
4	203·4	210·0	210·7	2·9	2·6	2·8
3	200·5	207·4	207·9	2·8	2·8	2·8
2	197·7	204·6	205·1	2·9	2·8	2·9
1	194·8	201·8	202·2			
Mean diffs. per ton ...				2·80	2·78	2·80

It was thus seen that the readings could be repeated with considerable accuracy. The above are some of the first readings taken, further experience in the use of the instrument led to still more uniform results.

Experimental Determination of Poisson's Ratio.

In these experiments the load on the specimen was in some cases varied from 1 to 5 tons, and in others from 1 to 3 tons. In the former the readings were taken at intervals of 1 ton, and in the latter at intervals of one-half ton. In every case the specimens were loaded several times previous to the readings being taken. In order to insure complete immunity from local strains (due to the method of gripping the specimen)

an unusually large distance was allowed between the ends of the specimen and the portion of it under observation.

The instrument used was carefully measured under a reading microscope and found to give a magnification of 4·106. The distance between the needle-points of the tilting mirror, measured in the same way, was ·2960 in. The normal distance from the mirrors to the line through the scale parallel to the tilting mirror was measured several times in different ways and the mean value taken as correct.

Under these circumstances the total magnification obtained was about 2800, but the exact value of course varied with the position of the telescope and scale.

Table II. contains the results of experiments on specimens of various materials.

The extensions and lateral contractions were measured simultaneously, an extensometer having been specially arranged so that both instruments could be attached to the specimen at the same time and used throughout the experiment.

The calculated means of the observed tensile and lateral strains for half-ton of load are given in the table, and the values of the modulus of elasticity and Poisson's ratio are obtained from these. The modulus is given in lbs. weight per sq. inch. It can be expressed in grammes weight per sq. cm. by multiplying the numbers in the table by 70·31.

TABLE II.

Material.	Strains per ½ ton of Load.		Young's Modulus.	Poisson's Ratio.	
	Tensile.	Lateral.			
Mild Steel.	·000,0464	·000,0128	$30·75 \times 10^6$	·276	
	·000,0469	·000,0130	30·46	·277	
	·000,0465	·000,0127	30·70	·273	
Sheffield Spindle Steel.	·000,0488	·000,0135	29·60	·277	
	·000,0505	·000,0142	28·33	·281	
	·000,0488	·000,0132	29·23	·271	
Wrought Iron.	·000,0531	·000,0144	29·08	·271	
	·000,0826	·000,0239	29·80	·289	Dia.=·761
	·000,0928	·000,0251	30·73	·270	,, =·707
Drawn Brass (Muntz Metal).	·000,1047	·000,0335	13·26	·320	
	·000,0978	·000,0345	14·23	·353	
	·000,0960	·000,0337	13·61	·351	
Drawn Copper.	·000,0770	·000,0262	17·61	·340	
	·000,0726	·000,0241	18·70	·332	
	·000,0822	·000,0255	16·88	·310	

The specimens were not annealed, and all the above were approximately one inch in diameter except in the cases noted in the table.

Cast Iron.—Two sets of cast iron specimens were carefully cast in a vertical position. The first was from a mixture of about equal parts of No. 1 Govan, No. 2 Govan, and Heavy Scrap, and the second from 3 parts Redcar Middlesborough, and one part Scrap.

The material was therefore in each case such as might be used for engine and machine castings.

The specimens were brought to a state of ease by continued application and removal of load until permanent set was completely eliminated.

The values given in the following table are those obtained from the elastic state of the bar, that is, the strains do not include permanent set.

<p align="center">TABLE III.—Cast Iron.</p>

Specimen.	Diameter.	Strains per $\frac{1}{2}$ ton of Load.		Young's Modulus.	Ratio, $\dfrac{\beta}{\alpha}$.
		Tensile.	Lateral.		
1st Series.					
No. 1	·752	·000,1552	·000,0374	$16·25 \times 10^6$	·241
No. 2	1·025	·000,0904	·000,0206	15·01	·228
No. 3	·752	·000,1556	·000,0420	16·21	·270
2nd Series.					
No. 1	·997	·000,0877	·000,0223	16·36	·254
No. 2	·990	·000,0947	·000,0245	15·37	·259
No. 3	·993	·000,0900	·000,0220	16·07	·244

A long series of observations was made for the purpose of determining whether there was any permanent set in the lateral strain. Readings were taken for loads varying by 0·1 of a ton up to 0·5 ton on a C.I. bar approximately 1 inch in diameter. The readings were in $\frac{1}{40}$ of an inch, and an estimation of $\frac{1}{10}$ of these was made. The total magnification was 2700. The bar had not been previously loaded, and it appeared to be perfectly elastic as regards lateral strain. On increasing the load up to 1·0 ton, however, there appeared to be a small amount of permanent set, and between 1·0 and 1·5 tons the set was very marked. It increased in amount up to 3 tons, which was the limit of load applied.

Table IV. is a comparison of the results of other experimenters with those given in this paper.

TABLE IV.

Material.	Value of Poisson's Ratio.		
	Bauschinger.	Stromeyer.	From Table II.
Mild Steel.........	·29	·273 to ·300	·271 to ·281
Wrought Iron ...	·26 to ·31	·279 to ·301	·270 to ·289
Brass Rod.........	....	·283 to ·357	·320 to ·351
Copper Rod	....	·325	·310 to ·340
Cast Iron	·16 to ·19 (Tension) ·32 to ·38 (Compression)	·148 to ·269	·228 to ·270

In conclusion I must acknowledge my indebtedness to my colleague, Mr. E. L. Watkin, M.A., for his valuable assistance in carrying out these experiments, which were made in the Engineering Laboratory at University College, Bristol.

LVI. *On the Distribution of Pressure around Spheres in a Viscous Fluid. By* S. R. COOK, *M.S., A.M., Former Fellow in Physics in the University of Nebraska, Instructor in Physics, Case School of Applied Science**.

[Plate XVIII.]

1. THE motion of a sphere in an incompressible frictionless fluid, at rest at infinity, has been discussed by Poisson, Stokes, Rayleigh, Kelvin, Koenig, and others. The solution in its present form was first given by Stokes in his celebrated paper "On Some Cases of Fluid Motion," read before the Cambridge Philosophical Society in 1843†.

On the principle that the mutual force acting between two adjacent elements of a fluid is normal to the surface which separates them, Stokes finds that the kinetic energy T of a sphere moving in an incompressible frictionless fluid at rest

† Camb. Trans. vol. viii. p. 184 ; Math. Papers, vol. i. p. 41.

at infinity is increased by an inertia term equivalent to one-half the mass of the fluid displaced times the square of the velocity:

$$2T = -\rho \iint \phi \frac{d\phi}{dr} ds = \tfrac{2}{3} \pi \rho a^3 u^2 . \quad \quad (1)$$

When the sphere moves in a straight line, its motion being accelerated, and there are no external forces acting on the fluid, the resultant pressure is equivalent to a force

$$-\tfrac{2}{3} \pi \rho a^3 \frac{du}{dT}, \quad \quad (2)$$

in the direction of motion. If the velocity of the sphere is constant, there being no external force, the force acting on the sphere is zero, the pressure is symmetrical with respect to any axis, and the sphere will move with uniform velocity through the fluid.

The problem of the motion of two spheres in a perfect fluid was discussed by Stokes in the paper already referred to, and a method for obtaining the solution was suggested. Later a solution was obtained by W. M. Hicks and presented to the Royal Society in 1879 in his paper " On the Motion of Two Spheres in a Fluid "[*].

Hicks finds that the kinetic energy T of two spheres moving in a perfect fluid may be expressed as a very simple function of their relative velocities u_1, u_2 :

$$2T = A_1 u_1^2 - 2B u_1 u_2 + A_2 u_2^2; \quad \quad (3)$$

and that the rate of change of the distance between the centres of gravity of the two spheres is given by the expression

$$\frac{\partial r}{\partial T} = \pm \sqrt{\frac{2T\rho - a^2}{A_1 A_2 - B}}, \quad \quad (4)$$

the positive or negative sign being taken according as the spheres are separating or approaching one another. The spheres will therefore move as though they repelled or attracted one another according as

$$\frac{\partial}{\partial r} \left\{ \frac{2T\rho - a}{A_1 A_2 - B} \right\}$$

is positive or negative. This condition does not depend on the relative motion of the two spheres at any time, but only on their distance apart and the ratio of the constant energy

* Phil. Trans. p. 455 (1880).

to the constant momentum. Since $K = \dfrac{a^2}{2T}$ is always positive
K is always positive, and $\dfrac{\partial r}{\partial T}$ tends to decrease, *i. e.* when
moving in line of centres, the spheres tend to repel each other. When the two spheres are moving perpendicular to the line of their centres, Hicks finds that for a perfect fluid the spheres tend to attract each other.

Koenig[*] solving the same problem finds that the mutual forces between two spheres moving in a perfect fluid are

$$X = -\frac{\frac{3}{2}\pi\rho a^3 b^3 u^2}{c^4}\sin\theta(1 - 5\cos^2\theta) \quad . \quad . \quad . \quad (5)$$

$$Z = -\frac{\frac{3}{2}\pi\rho a^3 b^3 u^2}{c^4}\cos\theta(3 - 5\cos^2\theta) \quad . \quad . \quad . \quad (6)$$

$$Y = 0,$$

where a and b are the radii, c the distance apart, and θ the angle which the line of centres makes with the direction of motion, Y vanishing on account of symmetry.

When $\theta = \dfrac{n\pi}{2}$, n being an integer,

$$X = -\frac{\frac{3}{2}\pi\rho a^3 b^3 u^2}{c^4}; \quad . \quad . \quad . \quad . \quad . \quad (7)$$

when $\theta = n\pi$,

$$Z = \frac{3\pi\rho a^3 b^3 u^2}{c^4}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

giving repulsion parallel and attraction perpendicular to the stream-lines.

As these results have been obtained on the assumption that the medium is a perfect fluid, it is not possible to obtain experimental data to test their validity. All known fluids are susceptible to changes of density, and possess internal friction. The kinetic energy of a system moving in them may, accordingly, be transferred to the medium itself, thereby necessitating the introduction of a term in the equation of motion that will represent this transfer of kinetic energy.

On the condition that the velocity of the sphere is small so that the square of the velocity may be neglected, Stokes first obtained the solution for a sphere in a viscous fluid in terms of the potential [†]

$$\psi = -\tfrac{1}{2}V\left\{1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\frac{a^3}{r^3}\right\} . \quad . \quad . \quad . \quad (9)$$

* Wied. *Ann.* Band xlii. pp. 356, 549 ; Band xliii. p. 43.
† Camb. Trans. ix. p. 8 (1850); Math. and Physical Papers, vol. iii. p. 56.

The expression for the resistance of a pendulum moving in a viscous fluid is, according to the same author [*],

$$\mathrm{F} = -\tfrac{2}{3}\pi\rho a^3 cn\sqrt{-1}\Big(1 + \frac{9}{ma} + \frac{9}{m^2 a^2}\Big)e\sqrt{-1}nt, \quad (10)$$

which, when the conditions for steady motion are applied, becomes

$$-\mathrm{F} = 6\pi\mu'\rho a\mathrm{V}, \quad \ldots \quad (11)$$

for the resultant force on a sphere parallel to the direction of motion.

These results are obtained on the assumption that there is no slip at the surface, and that the inertia term

$$u\frac{\partial u}{\partial \chi}$$

may be neglected in comparison with the viscous term

$$\mathrm{V}\nabla^2 u.$$

The general form of the results obtained by Stokes [†] from theory has been recently verified by Mr. H. S. Allen [‡]. Mr. Allen allowed air-bubbles of various size to escape from a small opening under water. The size of the bubble was varied until the velocity with which the bubbles rose in the water or other fluid became constant. The force on the sphere due to its motion in the viscous fluid could then be measured in terms of gravity. Mr. Allen also allowed bicycle bearing-balls to fall through viscous fluids, varying the diameter until constant velocity was obtained. From results thus obtained Mr. Allen concludes that for very small velocities the motion agrees with that deduced theoretically by Stokes.

When, however, the velocity is greater than a certain definite velocity given by the formula

$$\mathrm{V} = \tfrac{2}{9}\, ga^2 \frac{\sigma\rho}{\mu}\frac{\beta a + 3\mu}{\beta a + 2\mu'}, \quad \ldots \quad (12)$$

the resistance is proportional to the radius to the three-halves power, and when the velocities are considerably greater than the critical velocity the resistance follows the law deduced by Sir Isaac Newton:

$$\mathrm{R} = k\rho a^2\mathrm{V}^2. \quad \ldots \quad (13)$$

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[*] Math. and Physical Papers, vol. iii. p. 33.
[†] *L. c.* p. 4.
[‡] Phil. Mag. [5] vol. l. pp. 338, 519 (1900).

2. *The Method.*—The method of allowing air-bubbles to ascend or solid spheres to descend in a viscous fluid gives only the total resultant pressure on the sphere, and does not give the distribution of the pressure over the surface of the sphere. It occurred to the writer while experimenting with spheres in a Kundt-tube that the distribution of pressure around a sphere might be obtained by using a hollow sphere in which there was a small opening, the interior of the sphere being connected to a manometer.

3. *Apparatus.*—A glass sphere of uniform diameter was blown on a capillary tube. At a point in one of the equators of the sphere a small hole was drilled, and it was then mounted in a tube 160 cms. in length and 3·5 cms. in diameter, through which a constant flow of air was maintained. The arrangement in general is shown in Plate XVIII. fig. 1.

Great care was taken that all sharp edges which would tend to form surfaces of discontinuity around the opening *o* (fig. 2) were rounded *. The diameter of the capillary tube *c* leading to the manometer *m* (fig. 4) was small compared with the diameter of the sphere, being in general less than one-twentieth. The sphere was inserted into the tube through an opening in the side, which was so closed that the inner surface of the tube was smooth and continuous. A constant current of air was maintained in the tube by keeping the two ends of the tube at a constant difference of pressure, the end B (fig. 1) being open to the atmosphere while the end A was connected to a mercury manometer *m*, not shown in the figure. The air was furnished from a gas-reservoir maintained at constant pressure by means of weights.

4. *Methods of Determining the Pressure.*—The pressure in the interior of the spheres was determined by water-mano-meters (fig. 4) made from glass of uniform diameter and connected to the sphere through the capillary tube *c*, the difference between the levels of the two columns of water being read by a cathetometer reading to 0·1 mm. The manometers were so arranged that the difference between the pressure normal to the inner surface of the tube AB at the point at which the spheres were situated, and the pressure in the sphere could be determined. The difference of pressure between two spheres at any time could also be measured.

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* Von Helmholtz, " Ueber Discontinuirliche Flussigkeitsbewegungen," *Berl. Monatsber.* April 1868; Phil. Mag. Nov. 1868. (See Lamb's ' Hydrodynamics,' pp. 100 to 102, reference.)

5. *Distribution of Pressure around a Single Sphere.*—A glass sphere five millimetres in diameter with an opening two-tenths millimetre in diameter was mounted in the tube; the opening was in the equator whose plane was parallel to the direction of the stream-lines in the tube, and could be rotated through 360° in this plane (fig. 2). The pressure over unit surface of the sphere at all points in this equator could then be observed : and since by symmetry this plane is identical with any other equatorial plane parallel to the same straight lines, the total distribution of pressure around the sphere may be obtained by rotating the pressure-distribution curve obtained in this plane through 180° around an axis parallel to the axis of the tube.

The full-line curve *mm*, Plate XVIII. fig. 9, shows the distribution of pressure in a plane parallel to the stream-lines, in terms of the pressure normal to the surface of the tube.

In all the diagrams, unless otherwise stated, the curves of the observed pressures are plotted to a scale in which the pressure of one millimetre of water is represented by each 5 mm. circle measured from the double circle marked *a.p.* in the diagram.

6. *Pressure around two Spheres whose Line of Centres is Parallel to the Stream-lines.*—Two similar spheres of 5 mm. diameter were placed in the tube with the line of their centres parallel to the direction of flow, the distance apart of their surfaces being 1·5 cm. They were first placed with their openings up stream, making $\theta = 0$ (fig. 5). The openings were then rotated through an angle of 180°. Readings of the pressure normal to the surface of the spheres, as given by the water-manometer *m*, were taken for each 15°. The velocity of the air-current, as measured by the pressure at the ends of the tube, being the same for each reading.

In the following table columns 2 and 3 give the pressure normal to the surface of the sphere, in millimetres of water, for a normal pressure in the tube of three millimetres of water. Columns 4 and 5 give the same for a pressure in the tube of one and eight-tenths millimetres of water.

The pressure diagrams plotted from these readings are exhibited in fig. 10.

The general form for the pressure-distribution around Sphere A is similar to that for a single sphere. The distribution around B is slightly modified by the presence of A. The distributions for the two normal pressures (curves *mm* and *m'm'*) are similar.

TABLE I.

Angle θ.	Normal Pressure 3 mm.		Normal Pressure 1·8 mm.	
	Sphere A. P.	Sphere B. P.	Sphere A. P.	Sphere B. P.
1.	2.	3.	4.	5.
0°	5·5	4·5	3·5	2·0
15	5·3	4·2	3·3	2·0
30	4·7	3·7	3·1	1·7
45	4·3	3·5	2·5	0·9
60	3·4	2·3	1·5	0·5
75	2·9	2·3	1·6 ?	0·5
90	2·1	1·8	0·5	0·9 ?
105	2·4	2·4	0·5	0·6
120	2·5	2·5	0·7	0·9
135	2·5	2·5	0·7	1·1
150	2·5	2·5	0·6	1·0
165	2·5	2·5	0·8	1·2
180	2·5	2·5	0·8	0·5 ?

The two spheres were then moved until their distance apart was one-tenth the former distance, *i. e.* 1·5 mm., making the distance apart less than half the diameter of the spheres.

The following table obtains at the normal pressures 2·8 mm. and 1·4 mm. of water.

TABLE II.

Angle θ.	Normal Pressure 2·8 mm.		Normal Pressure 1·4 mm.	
	Sphere A.	Sphere B.	Sphere A.	Sphere B.
1.	2.	3.	4.	5.
0°	4·7	2·7	3·7	0·5
15	4·1	2·9	3·0	0·6
30	3·8	2·9	1·5 ?	0·6
45	2·9 ?	2·9	1·0	1·0
60	2·8	3·2	1·0	1·1
75	2·3	3·3	1·0	1·4
90	2·0	3·0	1·0	1·0
105	2·1	2·4	1·0	0·6
120	2·1	2·1	1·0	1·0
135	2·0	2·3	0·9	0·9
150	2·1	2·2	1·0	0·7
165	2·1	2·1	1·0	1·0
180	2·2	2·2	0·9	0·9

.The pressure-curves plotted from the readings for normal pressure 2·8 mm. are given by the full-line curve *mm*, fig. 11. The pressure-distribution around sphere A is quite similar to the pressure-distribution around a single sphere. The pressure-distribution around sphere B differs from that of a single sphere owing to the proximity of sphere A.

7. *The Pressure Distribution around two Spheres whose Line of Centres is Perpendicular to the Stream-lines.*—Two glass spheres of five millimetres diameter were mounted so that their line of centres was perpendicular to the direction of flow, the distance apart of their surfaces being 2 mm., Plate XVIII. fig. 3.

As the pressure-distribution around the two spheres is not symmetrical with respect to the horizontal and vertical planes containing the line of centres, the distribution was obtained in these two planes; the distribution in the horizontal plane is given in Table III.

Since the distribution of pressure is not symmetrical with respect to any plane, the opening was rotated through 360°, readings being taken for each 10° for the first 90° and for each 30° thereafter.

TABLE III.

Angle θ.	Pressure in mm. of water.	Angle θ.	Pressure in mm. of water.
0°	1·50	120°	0·05
10	1·40	150	0·15
20	1·20	180	0·22
30	1·20	210	0·15
40	1·20	240	0·15
50	1·05	270	0·05
60	0·60	300	0·50
70	0·30	330	1·10
80	0·15	360	1·50
90	0·05		

The distribution of the pressure according to the above data is exhibited in Plate XVIII. fig. 12, curve *mm*. As the curve about B would be the image of that about A the readings were only taken over A.

The curves of the distribution of pressure around these spheres differ from the curve around a single sphere in protruding slightly at a position between 30° and 60° from the line of centres on the side nearest the companion sphere.

The curves are plotted to a scale of one centimetre to one

millimetre of water pressure. The double circle *ap* being taken as atmospheric pressure.

The distribution of the pressure in a vertical plane is given in Table **IV**. The manometers were arranged to give the differential effect, the excess of the pressure on sphere **A** over the pressure on sphere **B** in the line of centres being recorded. The arrangement is shown in Plate XVIII. fig. 6.

TABLE **IV**.

Angle θ.	Difference in pressure in mm. of water. $P_A - P_B$.	Angle θ.	Difference in pressure in mm. of water. $P_A - P_B$.
0	0·30	190	0·25
10	0·40	200	0·20
20	0·40	210	0·15
30	0·50	220	0·05
40	0·34	230 to 310	0·00
50	0·30	320	0·05
60 to 120	0·00	330	0·15
130	0·30	340	0·20
140	0·34	350	0·25
150	0·50	360	0·30
160	0·40		
170	0·40		
180	0·30		

8. *The Distribution of Pressure for a Perfect Fluid.*—The velocity potential for a single sphere moving through a perfect fluid, at rest at infinity, with velocity u is

$$\phi = \tfrac{1}{2} u \frac{a^3}{r^2} \cos \theta, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where θ is measured from the direction of motion of the sphere. The pressure at any point of the sphere is

$$\frac{P}{\rho} = F(t) + \frac{d\phi}{dt} - \tfrac{1}{2} q^2, \quad . \quad . \quad . \quad (16)$$

where $F(t)$ is a function of the time.

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial r}\frac{dr}{dt} + \frac{\partial \phi}{\partial \theta}\frac{d\theta}{dt} \quad . \quad . \quad . \quad (17)$$

and

$$\tfrac{1}{2} q^2 = \frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial \theta}\right)^2. \quad . \quad . \quad . \quad (18)$$

Where the velocity is constant

$$\frac{\partial \phi}{\partial t} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

From (15)
$$\frac{\partial \phi}{\partial r} = -u \frac{a^3}{r^3} \cos \theta, \quad \ldots \quad \ldots \quad (20)$$

and
$$\frac{\partial \phi}{\partial \theta} = -\tfrac{1}{2} u \frac{a^3}{r^2} \sin \theta, \quad \ldots \quad \ldots \quad (21)$$

also
$$\frac{\partial r}{\partial t} = -u \cos \theta, \quad \ldots \quad \ldots \quad (22)$$

and
$$\frac{\partial \theta}{\partial t} = \frac{u \sin \theta}{r} \quad \ldots \quad \ldots \quad (23)$$

From (16) the pressure around a single sphere moving in a perfect fluid is

$$\frac{\mathrm{P}}{\rho} = \tfrac{9}{8} \cos^2 \theta - \tfrac{5}{8}. \quad \ldots \quad \ldots \quad (24)$$

The broken-line curve nn in Plate XVIII. fig. 9 exhibits the pressure of a perfect fluid around a single sphere when moving with constant velocity.

In order to obtain the pressure-distribution around two spheres in a perfect fluid we determine the velocity potential and solve equation (16). The velocity potential for two spheres moving in their line of centres may be obtained approximately by the theory of images *. Using only the terms in the expansion of the first image so far as $\left(\dfrac{a}{c}\right)^6$ we obtain for the velocity potential of two spheres moving in their line of centres, fig. 7,

$$\phi = \tfrac{1}{2} u \frac{a^3}{r^2} \cos \theta + \tfrac{1}{2} v \frac{b^3}{c^3} (c - r \cos \theta) \left\{ 1 + 3 \frac{r}{c} \cos \theta \right.$$
$$\left. + 3 \frac{r^2}{c^2} \frac{5 \cos^2\theta - 1}{2} + 5 \frac{r^3}{c^3} \frac{7 \cos^3\theta - 3 \cos \theta}{2} \right\}, \quad (25)$$

where a and b are the radii of the spheres, r the distance from the centre of sphere A, c the distance apart of the two spheres, and u and v are the respective velocities. For constant velocity $u = v$ and at the surface of sphere A, $r = b = a$ and

$$\frac{\partial \phi}{\partial r} = -u \cos \theta - \tfrac{1}{2} u \frac{a^3}{c^3} \cos \theta \left\{ 1 + 3 \frac{a}{c} \cos \theta + 3 \frac{a^2}{c^2} \frac{5 \cos^2\theta - 1}{2} \right.$$
$$\left. + 5 \frac{a^3}{c^3} \frac{7 \cos^3\theta - 3 \cos \theta}{2} \right\} + \tfrac{1}{2} u \frac{a^3}{c^3} \left(\frac{c}{a} - \cos \theta \right) \left\{ 3 \frac{a}{c} \cos \theta \right.$$
$$\left. + 6 \frac{a^2}{c^2} \frac{5 \cos^2\theta - 1}{2} + 15 \frac{a^3}{c^3} \frac{7 \cos^3\theta - 3 \cos \theta}{2} \right\}, \quad (26)$$

* Stokes, *l. c.* p. 1; Hicks, *l. c.* p. 2.

and

$$\frac{\partial \phi}{\partial \theta} = -\tfrac{1}{2}ua \sin \theta + \tfrac{1}{2}u \frac{a^3}{c^3} a \sin \theta \left\{ 1 + 3\frac{a}{c}\cos \theta + 3\frac{a^3}{c^3}\frac{5 \cos^2\theta - 1}{2} \right.$$
$$\left. + 5\frac{a^3}{c^3}\frac{7 \cos^3\theta - 3 \cos \theta}{2} \right\} - \tfrac{1}{2}u\frac{a^3}{c^3}(c - a \cos \theta) \left\{ 3\frac{a}{c}\sin \theta \right.$$
$$\left. + 3\frac{a^2}{c^2} 5 \cos \theta \sin \theta + 15\frac{a^3}{c^3}\frac{7 \cos^2\theta - 1}{2}\sin \theta \right\}. \quad . \quad (27)$$

The broken-line curve *nn* in Plate XVIII. fig. 11 exhibits the distribution of pressure as given by equation (16), to the approximation indicated, for two spheres moving with constant velocity in the line of their centres.

For two spheres whose direction of motion is perpendicular to the line joining their centres (fig. 8) to the same degree of approximation

$$\phi = \tfrac{1}{2}u\frac{a^3}{r^2}\cos \theta + \tfrac{1}{2}v\frac{b^3}{c^3}r \cos \theta \left\{ 1 + 3\frac{r}{c}\sin \theta + 3\frac{r^2}{c^2}\frac{5 \sin^2\theta - 1}{2} \right.$$
$$\left. + 5\frac{r^3}{c^3}\frac{7 \sin^3\theta - 3 \sin \theta}{2} + \dots \right\} \quad . \quad . \quad . \quad (28)$$

$$\frac{\partial \phi}{\partial r} = -u \cos \theta + \tfrac{1}{2}u\frac{a^3}{c^3}\cos \theta \left\{ 1 + 3\frac{a}{c}\sin \theta + 3\frac{a^2}{c^2}\frac{5 \sin^2\theta - 1}{2} \right.$$
$$\left. + 5\frac{a^3}{c^3}\frac{7 \sin^3\theta - 3 \sin \theta}{2} \right\} + \tfrac{1}{2}u\frac{a^3}{c^3}\cos \theta \left\{ 3\frac{a}{c}\sin \theta \right.$$
$$\left. + 6\frac{a^2}{c^2}\frac{5 \sin^2\theta - 1}{2} + 5\frac{a^3}{c^3}\frac{7 \sin^3\theta - 3 \sin \theta}{2} \right\}, \quad . \quad (29)$$

and

$$\frac{\partial \phi}{\partial \theta} = -\tfrac{1}{2}ua \sin \theta - \tfrac{1}{2}u\frac{a^3}{c^3}a \sin \theta \left\{ 1 + 3\frac{a}{c}\sin \theta + 3\frac{a^2}{c^2}\frac{5 \sin^2\theta - 1}{2} \right.$$
$$\left. + 5\frac{a^3}{c^3}\frac{7 \sin^3\theta - 3 \sin \theta}{2} \right\} + \tfrac{1}{2}u\frac{a^3}{c^3}a \cos \theta \left\{ 3\frac{a}{c}\cos \theta \right.$$
$$\left. + 3\frac{a^2}{c^2}\frac{5 \sin^2\theta - 1}{2} + 15\frac{a^3}{c^3}\frac{7 \sin^3\theta - 1 \sin \theta}{2} \right\}. \quad (30)$$

The broken-line curve *nn* in fig. 12 exhibits the approximate distribution of pressure for two spheres in a perfect fluid moving perpendicular to their line of centres.

For each set of spheres observations were made when the distance apart of the spheres was somewhat less than three times their radius. The ratio of the radius to the distance apart of the spheres used in computing the curves for the distribution of pressure in a perfect fluid was 1/3. All terms in the expansion of the second image will contain this ratio to the sixth and higher powers, but all terms in the first

image containing this ratio to a higher power than the sixth were neglected; that is, terms containing a factor less than 8.6×10^{-4} have been omitted in the computation of the broken-line curves in figs. 11 and 12.

9. *Comparison of the Distribution of Pressure for a Perfect Fluid with the Pressure obtained for a Viscous Fluid.*—For a single sphere moving with constant velocity in a perfect fluid at rest at infinity the curve of distribution of pressure is symmetrical with respect to each plane of the three rectangular axes whose origin is at the centre of the sphere. And hence the resulting force in any direction is zero. For the viscous fluid the curve is asymmetrical with respect to the plane perpendicular to the direction of motion, but symmetrical with respect to the line of motion; and the resultant force is such as would tend to bring the sphere to rest.

For two spheres moving in their line of centres in a perfect fluid the curves of distribution of pressure are asymmetrical with respect to the axial planes which are perpendicular to their direction of motion, the force on the inner hemisphere being the greater. The normal pressure and the resultant component pressures along the line of motion over the inner and outer hemispheres at different points are given in the following table for sphere B.

TABLE V.

Angle.	Inner hemisphere.	Angle.	Outer hemisphere.	Difference.	Resultant component.
0°	·5028	180°	·499	·0038	·0038
30	·2940	150	·220	·0540	·0468
60	— ·2550	120	— ·320	·0650	·0320
90	— ·5880	90	— ·5880	·0000	·0000

The resultant force on both spheres is tending to separate the spheres, *i. e*, gives repulsion. The results for the two spheres in a viscous fluid are exhibited in Table I., and it is evident that the two spheres would have a relative motion such that they would approach each other, *i. e.* attract.

For two spheres moving in a perfect fluid perpendicular to their line of centres the curves of distribution of pressure are asymmetrical with respect to a plane perpendicular to the line joining them, the pressure in the outer hemisphere being the greater.

The following table gives the normal pressures |and| the resultant component pressures over the outer and inner

hemispheres of the two spheres at corresponding points.   The curve being asymmetrical with respect to the line joining them, only the figures for the first quadrant are given.

<div align="center">TABLE VI.</div>

Angle.	Outer hemisphere.	Angle.	Inner hemisphere.	Difference.	Resultant component.
360°	·5000	0°	·5000	·0000	·0000
330	·2202	30	·1790	·0412	·0206
300	—·3480	60	—·4211	·0731	·0617
270	—·6340	90	—·7130	·0790	·0790

Table III. gives the results for a viscous fluid.   The curve (fig. 12) is asymmetrical with respect to both axial planes, and it is clear from the form of the curve that the pressure in the inner hemisphere is greater than the pressure on the outer hemisphere.   The pressure on the inner hemisphere is at 30°, 3·3 per cent. of the normal pressure at 0° in excess of the pressure at 360°, and at 60° it is 5·7 per cent. of the normal pressure greater than the corresponding pressure at 300°.

For a perfect fluid, therefore, two spheres moving with constant velocity perpendicular to the line joining their centres attract, and for a viscous fluid they repel.

I have shown in a former paper[*] that when two particles in a sound-wave are a certain critical distance apart they are attracted when their line of centres is parallel to the stream-lines and repelled when their line of centres is perpendicular to the stream-lines.   The spheres used in these experiments were relatively large compared with particles or sphere that would form flutings in a sound-wave.   The results, however, agree with the results obtained with the smaller sphere in a sound-wave.   I hope soon to be able to determine the pressure around spheres small enough to form flutings in a sound-wave.

The experimental work included in this paper was conducted under the direction of Dr. Brace in the Physical Laboratory of the University of Nebraska, and my sincere thanks are due to him for valuable suggestions during the progress of the experiments, and also for his assistance in determining the curves of distribution for a perfect fluid.

Physical Laboratory, Case School of Applied Science,
Cleveland, Ohio, April 23, 1903.

LVII. *On Electro-ethereal Theory of the Velocity of Light in Gases, Liquids, and Solids.* By Lord KELVIN, *O.M., G.C.V.O.* *

THIS communication is an advance-proof of the last five pages of Lecture XX., as written afresh for a long promised volume of twenty lectures given originally in the Johns Hopkins University of Baltimore, U.S.A., in October 1884, and now nearly ready for publication by the Cambridge University Press. It is founded on two recent contributions to electro-ethereal theory referred to as " Appendix D," and " Appendix A," previously published in the Philosophical Magazine (1902, 1st half-year, and 1900, 2nd half-year) under the titles " Aepinus Atomized," and " On the Motion produced in an Infinite Elastic Solid by the Motion through the Space occupied by it of a Body acting on it only by Attraction or Repulsion." The long title of Appendix A contains virtually a complete statement of the theory which constitutes its subject.

In App. A it is temporarily assumed, for the sake of a definite illustration, that the enormous variation of the ethereal density within an atom is due to a purely Boscovichian force acting on the ether, in lines through the centre of the atom and varying as a function of the distance. This makes no provision for vibrator or vibrators within an atom ; and, for the explanation of molecular vibrators, it only grants such molecular groups of atoms, as we have had for fifty years in the kinetic theory of gases, according to Clausius' impregnable doctrine of specific heats with regard to the partition of energy between translational and other than translational movements of the molecules. Now, in App. D, and in applications of it suggested in §§ 162–168 of Lec. XIX., we have foundation for something towards a complete electro-ethereal theory, of the Stokes-Kirchhoff vibrators † in the dynamics of spectrum-analysis, and of the Maxwell-Sellmeier explanation of dispersion.

§ 233. In our new theory, every single electrion within a mono-electronic atom, and every group of two, three, or more, electrions, within a poly-electronic atom, is a vibrator which, in a source of light, takes energy from its collision with other atoms, and radiates out energy in waves travelling through the surrounding ether. But at present we are not

* Communicated by the Author, having been read before Section A of the British Association, Southport.
† See Lec. IX. pp. 101, 102, 103.

concerned with the source ; and in bringing this last of our twenty lectures to an end, I must limit myself to finding the effect of the presence of electrionic vibrators in ether, on the velocity of light traversing it.

§ 234. The " fundamental modes " of which, in Lec. X., p. 120, we have denoted the periods by $\kappa, \kappa_{\prime}, \kappa_{\prime\prime}, \ldots$ are now modes of vibration of the electrions within a fixed atom, when the ether around it and within it has no other motion than what is produced by vibrations of the electrions. It is to be remarked, however, that a steady motion of the atom through space occupied by the ether, will not affect the vibrations of the electrions within it, relatively to the atom.

§ 235. To illustrate, consider first the simple case of a mono-electrionic atom having a single electrion within it. There is just one mode of vibration, and its period is

$$\kappa = 2\pi\sqrt{\frac{m}{c}} = \frac{2\pi}{e}\sqrt{m\alpha^3} \quad . \quad . \quad . \quad (199),$$

where α denotes the radius of the atom, e the quantity of resinous electricity in an electrion, and m its virtual mass ; and c denotes $e^2\alpha^{-3}$. This we see because the atom, being mono-electrionic, has the same quantity of vitreous electricity as an electrion has of resinous; and therefore (App. D, § 4) the force towards the centre, experienced by an electrion held at a distance x from the centre, is $e^2\alpha^{-3}x$; which is denoted in § 240 by cx.

§ 236. Consider next a group of i electrions in equilibrium, or disturbed from equilibrium, within an i-electrionic atom. The force exerted by the atom on any one of the electrions is $ie^2\alpha^{-3}D$, towards the centre, if D is its distance from the centre. Let now the group be held in equilibrium with its constituents displaced through equal parallel distances, x, from their positions of equilibrium. Parallel forces each equal to $ie^2\alpha^{-3}x$, applied to the electrions, will hold them in equilibrium * ; and if let go, they will vibrate to and fro in parallel lines, all in the same period

$$\frac{1}{\sqrt{i}}\frac{2\pi}{e}\sqrt{m\alpha^3} \quad . \quad . \quad . \quad . \quad (200).$$

This therefore is one of the fundamental modes of vibration of the group; and it is clearly the mode of longest period. Thus we see that the periods of the gravest vibrational modes of different electrionic vibrators are directly as the square roots of the cubes of the radii of the atoms and inversely as

* Compare App. D, § 23.

the square roots of the numbers of the electrions; provided that in each case the atom is electrically neutralized by an integral number of electrions. Compare App. D, § 6.

§ 237. I now propose an assumption which, while greatly simplifying the theory of the quasi inertia-loading of ether when it moves through space occupied by ponderable matter as set forth in App. A, perfectly explains the practical equality of the rigidity of ether through all space, whether occupied also by, or void of, ponderable matter. My proposal is that *the radius of an electrion is so extremely small that the quantity of ether within its sphere of condensation* (Lec. XIX., § 166) *is exceedingly small in comparison with the quantity of undisturbed ether in a volume equal to the volume of the smallest atom.*

This assumption, in connexion with §§ 164, 166 of Lec. XIX., makes the density of the ether exceedingly nearly constant through all space outside the spheres of condensation of electrions. This is true of space whether void of atoms, or occupied by closely packed, or even overlapping, atoms; and the spheres of condensation occupy but a very small proportion of the whole space even where most densely crowded with poly-electrionic atoms. The highly condensed ether within the sphere of condensation close around each electrion might have either greater or less rigidity than ether of normal density, without perceptibly marring the agreement between the normal rigidity of undisturbed ether, and the working rigidity of the ether within the atom. This seems to me in all probability the true explanation of what everyone must have felt to be one of the greatest difficulties in the dynamical theory of light;—the equality of the rigidity of ether inside and outside a transparent body.

§ 238. The smallness of the rarefaction of the ether within an atom and outside the sphere or spheres of condensation, around its electrions, implies exceedingly small contribution to virtual inertia of vibrating ether, by that rarefaction; so small that I propose to neglect it altogether. Thus if an atom is temporarily deprived of its electrion or electrions (rendering it vitreously electrified to the highest degree possible), ether vibrating to and fro through it, will experience no inertial or other resistance. Its presence will not be felt in any way by the ether existing in the same place. Thus the actual inertia-loading of ether to which the refraction of light is due, is produced practically by the electrions, and but little if at all perceptibly by the atoms, of the transparent body.

§ 239. For the present I assume an electrion to be massless,

that is to say devoid of intrinsic inertia, and to possess virtual inertia only on account of the kinetic energy which accompanies its steady motion through still ether. This is in reality an energy of relative motion; and does not exist when electrion and ether are moving at the same speed. See App. A *passim*, and equation (202) below.

§ 240. Come now to the wave-velocity problem and begin with the simplest possible case,—only one electrion in each atom. Consider waves of x-vibration travelling y-wards according to the formula (203) below. Take a sample atom in the wave-plane at distance y from XOZ. The atom is practically unmoved by the ether-waves; while the electrion is set a vibrating to and fro through its centre.

At time t, let x be the displacement of the electrion, from the centre of the atom (or its absolute displacement because at present we assume the atom to be absolutely fixed):

ξ the displacement of the ether around the atom):

ρ the mean density of the ether within and around the atom, being, according to our assumptions, *exactly* the same as the normal density of undisturbed ether:

n the rigidity of the ether within and around the atoms, being, according to our assumption, *very approximately* the same at every point as the rigidity of undisturbed ether:

N the number of atoms per unit of volume:

cx the electric attraction towards the centre of its atom, experienced by the electrion in virtue of its displacement, x:

m the virtual mass of an electrion:

E a volume $\dfrac{1}{N}$, of ether, having the centre of one, and only one, atom within it.

The equation of motion of E, multiplied by N, is

$$\rho \frac{d^2\xi}{dt^2} = n \frac{d^2\xi}{dy^2} - Ncx \quad . \quad . \quad . \quad . \quad (201) ;$$

and the equation of motion of the electrion within it, is

$$m \frac{d^2(x-\xi)}{dt^2} = -cx . \quad . \quad . \quad . \quad . \quad (202).$$

§ 241. The solution of these two equations for the regular regime of wave-motion is of the form

$$\xi = C \sin \omega\left(t - \frac{y}{v}\right); \quad x = C' \sin \omega\left(t - \frac{y}{v}\right) \quad . \quad . \quad (203),$$

where ω is given. Our present object is to find the

two unknowns C/C′ (or ξ/x), and v. By (203) we see that

$$\frac{d^2}{dt^2} = -\omega^2 \; ; \quad \frac{d^2}{dy^2} = -\frac{\omega^2}{y^2} \quad . \quad . \quad . \quad . \quad (204).$$

This reduces (201) and (202) to

$$\left(\rho - \frac{n}{v^2}\right)\xi = \frac{Nc}{\omega^2} x = Nm(x - \xi) \quad . \quad . \quad . \quad (205),$$

from which we find

$$\frac{x}{\xi} = \frac{m\omega^2}{m\omega^2 - c} = \frac{-\kappa^2}{\tau^2 - \kappa^2} \quad . \quad . \quad . \quad . \quad (206),$$

and

$$\frac{1}{v^2} = \frac{\rho}{n} + \frac{Nm}{n}\left(1 - \frac{x}{\xi}\right) = \frac{\rho}{n} + \frac{Nm}{n}\frac{\tau^2}{\tau^2 - \kappa^2} \quad . \quad . \quad (207).$$

The last members are introduced with the notation

$$\tau = \frac{2\pi}{\omega} \; ; \quad \kappa = 2\pi\sqrt{\frac{m}{c}} \quad . \quad . \quad . \quad . \quad (208);$$

where τ denotes the period of the waves, and κ the period of an electrion displaced from the centre of its atom, and left vibrating inside, while the surrounding ether is all at rest except for the outward travelling waves, by which its energy is carried away at some very small proportionate rate per period; perhaps not more than 10^{-6}. It is clear that the greater the wave-length of the outgoing waves, in comparison with the radius of the sphere of condensation of the vibrating electrion, the smaller is the proportionate loss of energy per period. (Compare with the more complex problem, in which there are outgoing waves of two different velocities, worked out in the Addition to Lec. XIV., pp. 190–219. See particularly the examples in pp. 217, 218, 219.)

§ 242. Look back now to the diagram of Lec. XII., p. 145, representing our complex molecular vibrator of Lec. I., pp. 12, 13, reduced to a single free mass, m; connected by springs with the rigid sheath, the lining of an ideal spherical cavity in ether. In respect to that old diagram, let x now denote what was denoted on p. 145 by $\xi - x$; that is to say the displacement of the ether, relatively to m. Thus in the old illustrative ideal mechanism, cx denotes a resultant force of springs acting on m: in the new suggestion of an electro-ethereal reality cx denotes simply the electric attraction of the atom on its electrion m, when displaced to a distance x from its centre. In the old mechanism it is the

reaction through the springs, from their force on *m*, by which *m* acts on the ether (always admittedly an unreal kind of agency, invoked only by way of dynamical illustration). In the new electric design, *m* acts directly on the ether, by inertia, in simple proportion to acceleration of relative motion.

It is interesting to see that every one of the formulas of §§ 240, 241 (with the new notation of *x*, in the old dynamical problem), are applicable to both the old and the new subjects: and to know that the solution of the problem in terms of periods is the same in the two cases, notwithstanding the vast difference between the artificial and unreal details of the mechanism thought of and illustrated by models in 1884, and the probably real details of ether, electricity and ponderable matter, suggested in 1900–1903.

§ 243. The interesting question of energy referred to in Lec. X., ll. 18–21 of p. 111 becomes more and more interesting now when we seem to understand its real triple character in (I) kinetic energy of pure ether, (II) potential energy of elasticity of ether, and (III) electric potential energy of mutual repulsions of electrions and of attractions between electrions and atoms. It is slightly and imperfectly treated in App. L. For the present I must leave it with much regret, to allow this Volume to be prepared for publication.

LVIII. *On the Magnetic Effect of Electric Convection.* *By* Victor Crémieu *and* Harold Pender *.

A DETAILED discussion of the present state of the question of electric convection has been recently given by Crémieu in the December number of the *Journal de Physique*. Contradictory results have been obtained : some seeming to prove the existence of a magnetic field around a moving charged body ; others seeming to prove the non-existence of such an effect.

In the same article, the conditions necessary for a proper experiment on electric convection have been clearly set forth. We shall recall that the realization of the following conditions is necessary :—

1. That the charge is actually carried along (*entrainée*) by the body in motion.

2. That this charge alone can act upon the apparatus

destined to detect the magnetic effect attributed to its *entrainement*.

3. That during its *entrainement* the charge undergoes no variation ; in particular, that the loss by leakage is limited to a very small fraction of the total charge carried.

In the hope of reconciling the contradictory results obtained by the various experimenters on this question of fundamental importance, M. H. Poincaré took the initiative in the fall of 1902 to bring about a collaboration between two of the experimenters whose results have invariably been in contradiction one to the other. In accordance with the suggestion of Lord Kelvin, Paris was chosen as the most favourable place for these joint experiments ; and M. E. Bouty gladly put at the disposal of the two investigators his laboratory at the Sorbonne.

Accordingly, invited by Messrs. Poincaré and Bouty, in the name of the University of Paris, Mr. H. Pender, whose experiments have invariably given positive results, went to Paris in January 1903 to work in collaboration with M. V. Crémieu, whose experiments have always been of a negative character. The Johns Hopkins University put at the disposal of Pender all the apparatus necessary, and obtained from the Carnegie Institution the funds necessary to defray the expenses of the journey. The expenses of the actual experimenting were met by the Institute of France.

We wish here to express to M. H. Poincaré and M. E. Bouty our gratitude for their initiative and liberality. Through their invaluable aid in advice and criticism, we have been able to realize in three months an almost complete programme of very delicate experiments. Thanks to the most excellent organization of the laboratory of M. Bouty, we have not once been delayed by lack of apparatus or of any physical means whatever.

Also it gives us great pleasure to express our gratitude for the liberalities of the Carnegie Institution, of the Johns Hopkins University, and of the Institute of France.

Repetition of Pender's Experiment.—Crémieu, in 1899–1900 [*], was led to repeat Rowland's experiment ; and, for considerations elsewhere developed, he modified considerably the method. Instead of the direct magnetic effect, he studied the electromagnetic induction of a moving charged body. The experiment gave negative results, which persisted during a long series of observations, in spite of

[*] Crémieu, *Thèse de Paris*, Gauthier-Villars, 1901.

many modifications suggested by objections from numerous
sources.

At the suggestion of Rowland, Pender took up the method
of Crémieu, improving it in some details.   His experiments
were carried on first in the Physical Laboratory of the
Johns Hopkins University, and later in the country, far
from the disturbances of any industrial centre.   The details
of these two series of experiments have been published
elsewhere *.   The results were invariably positive ; and in
the second series the observed and calculated values
accorded within less than 5 per cent.

Pender brought with him to Paris all his apparatus,
which we at once set up under conditions as nearly similar
as possible to the original disposition.   The only condition
we were unable to realize was the same degree of mechanical
and magnetic stability.   Nevertheless, we were able to find
again the qualitative effects of magnetic induction already
observed by Pender.   We verified with a sufficient degree
of approximation the proportionality of these effects with
the speed of rotation of the disks and with the surface-
density of the charge carried round.   We also found that
the effects of the two disks were superimposed or annulled
each the other in accordance with the theory of convection,
and that the effect obeyed the law of the distance between
the disks and induced coil.   As to the order of magnitude
of the effects observed, the deflexions varied from 38 to
800 mm. on a scale placed 4 metres from the galvanometer.

Repetition of Crémieu's Experiment.—Crémieu at the same
time mounted his induction experiment, the conditions
being almost identical with those of his original experiment.
However, the galvanometer previously employed had been
destroyed ; so we were obliged to use another.   Also the
conditions of stability were not so good as in the original
installation.

Although the effects due to charging and discharging
the disk at rest were not entirely eliminated, we obtained
with a sufficient degree of certainty the same negative
results as in the original experiment.

Verifications in Crémieu's Experiment. — In these two
series of experiments we employed continuous metallic
disks turning between continuous parallel condensing plates.
In the course of the repetition of Pender's experiment, we
observed the following effect.   When a voltage was em-
ployed sufficiently high to cause sparks to spring across

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* Phil. Mag. ser. 6, vol. ii. p. 179 (1901), *et loc. cit.* vol. v. p. 34 (1903).

from the disks to the condensing plates, the magnetic effect observed fell practically to zero; whereas the potential of the disk, as measured by an electrometer connected thereto, showed a decrease of less than $\frac{1}{10}$ of the value observed just before the sparks began. In Crémieu's apparatus it was impossible to see such sparks if they should occur, as the disk turned in a completely closed box of cast iron. However, in spite of the fact that the disk and condensing plates were covered with a thin layer of caoutchouc, it was possible that such sparks might occur, as the distance between these plates was very small.

Only a galvanometric measurement of the quantity of electricity taken by the disk at each charge, or the quantity given up at each discharge, could decide this point. We made these measurements, employing voltages from 1000 to 5500 volts. For 1000 volts, it is certain that no spark can occur across 3·5 mm. of air. As the voltage is increased, one should obtain currents proportional to the voltage as long as no sparks occur. If such sparks take place, the charging current should increase more rapidly, the discharge current less rapidly, than the voltage. Measurements showed that up to 5500 volts, the maximum potential employed, there was a strict proportionality between the charge and voltage.

The cause of Crémieu's negative results was not then in this.

Verifications in Pender's Experiments.—In the series of experiments on open currents made by Crémieu * in 1902, with the aid of M. J. Javal, certain peculiar magnetic effects were observed in the neighbourhood of a node of electrical oscillations, such as is formed by the turning disk in the convection experiments. The conditions for the production of these effects are so far similar to the conditions in Pender's experiment that one might believe that the effects observed by Pender were due to such oscillations, and not to convection itself.

This hypothesis was rendered still more plausible by certain magnetic effects observed when the disks are charged or discharged at rest. These effects, though extremely irregular, are capable of giving to the galvanometer connected with the induced coil deflexions of more than 100 mm. Moreover, these effects are considerably augmented by the smallest hole in the electric screen of tinfoil which protects the coil. Again, as the tinfoil is wrapped around the winding of the coil only, thus leaving the centre of the coil

* *Comptes Rendus,* vol. cxxxv. p. 27 (1902), and *Journ. de Phys.* Dec. 1902.

open, any electric oscillation through the coil will suffer little damping.

To ascertain if such oscillations played any part, we placed Pender's coil in a box of brass 2 mm. thick, entirely closed. This diminished considerably the perturbations caused by charging and discharging the disk at rest; but the magnetic effect due to the movement remained of the same order to within 10 per cent., *i. e.* as close as we could observe under the unstable conditions of the experiment.

However, in Crémieu's negative experiments, the iron box in which the disk turns can arrest *every* magnetic oscillation tending to traverse the coil, as this box constitutes a perfect screen, which is not the case with the brass box in which we inclosed Pender's coil. In fact, it is well known that damped waves, such as those which occur around a body whose charge is rapidly varying, traverse without considerable alteration even very thick conducting screens. To verify this point, it would therefore have been necessary to put a magnetic screen around Pender's coil.

The following method of procedure, however, is much simpler. If the effects observed by Pender are due to an oscillatory phenomenon, they must certainly be modified by any change in the circuits serving to charge or discharge the disks.

We placed in these circuits liquid resistances. They did not cause the effect to disappear; they merely diminished considerably the perturbations at rest.

Again, we placed in parallel with the disks a variable capacity. The capacity of the disks was 200 c.g.s. electrostatic units. The capacity in parallel was an air-condenser, and could be varied by sixths from 166 to 1000 c.g.s. This time the effect in movement was considerably diminished. The diminution was approximately proportional to the capacity in parallel with the disks. At the same time we noticed a diminution of the potential assumed by the disks. (In all our experiments we used for the source of charge M. Bouty's high-potential storage-battery, capable of giving as much as 14,000 volts.) This potential was measured by an electrometer connected automatically to the disks at the moment when they were charged. However, there was not a proportionality between the diminution of voltage and diminution of the deflexion. The deflexion dropped to one third its original value, whereas the decrease in voltage was only 10 per cent. However, we discovered that the disks also no longer became completely discharged, and that the charge remaining on the disks was proportional to the capacity in parallel.

These facts led us to study the duration and number
of contacts necessary to charge or dis-
charge a condenser completely. A cylin-
drical rod T (fig. 1) of ebonite, 120 cm.
long, was placed vertically in such a
manner that a brass ring B, 1 cm. high,
could fall freely along T. At 10 cm.
above the lower end of the rod were
placed two metallic brushes, C and D,
opposite each other. The brass ring, B,
in falling made contact between C and D
for any desired interval of time, depending
upon the length of the drop. The con-
denser employed was the same as that put
in parallel with the disks. The source of
electricity was the high-potential battery
of M. Bouty, the capacity of which can be
considered infinite in comparison with that
of the condenser.

Fig. 1.
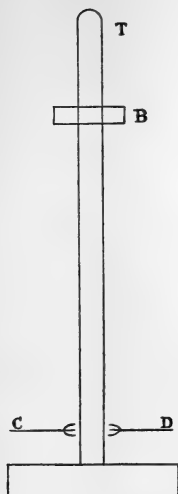
The experiment was made as follows :—
First was measured the potential of one
pole of the battery when the other was
earthed. One plate of the condenser was connected to
the earth and the other connected to the brush D. The
brush C could be connected either to the free pole of the
battery or to the earth. First, let C be connected to the
battery, and let the ring fall; contact will be established
between C and D as the ring passes. The duration of the
contact could be varied between $\frac{1}{10}$ and $\frac{1}{90}$ of a second.
After the fall the potential communicated to the condenser
was measured by an electrometer connected permanently
thereto. Under these conditions, we observed the fol-
lowing :—

1. The potential assumed by the condenser was always
inferior to that of the battery. This difference was inversely
proportional to the time of contact and directly proportional
to the capacity of the condenser.

Next, the condenser was charged by a prolonged contact
with the battery, and then insulated; C was connected to
the earth, and the ring B dropped. The electrometer con-
nected with the condenser then indicated if the condenser
retained a charge after contact. We observed the following:—

2. The condenser always retained a residual charge, the
value of which was directly proportional to the capacity
of the condenser and inversely proportional to the duration
of the contact.

However, the difference between the potential of the battery

and that taken by the condenser after contact was much greater, all else being the same, than the residual potential after the discharge.

These facts explain sufficiently the diminution of the effects of induction of the disks in motion when a capacity is put in parallel with the disks. The presence of this capacity decreases the variation of the charge at each reversal of the commutator, and, in consequence, the effects of induction, which are proportional to this variation; further, the indications of the electrometer, which gives the maximum potential assumed by the disks, cannot be affected to the same degree.

Consequently, it appears that the effects observed in Pender's experiment are due to the charged disks in movement, as demanded by the theory of electric convection.

Verifications without fixed Condensing-plates.—To eliminate entirely the questionable role of the fixed condensing-plates, we made the following modification :—

Pender's two disks, diameter 31 cm., were placed opposite each other, 1 cm. apart, and arranged to turn in opposite directions. Concentric with the two disks was placed a coil 34 cm. internal diameter and 3 cm. thick. The coil, which was entirely inclosed in a brass sheath, consisted of 1300 turns of copper wire, with a total resistance of 60 ohms. The commutator was arranged so as to connect this coil to the galvanometer synchronously with the charging and discharging of the turning disks. The deflexion of the galvanometer under these conditions was much less than in the former experiments, on account of the considerable decrease in the capacity of the two disks thus arranged. In spite of that, however, the deflexion was sufficiently large to be measured, and the agreement between the observed and calculated deflexion was quite satisfactory.

Here we stopped our experiments with the induction method. These experiments can, in fact, give no information in regard to the nature of the magnetic field produced by the movement of the two disks, whereas it is essential to know whether this field is permanent with the rotation, or is merely an instantaneous field produced at the moment of charging or reversing the charge, resulting from magnetic perturbations thereby set up.

Experiments on the Direct Magnetic Effect.—First, we utilized the same arrangement as in the preceding induction experiment. The coil was replaced by a delicate astatic system inclosed in a metallic tube so arranged that the lower needle of the system was 1 cm. above the upper edge of the disks, in a plane equally distant from the two. The tube was protected from the air-currents caused by the rotating disks by a large

sheet of mica properly placed. The sensibility of the system was determined by a test-coil. The reversal of a current of 10^{-4} ampere in this coil gave a deflexion of 20 mm. on a scale 4 metres distant. The convection-current possible to realize in this experiment was only $2 \cdot 5 \times 10^{-5}$ ampere, and in consequence the deflexion expected was less than 5 mm. Hence the experiment was purely qualitative. However, there was no doubt as to the effect. At each reversal of the sign of charge upon the turning disks the system was deflected in the expected direction, but it was impossible to say, on account of the unsteadiness of the needle, whether the deflexion was permanent or only an impulse.

It is to be noted, that in spite of the smallness of this result, it is nevertheless of great importance. Any perturbing effects due to the fixed condensing-plates or to currents which might circulate in the moving disks are entirely eliminated ; because the first are suppressed, and the second would destroy each other, as the two disks turn in opposite directions.

In the above experiment only a small part of the disk is effective. The needle is in a very unfavourable position, because the equipotential surfaces of the magnetic field are very sharp over the edge of the disk, and further, it is poorly protected against the air-currents, which produce a considerable unsteadiness. These considerations led us to return to Rowland's original method.

A single disk turning in a horizontal plane was inclosed in a box of ebonite, the inside of which, covered with tinfoil, formed the condensing-plates, thus increasing considerably the capacity of the moving disk. This arrangement allowed us to put the astatic system very near the disk, in a region where the magnetic field is horizontal and quite constant ; and, finally, the effects of the air-currents were entirely eliminated. We were therefore enabled to give the needle a much longer period, without having to fear as much the mechanical perturbations which might be produced during the interval of time necessary for the system to take its complete swing.

This time the deflexions obtained were considerable, 20 to 30 mm., of the order and in the direction expected. But it was still impossible to decide with certainty whether the deflexions were permanent or simply impulses.

Pender observed in his former experiments [*], in which a perfect stability was realized, deflexions which were undoubtedly permanent, but he did not know at that time of the magnetic effects which are produced in the neighbourhood

[*] Phil. Mag. vol. v. Jan. 1903, p. 34.

of nodes of electric oscillations such as formed by the disks, a description of which Crémieu has recently given *. He was therefore not absolutely certain that the deflexions he observed were not due to such oscillations, the effects of which might be easily confused with those due to a permanent magnetic field presumably due to the convection-current. These considerations led us to modify both our mode of operation and the system used to detect the magnetic field.

New Mode of Operation.—In all the experiments performed up to the present by other experimenters and ourselves, the effect observed was always that resulting from charging or reversing the sign of the charge on the disks already in rotation. Of course care was always taken that there was never an effect produced by charging or reversing the charge on the disks at rest, which condition can be realized by properly inclosing the astatic system (or galvanometer-circuit) in a suitable conducting screen. However, the action of such a screen is both electrostatic and electromagnetic. Besides, it is well known that in the neighbourhood of a node of electrical oscillations, there are produced rapidly damped magnetic waves capable of demagnetizing a magnet; and such a demagnetization can result in a permanent change of the position of equilibrium of the astatic system. Moreover, it is unknown what effect the movement of the body forming the node for these oscillations can have upon the oscillations themselves. Consequently one cannot assume *a priori* that a screen which is sufficient to suppress any effect of these oscillations when the disk is at rest will also be sufficient when the disk is in motion.

The following mode of operation appears to us to avoid all these inconveniences:—First, charge the disk at rest, care being taken that this operation produces no effect on the magnetic system. Then insulate the disk from the charge-source and set it in motion. The magnetic system should then, if the effect of convection exists, take a deflexion increasing with the velocity, permanent for any given velocity, and returning to zero at the moment the disk is stopped. An electrometric measurement of the potential of the disk before and after movement will show if there has been any sensible leakage.

As ordinary astatic systems are poorly suited for such an experiment, we attempted to construct a system of a different kind.

Experiments with New Magnetic Systems.—It is difficult to realize, in most laboratories, a magnetic stability sufficient to

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* *Comptes Rendus,* vol. cxxxv. p. 153 (1902).

allow of the measurement of magnetic fields as low as 10^{-6} c.g.s.—the order of the field in most convection experiments. Further, from the nature of its construction, a sensitive astatic system always tends to drift in a certain direction, due to the slow and unequal demagnetization of the needles which compose it. Another cause of instability is that the direction assumed by such a system is determined by the difference between the earth's field and that of the compensating magnets: consequently any slight variation in either of these fields will produce a considerable change in the position of equilibrium of the system. In particular, if the field to be studied is produced by oscillatory phenomena more or less damped, astatic systems become practically useless, since the demagnetizations which accompany these damped oscillations not only affect the system itself, but also the directing magnets. Finally, in the case where the field to be studied is very feeble, it seems advantageous to increase the effect by using strong magnetic poles; but it is well known that the sensibility of an astatic system is independent of the moments of the magnets which form it.

All these inconveniences can be avoided, without decreasing the sensibility, by employing an extremely simple system. The system consists essentially of a light horizontal beam carrying at one end a vertical magnet and at the other a non-magnetic counterweight of brass. The whole is supported by a long fine metallic wire fixed to the centre of the beam. If the magnetic axis of the magnet is exactly vertical, the couple to which this magnet is submitted due to the earth's field will produce no action on the torsion wire. The beam will therefore take up a position of equilibrium due only to the torsion of the wire which supports it. To adjust the system to this condition, the magnet is first replaced by a non-magnetic needle of the same form and weight, and the period of oscillation of the system determined. The magnet is then replaced. In general the period of oscillation becomes much shorter, and the beam takes up a different position of equilibrium. However, by adjusting the counterweight an exact verticality of the magnetic axis of the magnet can soon be obtained, which is indicated by the system again taking the same period of oscillation as when the magnet was replaced by the non-magnetic needle. A magnetic system is then realized which is directed only by the torsion of the wire.

With systems of this kind we have obtained very great sensibilities. For example, for a magnet-pole of 20 c.g.s., supported by a silver wire 95 cm. long and ·025 mm. in diameter, we obtained a deflexion of 34 mm. on a scale 2 metres distant for a variation of the field of 10^{-6} c.g.s. The

period of oscillation was about 60 seconds. When this system
was placed near the moving disk in a suitable screen, it was
absolutely unaffected by the starting of the motor and the disk.

However, an unexpected effect prevented us from obtaining
what we had hoped. At the end of about two minutes'
rotation of the disk, over which the system was placed, the
system began slowly to deflect, the deflexion increasing as
the time went on, thus rendering all observations impossible.
At first we attributed this deflexion to air-currents set up in
the screen containing the system by the heating of the con-
densing-plate over the disk. This plate became quite warm
on account of the violent vibrations caused by the rapid
motion of the disk. But the "dummy" system with the
non-magnetic needle gave no such deflexion ; hence the
effect could not be attributed to air-currents. We finally
concluded that the cause of this effect was the following:—
The rotation of the disk immediately below the pole of the
magnet causes Foucault currents to be set up in the gilded
surface, and the reaction of these on the magnet produces
either a slow demagnetization or a change in the position of
the magnetic axis of the needle with respect to its geometric
axis. However, be the cause what it may, we were forced
to abandon the use of such systems.

Returning to the ordinary systems, we constructed a very
sensitive one in the following manner :—Three pairs of very
fine steel needles, magnetized to saturation, were arranged
upon a thin sheet of mica as indicated in fig. 2. This system
was 6 cm. long and weighed about 500 mg. It was
suspended inside a metallic tube from a silk fibre
about a metre long. A test current-sheet con-
structed in such a manner as to give a distribution
of current similar to that which is produced by
the disk, was used to determine the sensibility of
the system. When this test-sheet was put in
place of the disk and a current of 10^{-4} ampere
sent through it, the system was deflected 120 mm.
This sensibility was amply sufficient, since the
convection-current realized in the experiments was
between 3 and 5×10^{-5} ampere.

But even with this system we were not able
to realize exactly the experiment which we have
described. In fact, the micanite disks which we
employed are magnetic ; and very unequally so
at different points. Consequently the system
would take different positions of equilibrium de-
pending upon the part of the disk immediately
under it. However, if the disk is given a velocity such that

Fig. 2.
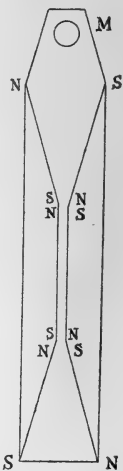
it makes at least one complete turn during the time required for a complete oscillation of the system, the latter will take a mean position of equilibrium which will not change when the velocity is augmented. We therefore proceeded as follows. The disk was given a slow rotation, the equilibrium position of the system noted, the disk then charged, and the equilibrium position again noted. Thanks to the liquid resistance in the charging circuit, the change in the equilibrium position was too small, if any, to notice. Then the speed of rotation was increased to its maximum, maintained at such for some time, and the position of equilibrium again noted ; finally, the velocity was reduced to its first value and another reading taken.

In this way we assured ourselves of the following :

There was a deflexion of the system when the velocity was increased, in the direction demanded by the theory of electric convection.

The deflexion was permanent.

It accorded quantitatively with the calculated value to within 10 or 20 per cent.

From all the foregoing results, corroborated by those previously obtained by Pender, we can conclude that :

A charged disk, having a continuous metallic surface, turning in its own plane between two fixed condensing plates, parallel to this plane, produces a magnetic field in the direction and of the order required by the theory of electric convection.

There now remained two questions for us to solve :

1. Are the magnetic effects thus obtained due to an actual *entrainement* of the charge by the moving metallic surfaces, or can they be attributed in some way to conduction-currents, open or closed, produced by the relative motion of the disk and the condensing plates ?

2. What cause concealed from Crémieu the effects observed by Pender in his previous experiments and also observed by us in common in the experiments just described?

Experiments on the Entrainement *of the Charge by Continuous Metallic Surfaces.*—To answer the first question, we first undertook an experiment the idea of which is due to Helmholtz, and the first attempt at realization to Rowland. Consider a plane ring, continuous and conducting, represented schematically (fig. 3) by the circle NRMP. This ring can turn in the direction of the arrow under a fixed condensing-plate, represented by the arc SS_1, covering a portion of the moving ring. Let the arc covered by this condensing-plate be $1/n$ of the whole circumference, let ρ be the distance between the planes of the two plates NRMP and SS_1, γ the linear velocity

of any point on the ring NRMP. Suppose SS$_1$ charged to a
negative potential K, and NRMP connected permanently to
the earth. As each element *ds* of NRMP arrives opposite S
it will become charged by influence. If the hypothesis above
stated is true, this element should carry its charge along

Fig. 3.
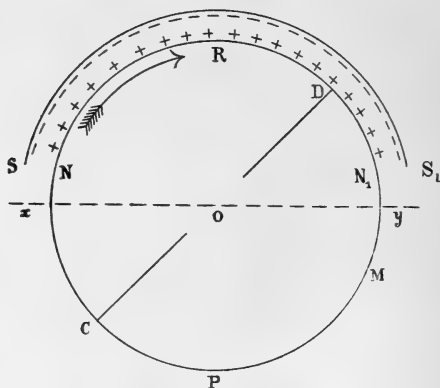
with it. In other words, this charge will be displaced with
reference to the electrostatic field between NRMP and SS$_1$,
which remains fixed in space. When *ds* arrives opposite S$_1$,
the charge on it is no longer retained by influence. As this
is true for all elements of NRMP, there will result from the
movement a constant difference of potential between the two
points N and N$_1$ on the moving disk opposite S and S$_1$, con-
sequently conduction-currents will be set up in the ring, and
will be distributed according to the relative resistances of the
sectors NRN$_1$ and NPN$_1$.

A magnetic needle placed above R will therefore be sub-
jected to a magnetic field due to two causes : first, that due
to a convection-current in the direction of the arrow ; and
second, that due to a conduction-current in the opposite
direction. If the needle is placed over P, it will be acted
upon by the conduction-current only. Moreover, the sum
of the conduction-currents in NRN$_1$ and NPN$_1$ should be
equal to the convection-current.

Let σ be the surface-density of the charge on NRMP,
then the intensity of the convection-current due to a ring
1 cm. wide is

$$i = \sigma\gamma.$$

But

$$\sigma = \frac{K}{4\pi\rho}.$$

Hence
$$i = \frac{K\gamma}{4\pi\rho}.$$

Let r be the total ohmic resistance of the conducting ring. Then the resistance of NRN_1 is $\frac{r}{n}$ and that of NPN_1 is $r\frac{n-1}{n}$. The conduction-current in NRN_1 is evidently

$$-i_1 = -i\left(\frac{n-1}{n}\right).$$

The magnetic field at a distance above R great with reference to ρ will be due to the difference between i and i_1, that is, to

$$i\left(1 - \frac{n-1}{n}\right) = +\frac{i}{n}.$$

Similar reasoning shows that the field over P is proportional to $-\frac{i}{n}$.

Hence, a magnetic system placed over P or R will show magnetic effects equal and in opposite directions.

Rowland tried this experiment, but the smallness of the effect expected prevented his obtaining satisfactory results.

He tried the experiment, giving $\frac{1}{n}$ the value $\frac{1}{2}$. The conducting ring was thin gold-leaf on an ebonite disk, the width of the ring being 5 cm. and the mean radius 14·5 cm. The calculated intensity of the convection-current was 4×10^{-5} ampere. The deflexions of an astatic system placed over P were in the direction expected and accorded quantitatively with the calculated deflexion within the usual approximation in our experiments, *i. e.*, 10 to 15 per cent.

However, by slightly modifying the experiment we were able to measure the intensity of the convection with great precision. If the two points on the disk immediately under S and S_1 are connected to a galvanometer the resistance of which is of the same order as that of the halves of the ring NRMP, there will flow through the galvanometer a current easily measurable, since it will be a considerable fraction of the convection-current, that is to say, in this experiment, of the order of 10^{-5} ampere. The gilding on the disk can be made extremely thin, having a resistance from 4 to 6 ohms per square centimetre of surface. The resistance of one-half the ring on our disk was 6 ohms. We employed a galvanometer of the d'Arsonval type (Hartmann and Braun) having

a resistance of 4·38 ohms and sensitive to 10^{-7} ampere. Hence for a convection-current of 4×10^{-5} ampere, the current flowing through the galvanometer will be of the order of $1·7 \times 10^{-5}$ ampere, and the resultant deflexion about 170 mm., a deflexion susceptible of great accuracy of measurement.

The first experiment was to place two fixed metallic brushes so as to rub on the ring under the two points S and S_1. However, the friction of the brushes against the uncharged disk was sufficient to produce a deflexion of about 100 mm. Moreover, the gold was rapidly rubbed away and the resistance therefore rapidly increased, so that all we could obtain from this method of procedure were rough qualitative results.

But, making yet another slight modification, we were enabled to make quite accurate measurements. This was accomplished by employing the following exceedingly simple and effective contacts. At the extremities AA of the moving axle (fig. 4), coinciding with the axis of rotation, were fixed

Fig. 4.
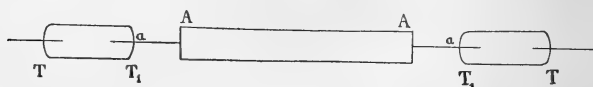
two very fine copper wires insulated from the axle. Two small glass tubes TT_1, all but sealed up at T_1 so that the opening there was just a little larger than the diameter of the wire aa, were so placed that the wires turned in the openings T_1 without touching the walls of the tube. Before being placed in position these tubes were filled with mercury and electrodes sealed in the outer ends. These contacts proved extremely satisfactory—their resistance when the axle was in motion remained practically the same as when the axle was at rest, an hour's running producing an increase of resistance of less than five hundredths of an ohm.

To return to the convection experiment. Two points C and D, diametrically opposite, were connected permanently to the wires aa, which in turn were connected through the contacts to the galvanometer. If now SS_1 is charged and the disk set in motion, there will be produced in the disk conduction-currents distributed as explained above. But as C and D move with the disk, between these two points the difference of potential will be alternating, and consequently in the galvanometer there will be an alternating current and

hence no deflexion. However, if in the galvanometer circuit is placed an interrupter actuated by an eccentric on the axle of the disk, arranged so as to close the galvanometer circuit during only half a revolution of the disk, a uni-directed current can be obtained. If the eccentric is adjusted so as to close the galvanometer-circuit while CD turns from PR to RP, the difference of potential between C and D will pass from 0 to 0 through a maximum corresponding to the moment when CD is in the position *xy*. The current through the galvanometer will then be uni-directed, and a deflexion should be obtained. If the eccentric is shifted 90° so as to close the galvanometer-circuit a quarter of a period later, this deflexion should become nil, because the difference of potential between C and D passes from the value $-e$ to $+e$ during the half period of contact. Between these two positions of the eccentric all intermediate deflexions should be obtainable.

This experiment proved unusually satisfactory. The maximum deflexion was quite large and the measurements agreed within 2 per cent. with the calculated values, which, considering the numerous approximations which must necessarily be made in determining the various constants (surface-density of the charge, relative resistances of the various parts of the circuit, speed of rotation), is all that could be desired. From the results we may therefore conclude that :

A charged metallic surface moving in its own plane in the presence of fixed parallel metallic surfaces carries its charge along with it.

Experiments with Sectored Disks. Crémieu's "Open Current" Method.—Having thus solved in the affirmative the question of the existence of a magnetic effect produced by the rotation of charged continuous disks, and having verified directly the *entrainement* of the charge, we returned to the experiments made with sectored disks.

During the past year, Crémieu [*] obtained with disks formed of insulated sectors magnetic effects which were quite irregular and without any quantitative relation to the intensity of the convection-current, which, in these experiments, was measured directly. The apparatus consisted of a core of ebonite 24 cm. in diameter carrying 18 sectors of micanite 13 cm. long, separated from each other by 2 cm. of air. Only the outer portion of the sectors was gilded, the gilt covering a width of 5 cm. The moving sectors M (fig. 5) passed between two charged fixed sectors S, and at the same

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[*] *Comptes Rendus,* vol. cxxxvi. p. 27 (1902).

time touched a brush A connected to the earth; they there-
fore became charged by influence. They then passed from
under the brush A and the fixed sectors S and came around

Fig. 5.
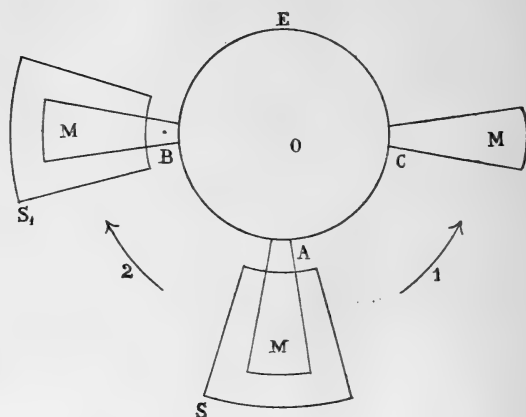
under the astatic system. Beyond the system they met a
second brush B connected to the earth, thus becoming dis-
charged. By placing a galvanometer between A or B and
the earth, one can measure the charging current or the
current resulting from the discharge.

Certain phenomena, first observed by Crémieu, of solid
dielectrics, to which we shall return further on, led us to
employ in our joint experiments micanite sectors entirely
gilded, so as to avoid all action of the penetration of the
charge into the naked micanite. The astatic system was
suspended in a metallic tube connected to the earth. To
protect this tube from the electrostatic action of the moving
sectors, which were at a very high potential, Crémieu employed
a paraffined tube of mica fixed directly on the metallic tube.
To this disposition there are two objections. Since the
bottom of the tube is small relative to the area of the moving
sector, at the moment when the sector comes under the tube
there is a considerable increase of capacity of that portion
of the sector directly under the tube. Consequently to this
point there will be a flow of charge by conduction-currents,
distributed in a manner impossible to calculate, which may
act upon the system. Secondly, in consequence of the
peculiar phenomena of the dielectric to which reference has
been made, the mica protector of the tube is submitted to a
penetration of charge from which may result considerable
perturbations. To avoid these inconveniences, in some

measure at least, we left the metallic tube bare, but interposed between it and the moving sectors a large piece of paraffined ebonite, which touched neither the tube nor the sectors.

In these conditions the passage of the charged sectors under the astatic system produced deflexions qualitatively in accord with the effects expected from the theory of convection and properly varying with the density of the charge and the velocity. As to a quantitative agreement, that was not very satisfactory, but in this form of experiment the distribution of the charge on the sectors is too indefinite to allow of even a rough approximation. Again, as the sectors rotate in the open air, it is impossible to eliminate completely the effects of the air-currents, which render the system too unsteady for accurate readings to be possible.

There now remained for us to find the cause of the repeatedly negative results of Crémieu's experiments.

Essential Difference between the Negative and Positive Experiments.—An analysis of the details of the negative experiments led us to see that they all differed from the positive experiments in this, that in the negative experiments the moving charged surfaces and the condensing-plates, when there were any, were always covered with a thin layer of some solid dielectric, usually caoutchouc. In his experiments Crémieu found it desirable to realize as high a surface-density of the charge as possible. To do this, he placed the turning disk and the fixed condensing-plates as near each other as possible ; the caoutchouc coverings were for the purpose of preventing sparks between plates and disk. *A priori*, one could see no inconvenience resulting from the presence of the caoutchouc ; in fact, one would naturally suppose that it would prevent leakage and assure a more perfect *entrainement* of the charge, for, as the charge is carried on the surface of the dielectric in contact with the conductor, any slipping which might occur would be rendered less probable.

To test the truth of these considerations, we tried the effect of covering the moving sectors in the above experiment with a layer of caoutchouc. There resulted a considerable diminution of the magnetic effects, without, however, the intensity of the convection-current, as measured by the quantity of charge going on or leaving the sectors (measured as above described), showing a corresponding diminution. Moreover, the nature of the deflexions of the astatic system changed. At first quite distinct for the two signs of the charge, they rapidly became smaller, and at the end of several reversals became scarcely perceptible, especially when the moving sectors were charged positively. For the negative

charge the diminution was less. Further, the deflexions ceased to be proportional to the potential of the fixed sectors for voltages above 2000 volts. Further, the charge going on the sectors became less than it should be according to calculation, and above a certain definite voltage the charging current remained practically constant, no matter how much the voltage was increased.

These facts made clear the cause of the negative results. Moreover, in 1900, in his first experiments on open currents, Crémieu observed a part of these phenomena. However, as he was eager to complete his experiments on convection, he did not stop to investigate completely these effects.

We now undertook a systematic study of the role of the dielectric, but, in consequence of the short time at our disposal, we have not been able to carry this study very far. As of first importance from the point of view of the theory of convection we verified the following points :—

1. When the continuous disks were covered with caoutchouc, the magnetic effects diminished and presented the dissymmetries of sign observed with the sectored disk. In these experiments the charge on the disks can be determined only by measuring the potential. These measurements showed that there was the same lack of proportionality between the potential and the magnetic effect observed.

2. A thin sheet of paraffined mica fixed on the surface of the disks produced the same effects as the caoutchouc.

3. In Pender's induction experiment we found that the presence of the mica diminished considerably the effects observed. For example, in one series of measurements we obtained the following deflexions :—

Disks bare, condensing-plates bare   . . . . 140 mm.
Disks covered with mica, condensing-plates bare 100 ,,
Disks and condensing-plates covered with mica   15 ,,

These facts show clearly the experimental cause of Crémieu's experiments, but they do not *explain* them. Indeed, the role of the dielectric seems difficult to unravel from any *a priori* considerations.

From the phenomena observed in 1900 and in our joint experiments it seems legitimate to conclude that, when solid dielectrics are submitted to considerable penetrations of charge, they act as if they suppressed the electrostatic influence between the conductors they separate. Moreover, for each kind of dielectric the penetration of charge and the suppression of electrostatic influence resulting therefrom occur at a certain well-marked critical voltage of the charged

conductor, being independent of the value of the electrostatic field produced by this voltage. For mica this critical value is in the neighbourhood of 4300 volts; for sulphur between 8000 and 12,000 volts; for ebonite, glass, and paraffin the critical voltage is above 12000 volts.

In the negative convection experiments the action of the charge on the metallic surfaces was rendered nil by the screening effect of the dielectric, or possibly by the neutralizing effect of the very considerable charges absorbed by the dielectric. But this is only a supposition,—only a detailed systematic study of these dielectric effects can clear up this very delicate point.

Study of the Amount of Charge carried round under various Conditions in the " Open Current " Experiments.— To complete our researches we undertook a study of the apparent amount of charge carried round in the open-current experiments, with the object, first, to make sure that the third fundamental condition laid down at the beginning of this article was realized, and, secondly, to clear up a little more the role of the dielectric.

The apparatus employed (fig. 5, p. 458) was the disk with an ebonite core, OE, carrying 18 peripheral sectors, M. These passed between the fixed sectors S and at the same time touched the brush A ; then, according to the direction of the rotation, they either became directly discharged by the brush C, or first passed between the supplementary sectors S_1 and then were discharged at C. The sectors S_1 formed with each sector M a capacity equal to that formed by the sectors S with M. These sectors S_1, connected to an electrometer, permitted us to see what takes place, of an electrostatic nature, in the air around the moving charged sectors. An experiment consisted in measuring galvanometrically the quantity of charge taken up by the sectors as they came under A, and the quantity of charge given up at C ; also the reading of the electrometer connected to S_1 was noted. The results were as follows :—

1. *Moving Sectors* M *bare; Fixed Sectors* S *bare.* — (*a*) Whatever was the sign of the charge, the *discharge-current* was almost invariably *superior* to the *charging-current*, the excess being variable, but in some cases as high as 15 per cent. This dissymmetry was the greater when the moving sectors were positive.

(*b*) On the sectors S_1 the following peculiarities were observed :—Suppose the disk in rotation in the direction of the arrow 2, the sectors S_1 connected to the electrometer, and all the rest of the apparatus to earth. No matter what was the velocity of rotation, no deflexion of the electrometer

was observed, as should be expected. However, when the
sectors S were charged so that the moving sectors passed
under S_1 carrying their charge, the electrometer immediately
took a sudden deflexion, *which gradually increased*, the
more rapidly the faster the rotation. The sign of the charge
on the electrometer was the same as that of the charge
on the moving sectors, while the charge on the sectors S_1
was of the opposite sign, as should be expected. However,
if the sectors S and M were then earthed, the deflexion of
the electrometer did not fall to zero, but indicated a residual
charge on the sectors S_1, which charge was of the *same* sign
as that on the moving sectors.

These facts seem to show that the rotation of the charged
sectors produced in the surrounding air electricity of the
same sign as the charge they carry. In accord with this
assumption is the above fact that the charge-current is in-
ferior to the discharge-current. In this connexion attention
may be called to the fact that the loss of charge from a
body in air is considerably less when the body is in rapid
movement than when the body is at rest. This fact was first
observed by Matteucci *, under most excellent conditions.

For the present, we merely state these facts without
attempting an interpretation. It is impossible to say whether
they should be attributed to the movement of the body itself,
or rather to sudden variations in the field caused by the
violent shocks given to the air by the rapid motion of the
sectors. It should be added here that we tried to detect a
possible production of charge in the surrounding air by
placing in the air between the charged disk and earthed
condensing-plate small metallic brushes connected to an
electrometer. The electrometer showed no deflexion when
the disk was set in rotation.

2. *Moving Sectors* M *bare, Fixed Sectors* S *covered with
Mica.*—We found the same dissymmetry between the charge
and discharge current as in the preceding case, and in the same
direction. Further, for voltages above 3000 volts on the
fixed sectors S, the charging-current did not increase pro-
portionally to the voltage on the fixed sectors. When this
voltage reached about 4500 volts, the charging-current
appeared to attain a maximum, which could not be exceeded
even for a very great increase of voltage.

Again, when the inducing sectors S were earthed after
having been submitted to a charge for several seconds, the
moving sectors continued to take up a charge as they passed
under S, as indicated by a current continuing to flow through

* *Ann. de Chimie et de Phys.* 3rd series, vol. xxviii. p. 385.

the galvanometer, but of the opposite sign to that previously taken, and this effect continued even for several hours thereafter with no appreciable diminution. These phenomena were extremely irregular, the size of the supplementary currents seeming to depend upon the previous state of the mica and the duration of the experiments.

At the sectors S_1 the same phenomena were observed as in the preceding case.

3. *Moving Sectors and Fixed Sectors covered with Mica.*— As long as the potential of the fixed sectors S was below 2000 volts no anomalies were observed. Above 2000 volts the charge-current became considerably greater than the discharge-current. This dissymmetry attained even as much as 30 per cent. In the neighbourhood of 4500 volts the proportionality between voltage and charge-current ceased, as in the preceding case. There were also supplementary currents after the charging sectors were earthed. These could attain as much as 50 per cent. of the value of the currents when the sectors were charged. But there was this peculiarity, that for the moving sectors charged positively they were in the opposite direction to the normal currents, while for the moving sectors charged negatively they were in the same direction as the normal currents. *The sectors* S_1 *took up no charge by influence.* Even more so than in the former case did all these irregularities depend upon the previous state of the dielectric and the duration of the experiments.

All this shows how complex the phenomena are, and that only a prolonged study can fully explain them. This much seems certain, that there is a penetration of charge in another sense from that usually understood by the expression. That which seems to prove this conclusively is that, in the third case above, the supplementary currents are obtained only when the two brushes A and C both make contact with the sectors. If either of the brushes is suppressed, there is no longer a supplementary current. If the supplementary currents are due merely to the ordinarily considered penetration of charge, this charge would gradually flow out through the brush left in contact, and therefore produce a current in the galvanometer, without the presence of the second brush being necessary. It seems to us that the mechanism of the supplementary currents is analogous to an electrophorus, the mica on the sectors being in some way electrolysed.

It is essential, in any case, from the point of view of the accepted theories, to know exactly why the dielectric diminishes or suppresses the magnetic effect, and what is the

nature of the charge after its penetration into the mass of the
dielectric.

Conclusions.—All that can be said at present of these acces-
sory phenomena is that they do not permit us to affirm with
certainty that the third fundamental condition of a correct
experiment on convection is exactly fulfilled, particularly
with sectored disks.   But the following conclusions are
legitimate :—

1. A charged continuous metallic disk turning in its own
plane opposite fixed condensing-plates carries its charge
with itself.

2. The *entrainement* of this charge produces a magnetic
field in the direction demanded by the assumption of a
magnetic effect due to electric convection, and in accord with
the calculated value to 10 per cent.

3. Charged sectors moving in their own plane, without the
presence of any condensing-plates, produce a magnetic effect
in the direction and of the proper size demanded by this
same assumption.

It is not for us to say whether these effects are really due
to electric convection in the sense in which Faraday and
Maxwell understood this expression, nor to decide whether
they are in accord with the fundamental hypotheses of the
accepted theories.

Physical Laboratory of the Sorbonne,
    Paris, April 15, 1903.

LIX. *On the Theory of Refraction in Gases.   By* GEORGE
W. WALKER, *M.A., A.R.C.Sc., Fellow of Trinity College,
Cambridge**.

[Plate XIX.]

THE problem involved in the explanation of the properties
    of refraction and dielectric susceptibility in gases, has
occupied my attention for several years.   The subject is of
extreme importance on account of the evidence which a
satisfactory theory could give of the constitution of the mole-
cules of a gas.

More than two years ago I suggested a theory of refraction
in gases†; but while the present theory is an extension of
the former, careful consideration of more experimental
evidence has led me to modify some of the views expressed
there.

Several theories in terms of the electromagnetic theory of
light have been proposed ; but while these explain some of

the facts they are only partial representations of the whole facts, and quite at variance with some features of the phenomena.

(1) *Summary of Experimental Evidence.*

Before proceeding to discuss these theories and to frame a new theory, it is desirable to summarize as briefly as possible the experimental facts in our possession.

First.—The Dielectric Constant of Gases.

Boltzmann[*] and Klemenčič[†] made measurements of the value of K for several gases at ordinary temperatures. They found that $K - 1$ was very nearly proportional to the pressure. They did not, however, measure the effect of temperature.

Baedecker [‡] has measured the effect of temperature on the value of K for some of the denser gases in which $(K - 1)$ is much greater than $(\mu^2 - 1)$; and I have shown[§] that his results for sulphur dioxide and ammonia agree extremely well with the law $K - 1 \propto \dfrac{P}{\theta^2}$, P being the pressure and θ the absolute temperature.

Second.—The Refractive Index of Gases.

Perhaps the most extensive measurements up to the present time have been made by Mascart[||], Ketteler[¶], and Lorenz[**]. Their results show that $\mu^2 - 1$ is very nearly proportional to the pressure.

Mascart is the only observer who measured the effect of temperature for a number of gases. He found that the effect of temperature was greater than that indicated by the theoretical law $\mu^2 - 1 \propto \dfrac{P}{\theta}$.

Recently the present writer [††] measured with great care the temperature effect on refraction in several gases, the range of temperature being much greater than that used by Mascart. The values obtained were in every case less than those obtained by Mascart, but they still show a divergence from the law $\mu^2 - 1 \propto \dfrac{P}{\theta}$. In the case of hydrogen $\mu^2 - 1$ diminished less with increasing temperature than the theoretical law indicates.

* *Annalen der Physik*, clv. p. 403 (1875).
† *Wien. Berichte*, xci. p. 712 (1885).
‡ *Zeitschrift Phys. Chem.* vol. xxxvi. p. 305. § *L. c. ante.*
|| *Annales de l'école normale*, vi. p. 9 (1877).
¶ *Annalen der Physik* (Pogg.), cxxiv. (1865).
** *Annalen der Physik*, xi. (1880).
†† Phil. Trans. cci. p. 435 (1903).

The observers already mentioned measured the amount of dispersion, and in every case the refractive index increased with the frequency. They found no variation in the dispersive power with temperature ; but as the whole amount of dispersion in the visible spectrum is very small and not much in excess of possible experimental errors, it is doubtful whether this can be taken as more than an approximate statement of the truth.

It appears that the presence of a spectral line only produces a measurable effect on the refractive index at points in the immediate vicinity of it, and the experiments show that dispersion is controlled by something of a general nature and common to all gases.

It is well known that all gases have intense absorption of ultra-violet light*, this fact being intimately connected with ionization. Again, there is evidence that several, if not all, gases have absorption in the infra-red †.

We have next to note that while for some gases the value of $\mu^2 - 1$ agrees with $K - 1$, this does not hold for all. In some of the denser gases, such as SO_2 and NH_3, the value of $K - 1$ is much greater than the value of $\mu^2 - 1$. I have already referred to the fact that in such cases the temperature coefficients are very different.

The deviation from the law $\mu^2 - 1 \propto \dfrac{P}{\theta}$, as regards pressure,

is extremely small, and has been satisfactorily explained by Mascart as due to the deviation from Boyle's law.

(2) *Demands on a Correct Theory.*

It appears that a satisfactory theory of refraction must account for the following general features :—

(A) That the refractive index very nearly obeys the Gladstone and Dale Law $\mu^2 - 1 \propto \rho$, where ρ is the density, with small variations positive or negative as regards temperature.

(B) That the refractive index increases regularly with the frequency, and that the dispersion is substantially independent of temperature.

(C) That there is absorption in the ultra-violet, and possibly also in the infra-red.

(D) That in some cases the value of $K - 1$ may be much greater than the values of $\mu^2 - 1$ in the visible spectrum, and that in such cases the law $K - 1 \propto \rho/\theta$ should be nearly true.

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* Mascart, *Traité d'optique*, t. iii. p. 373.

† Langley, Phil. Mag. ii. p. 119 (1901) ; also Rubens and Aschkinass, *Annalen der Physik*, lxiv. p. 584 (1898).

(3) *Present Theories.*

The theory first given by Lorentz[*], and the more general theory of Voigt[†], both depend on the hypothesis that the molecule may be regarded as composed of a greater or less number of electrical doublets, and includes the view that an atom may be regarded as consisting of a positively charged particle and a large number of comparatively small negatively charged particles, there being no charge on the whole.

The doublets are supposed to be capable of vibrating in a number of definite periods, which correspond to spectral lines. The periods must therefore be regarded as independent of temperature.

The result readily follows that

$$\mu^2 - 1 = k\rho f(p),$$

where k is a constant depending on the gas, ρ is the density, and $f(p)$ is a function of the frequency of the waves and of the free periods of vibration of the molecule.

This view is therefore capable of explaining the general features which I have called (B) and (C), but cannot be made to explain (A) and (D).

Before passing to a modified theory, it may be pointed out that a formula of the preceding form may be obtained by regarding the molecules simply as obstacles which differ in their properties from the surrounding æther. If we take the formulæ for the waves scattered from a spherical obstacle[‡], and use Lord Rayleigh's method[§], we readily obtain

$$\mu^2 - 1 = 3N \cdot T \frac{\{4^2\pi^2 k^2 + p^2(2K_0 + K_1)(K_1 - K_0)\}}{\{4^2\pi^2 k^2 + p^2(2K_0 + K_1)^2\}},$$

where N is the number of particles per unit volume,

T	„	volume of a particle,
K_0	„	dielectric constant for the æther,
K_1	„	„ „ particle,
k	„	electrical conductivity of the particle,
$p/2\pi$	„	frequency of the waves.

A like result holds for particles of any shape, provided the dimensions of the particle are small compared with a wavelength.

* *Annalen der Physik,* ix. p. 641 (1880) ; also ' *Théorie Elect. de Maxwell.'*
† *Ibid.* lxvii. p. 345 (1899).
‡ Quart. Journ. Math. No. 121, p. 36 (1899).
§ Phil. Mag. (5) xlvii. p. 375 (1899).

For silver, which is the best conductor we know,

$$k \text{ is of order } 10^{-3}; \quad \therefore \ k^2 \text{ is of order } 10^{-6}.$$

For the Na line,

$$p \text{ is of order } 10^{15},$$

$$K_0 \text{ or } K_1 \text{ is of order } 10^{-21}.$$

Hence $\qquad p^2 K_0{}^2 \text{ is of order } 10^{-12}.$

Hence, if the particles conduct as well as silver, it is clear that the coefficient of $3N \cdot T$ preserves the value unity from $p = 0$ to far beyond the violet.

I think that we may conclude that the term is capable of accounting for refraction following Gladstone and Dale's law; but, on account of the extremely small rate of change with frequency, it is incapable of practically accounting for dispersion.

(4) *Proposed Theory.*

I propose now to modify the supposition that the motion of the molecule in those coordinates which are affected by electrical forces is independent of temperature. It will conduce to clearness if we select the simplest molecule capable of giving the effects we require, and afterwards consider a more complicated molecule.

I therefore select a molecule which consists of a particle of mass m_1 with a positive charge e and a particle of mass m_2 with a negative charge $-e$, and suppose that the force between them is the ordinary electrostatic attraction.

It is necessary to suppose that the two particles are closely associated, because very large electrical forces are required to produce ionization.

Since the particles are supposed to be of finite size, there will be a minimum distance within which the particles cannot approach. Let us call this distance between the centres r. We shall therefore suppose that the two particles describe circles about their common centre of inertia, and do not vibrate radially. There will be a pressure between the particles equal to the difference between the electric attraction and the centrifugal force. It is clear that when such a molecule is under the influence of external electrical forces, the effective control against those forces is the rotational energy of the molecule, which plays the part of potential energy. On averaging for all the molecules,

we shall thus be concerned with the mean rotational energy; and this, there is considerable reason to believe, is proportional to the absolute temperature. We have to note, however, that the two particles can remain in the above state only as long as the pressure between the particles does not vanish. As soon as the angular velocity attains the value Ω given by

$$\frac{m_1 m_2}{m_1 + m_2} \Omega r = \frac{e^2 V_0^2}{r^2},$$

the particles must separate. For higher values of the rotational energy we shall have elliptic or even hyperbolic or parabolic orbits.

In the gas a molecule may by collision attain a value of rotational energy which is too great for the particles to remain in contact. We must therefore have a small proportion of dissociated molecules on average—small, because of the comparatively high values of rotational energy required.

Further, the period of any elliptic orbits must be longer than the period $\frac{2\pi}{\Omega}$. The period thus defined is quite independent of temperature, and it seems to me ought to be identified with a spectral line, there being considerable evidence that luminosity in a gas is always associated with ionization.

We have now to calculate the effect of electrical waves on the motion of the molecule.

Let the coordinates of the centre of m_1 be x_1, y_1, z_1; and ,, ,, ,, m_2 be x_2, y_2, z_2; and let r be the distance between their centres.

Further, let the incident plane waves be represented by

Electrical force, $X = X_0 \cos (pt - kz)$,

Magnetic force, $M = \dfrac{k}{p} X_0 \cos (pt - kz)$, $\quad \dfrac{p}{k} = V_0$.

We shall suppose the motion to be only slightly disturbed from what it would be in the absence of waves. The linear velocities of the c.g. of the molecule are small compared with V_0, and r is small compared with the wavelength. Neglecting squares of these ratios, the equations of motion are

$$m_1 \ddot{x}_1 - \mathrm{P}\frac{(x_1 - x_2)}{r} = e\mathrm{X}_0 \cos pt,$$

$$m_2 \ddot{x}_2 + \mathrm{P}\frac{(x_1 - x_2)}{r} = -e\mathrm{X}_0 \cos pt,$$

$$m_1 \ddot{y}_1 - \mathrm{P}\frac{(y_1 - y_2)}{r} = m_2 \ddot{y}_2 + \mathrm{P}\frac{(y_1 - y_2)}{r} = 0,$$

$$m_1 \ddot{z}_1 - \mathrm{P}\frac{(z_1 - z_2)}{r} = m_2 \ddot{z}_2 + \mathrm{P}\frac{(z_1 - z_2)}{r} = 0,$$

where $\left(\mathrm{P} + \dfrac{e^2 \mathrm{V}_0^2}{r^2}\right)$ is the pressure between the two particles.

We thus get

$$m_1 \dot{x}_1 + m_2 \dot{x}_2 = \text{const.},$$
$$m_1 \dot{y}_1 + m_2 \dot{y}_2 = \text{const.},$$
$$m_1 \dot{z}_1 + m_2 \dot{z}_2 = \text{const.};$$

so that the motion of the c.g. is independent of the waves.

For the rotational motion we get

$$\ddot{\xi} - \mathrm{P}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\xi}{r} = e\mathrm{X}_0\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\cos pt,$$

$$\ddot{\eta} - \mathrm{P}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\eta}{r} = 0,$$

$$\ddot{\zeta} - \mathrm{P}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\zeta}{r} = 0,$$

where

$$x_1 - x_2 = \xi, \quad y_1 - y_2 = \eta, \quad z_1 - z_2 = \zeta,$$

and

$$\xi^2 + \eta^2 + \zeta^2 = r^2.$$

Now r is constant.

Hence

$$\xi\dot{\xi} + \eta\dot{\eta} + \zeta\dot{\zeta} = 0$$

and

$$\xi\ddot{\xi} + \eta\ddot{\eta} + \zeta\ddot{\zeta} + \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = 0.$$

Thus we get

$$\dot{\xi}\ddot{\xi} + \dot{\eta}\ddot{\eta} + \dot{\zeta}\ddot{\zeta} = e\mathrm{X}_0\left(\frac{1}{m} + \frac{1}{m_2}\right)\cos pt\,\dot{\xi}.$$

$$\therefore \quad \tfrac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) = \tfrac{1}{2}r^2(\dot{\vartheta}^2 + \sin^2\!\vartheta\,\dot{\phi}^2)$$

$$= \text{const} + \int e\mathrm{X}_0\!\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\cos pt\,\dot{\xi}\,dt$$

$$= \tfrac{1}{2}r^2\dot{\psi}^2, \text{ say,}$$

where ϑ and ϕ are the angular coordinates.

In the undisturbed motion we have

$$\xi = \alpha\cos(\omega t - \epsilon_1), \quad \eta = \beta\cos(\omega t - \epsilon_2), \quad \zeta = \gamma\cos(\omega t - \epsilon_3),$$

where

$$\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = \text{const.} = \omega^2 r^2 = r^2(\dot{\vartheta}^2 + \sin^2\!\vartheta\,\dot{\phi}^2);$$

and hence, neglecting squares of X_0, we get

$$\tfrac{1}{2}r^2(\dot{\vartheta}^2 + \sin^2\!\vartheta\,\dot{\phi}^2) + e\mathrm{X}_0\!\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\!\left\{\frac{p\dot{\xi}\sin pt}{\dot{\psi}^2 - p^2} + \frac{\dot{\psi}^2\xi\cos pt}{p^2 - \dot{\psi}^2}\right\} = \text{const.} = \mathrm{E}$$

in the disturbed motion.

In the undisturbed motion the usual Boltzmann-Maxwell law of the distribution of velocities gives us

$$\epsilon^{-\frac{\omega^2}{\bar{\omega}^2}}\,\omega\,d\omega,$$

where $\bar{\omega}^2$ is the mean value of ω^2.

Now in the disturbed motion E is independent of time; so that with the same restrictions as in the ordinary Boltzmann-Maxwell law we have

$$\dot{\psi}\,\delta\dot{\psi}\,\epsilon^{-\frac{1}{\bar{\omega}^2}\left\{\dot{\psi}^2 - \frac{2e\mathrm{X}_0\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}{r^2}\left(\frac{p\dot{\xi}\sin pt}{p^2 - \dot{\psi}^2} - \frac{\dot{\psi}^2\xi\cos pt}{p^2 - \dot{\psi}^2}\right)\right\}}$$

as a permanent distribution-law independent of time. Although t is involved explicitly, $\dfrac{d\mathrm{E}}{dt} = 0$.

Neglecting squares of X_0, we may write this

$$\dot{\psi}\,\delta\dot{\psi}\,\epsilon^{-\frac{\dot{\psi}^2}{\bar{\omega}^2}}\left\{1 + \frac{2e\mathrm{X}_0\!\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}{r^2\bar{\omega}^2}\!\left(\frac{p\dot{\xi}\sin pt}{p^2 - \dot{\psi}^2} - \frac{\dot{\psi}^2\xi\cos pt}{p^2 - \dot{\psi}^2}\right)\right\}$$

This law leads, as we might expect, to periodic orientation of the axes of the molecules. It appears to me that the validity is not affected by the rapidity of the waves. The above distribution having once obtained, continues to represent the distribution, since E is constant throughout the

free path of a molecule ; and, as can be proved, the multiple
product of the differentials of coordinates and velocities is
also constant. The only point which we must take account
of is that the time over which we observe the effects produced
should be large compared with the time it takes for a
disturbance from this state to subside.

We have now to find the sum of $\dot{\xi}$ for all the molecules.

$$\Sigma\dot{\xi} = \Sigma\dot{\psi}\,\delta\dot{\psi}\,\epsilon^{-\frac{\dot{\psi}^2}{\bar{\omega}^2}}\left\{\frac{2e\mathrm{X}_0\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}{r^2\bar{\omega}^2}\frac{p\dot{\xi}^2\sin pt}{p^2-\dot{\psi}^2}\right\},$$

since the sum of the other terms vanishes on summation for
all the molecules.

Also

$$\Sigma\dot{\xi}^2 = \Sigma\dot{\eta}^2 = \Sigma\dot{\zeta}^2 = \tfrac{1}{3}\Sigma r^2\dot{\psi}^2.$$

Hence, neglecting squares of X_0, we find that

$$\Sigma\dot{\xi} = \mathrm{X}_0\,p\sin pt\,\frac{2e\mathrm{N}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}{3\bar{\omega}^2}\frac{\displaystyle\int\frac{\dot{\psi}^2\epsilon^{-\frac{\dot{\psi}^2}{\bar{\omega}^2}}}{p^2-\dot{\psi}^2}d\dot{\psi}^2}{\displaystyle\int\epsilon^{-\frac{\dot{\psi}^2}{\bar{\omega}^2}}d\dot{\psi}^2}.$$

We are at liberty to integrate from 0 up to the critical
value Ω at which the molecule splits up.

We have

$$\int_0^{\Omega^2}\epsilon^{-\frac{\dot{\psi}^2}{\bar{\omega}^2}}d\dot{\psi}^2 = \bar{\omega}^2\left\{1-\epsilon^{-\frac{\Omega^2}{\bar{\omega}^2}}\right\}.$$

Now $\dfrac{\Omega^2}{\bar{\omega}^2}$ is a comparatively large number, and hence
$\epsilon^{-\frac{\Omega^2}{\bar{\omega}^2}}$ is practically zero. $\bar{\omega}^2$ is thus the mean value of ω^2,
and is proportional to the absolute temperature.

Hence we get

$$\Sigma\dot{\xi} = \mathrm{X}_0\,p\sin pt\,\frac{2e\mathrm{N}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}{3\bar{\omega}^2}\frac{\displaystyle\int_0^{\Omega^2}\frac{x\epsilon^{-\frac{x}{\bar{\omega}^2}}}{x-p^2}\frac{dx}{\bar{\omega}^2}}{\left(1-\epsilon^{-\frac{\Omega^2}{\bar{\omega}^2}}\right)}.$$

We are now in a position to calculate the contribution to
refraction.

The equation of propagation of the waves is

$$\frac{1}{V_0{}^2}\frac{\partial X}{\partial t} = -\frac{\partial M}{\partial z}-4\pi\Sigma e\dot{x}$$

$$= -\frac{\partial M}{\partial z}-4\pi e\dot{\Sigma}\dot{\xi}.$$

Hence we get

$$\mu^2-1 = \frac{8\pi}{3}\,e^2 V_0{}^2\frac{N\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}{\overline{\omega}^2}\int_0^{\Omega^2}\frac{x}{x-p^2}\epsilon^{-\frac{x}{\overline{\omega}^2}}\frac{dx}{\overline{\omega}^2}\cdot\frac{1}{1-\epsilon^{-\frac{\Omega^2}{\overline{\omega}^2}}}.$$

Now

$$\int_0^{\Omega^2}\frac{x}{x-p^2}\,\epsilon^{-\frac{x}{\overline{\omega}^2}}\frac{dx}{\overline{\omega}^2} = \left(1-\epsilon^{-\frac{\Omega^2}{\overline{\omega}^2}}\right)+\frac{p^2}{\overline{\omega}^2}\int_0^{\Omega^2}\frac{\epsilon^{-\frac{x}{\overline{\omega}^2}}dx}{x-p^2}.$$

Hence, taking the principal value of the integral, we get

$$\mu^2-1=\frac{8\pi}{3}\frac{e^2 V_0{}^2 N}{\overline{\omega}^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\frac{\left\{1-\epsilon^{-\frac{\Omega^2}{\overline{\omega}^2}}-\frac{p^2}{\overline{\omega}^2}\epsilon^{-\frac{p^2}{\overline{\omega}^2}}\,\mathrm{E}_i\,\frac{p^2}{\overline{\omega}^2}+\frac{p^2}{\overline{\omega}^2}\epsilon^{-\frac{p^2}{\overline{\omega}^2}}\,\mathrm{E}_i\,\frac{p^2-\Omega^2}{\overline{\omega}^2}\right\}}{1-\epsilon^{-\frac{\Omega^2}{\overline{\omega}^2}}}$$

where

$$\int_0^{\infty}\frac{e-z\,dz}{z-x} = -e^{-x}\mathrm{E}_i(x)$$

and

$$\mathrm{E}i(x)=\gamma+\tfrac{1}{4}\log x^4+\Sigma_1^{\infty}\frac{x^n}{n\,.\,n\,!},$$

and γ is Euler's constant.

(5) *Digression on the Functions involved.*

The function $\mathrm{E}_i(x)$ involved has been discussed in Lamb's 'Hydrodynamics,' p. 397, and by Schlömilch [*]. Recently, Mr. E. W. Barnes [†] has discussed the form of the asymptotic expansion; and I am indebted to him for considerable assistance in examining the general form.

I found it necessary to calculate the numerical value of several functions involving $\mathrm{E}_i(x)$. The results are given in the following Table, in which $\mathrm{E}(x)$ stands for $x\epsilon^{-x}\mathrm{E}_i(x)$.

[*] Crelle's Journal, t. xxxiii. p. 316.
[†] Phil. Trans. cxcix. p. 447 (1902).

x	$E_i(x)$	x_ϵ	$E(x)$	$1-E(x)$	$x(1-E(x))$	$x_\epsilon^{-x}E_i(x-5)+\epsilon^{-5}$	$x^2(1-Ex)+xEx$
$-\infty$	0	0	$+1$	0	-1	0	-1
-10	$\vdots$	$+\cdot000045$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-5	$\vdots$	$+\cdot006738$					
-2	$-\cdot00075$	$+\cdot135331$					
-1	$-\cdot048186$	$\cdot367989$					
$-\cdot5$	$-\cdot219384$	$\cdot636987$					
$-\cdot2$	$-\cdot559774$	$\cdot826140$					
$-\cdot1$	$-1\cdot222651$	$\cdot904838$					
$-\cdot01$	$-1\cdot822924$	$\cdot990049$					
$-\cdot001$	$-4\cdot037930$	$\cdot999000$					
-0	$-6\cdot331540$						
0	$-\infty$	$+1$	0	1	0	0	0
$+\cdot001$	$6\cdot329540$	$1\cdot00100$	$-\cdot00632$	$+1\cdot00632$	$+\cdot001006$	$\vdots$	$+\cdot000293$
$+\cdot01$	$4\cdot017930$	$1\cdot01005$	$-\cdot03977$	$+1\cdot03977$	$+\cdot010397$	$\vdots$	$+\cdot003215$
$+\cdot1$	$1\cdot622814$	$1\cdot105170$	$-\cdot14683$	$+1\cdot14683$	$+\cdot114683$	$\vdots$	$+\cdot018276$
$+\cdot2$	$\cdot821763$	$1\cdot21046$	$-\cdot13577$	$+1\cdot13577$	$+\cdot227155$	$\vdots$	$+\cdot28622$
$+\cdot5$	$\cdot454218$	$1\cdot570203$	$+\cdot14488$	$-\cdot85512$	$+\cdot427660$	$\vdots$	$+1\cdot3182$
$+1$	$1\cdot895114$	$2\cdot718281$	$+\cdot69737$	$-\cdot30263$	$+\cdot30263$	$\vdots$	$+\cdot09732$
$+2$	$4\cdot95423$	$7\cdot38924$	$+1\cdot34090$	$-\cdot34090$	$-\cdot68180$	$\vdots$	$+1\cdot25516$
$+3$	$9\cdot93383$	$20\cdot0144$	$+1\cdot48372$	$-\cdot48372$	$-1\cdot45116$	$-\cdot007221$	$-2\cdot07660$
$+4$	$19\cdot63087$	$54\cdot604$	$+1\cdot43793$	$-\cdot43793$	$-1\cdot75172$	$-\cdot016068$	
$+4\cdot5$	$\vdots$	$\vdots$				$-\cdot07243$	
$+5$	$40\cdot18527$	$148\cdot41$	$+1\cdot35383$	$\cdot35383$	$-1\cdot76915$	$-\infty$	$-2\cdot3616$
$+5\cdot2$	$\vdots$	$\vdots$				$-\cdot03485$	
$+5\cdot5$	$\vdots$	$\vdots$				$+\cdot02643$	
$+6$	$85\cdot9897$	$403\cdot44$	$+1\cdot27872$	$\cdot27872$	$-1\cdot67232$	$+\cdot02831$	$-2\cdot1814$
$+7$	$\vdots$	$\vdots$				$+\cdot03173$	
$+8$	$440\cdot3798$	$2980\cdot9$	$+1\cdot18183$	$\cdot18183$	$-1\cdot45464$	$+\cdot02665$	$-1\cdot8232$
$+9$	$\vdots$	$\vdots$				$+\cdot02179$	
$+10$	$2492\cdot228$	22026	$+1\cdot13146$	$\cdot13146$	$-1\cdot3146$	$+\cdot01824$	
$+11$	$\vdots$	$\vdots$				$+\cdot01585$	
$+15$	$\vdots$	$\vdots$				$+\cdot01143$	
$+\infty$	$+\infty$	$+\infty$	$+1$	0	-1	0	-1

The curves required for the discussion are shown in the diagrams (Pl. XIX.), in which the abscissæ represent values of $\dfrac{p}{\omega}$.

We may first note that $\mathrm{E}_i(x)$ is infinite for $x=0$, but since $\underset{x=0}{\mathrm{Lt}}\, x \log x$ is zero, the function $\mathrm{E}(x)$ is finite for $x=0$. It is also finite for all positive values of x, converging to 1 when $x=\infty$.

In the function

$$\frac{p^2}{\omega^2}\,\epsilon^{-\frac{p^2}{\omega^2}}\,\mathrm{E}_i\left(\frac{p^2-\Omega^2}{\omega^2}\right)$$

which is drawn for $\dfrac{\Omega^2}{\omega^2}=5$, it will be observed that the infinity at $p=\Omega$ is of extraordinary sharpness, and for larger values of $\dfrac{\Omega^2}{\omega^2}$ the sharpness would be still more marked. We have already pointed out that $\dfrac{\Omega^2}{\omega^2}$ must be a large number. Thus μ^2-1 is adequately represented numerically by

$$\frac{8\pi}{3}\frac{e^2\mathrm{V}_0^2\mathrm{N}}{\omega^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left\{1-\mathrm{E}\left(\frac{p^2}{\omega^2}\right)\right\}$$

except in the immediate vicinity of the point $p=\Omega$. Just at this point we should get negative values for μ^2, or in other words absorption must occur.

We may now justify the selection of the principal value of the integral. In any physical problem the denominators are prevented from vanishing by frictional terms in the equations of motion. In this case radiation produces the result. Thus, instead of the integral

$$\int_0^\infty \frac{e^{-\frac{x}{\omega^2}}\,dx}{x-p^2},$$

we should get

$$\int_0^\infty \frac{(x-p^2)\epsilon^{-\frac{x}{\omega^2}}}{(x-p^2)^2+b^2}\,dx$$

where b is a small quantity depending on radiation. By Lorentz's method we find that

$$b=\frac{e^2}{3}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\frac{p^3}{\mathrm{V}_0}.$$

If we put

$$S_1 = \gamma + \log \frac{(p^4+b^2)^{\frac{1}{2}}}{\overline{\omega}^2} + \Sigma_1^\infty \frac{(p^4+b^2)^{\frac{n}{2}} \cos n\vartheta}{\overline{\omega}^{2n}} \frac{\cos n\vartheta}{n \cdot n!}$$

$$S_2 = \vartheta + \Sigma_1^\infty \frac{(p^4+b^2)^{\frac{n}{2}} \sin n\vartheta}{\overline{\omega}^{2n}} \frac{\sin n\vartheta}{n \cdot n!} ,$$

where $\tan \vartheta = \dfrac{b}{p^2}$, we find that

$$\int_0^\infty \frac{\epsilon^{-\frac{x}{\overline{\omega}^2}}(x-p^2)}{(x-p^2)^2+b^2} dx = -\epsilon^{-\frac{p^2}{\overline{\omega}^2}} \left\{ S_1 \cos \frac{b}{\overline{\omega}^2} + S_2 \sin \frac{b}{\overline{\omega}^2} - \pi \sin \frac{b}{\overline{\omega}^2} \right\}.$$

Now for Na light b is of order 10^{21}, while p^2 is of order 10^{30}.

We shall find later that $\overline{\omega}^2$ is of about the same order as p^2. Hence with great numerical accuracy to far beyond the violet we may conclude that

$$\int_0^\infty \frac{\epsilon^{-\frac{x}{\overline{\omega}^2}}(x-p^2)}{(x-p^2)^2+b^2} dx = -\epsilon^{-\frac{p^2}{\overline{\omega}^2}} E_i\left(\frac{p^2}{\overline{\omega}^2}\right).$$

(6) *Generalization.*

We have now seen that the molecules of the type selected contribute to refraction a term of which an adequate representation is

$$\mu^2 - 1 = \frac{8\pi}{3} \frac{e^2 V_0^2 N}{\overline{\omega}^2} \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \left\{ 1 - E\left(\frac{p^2}{\overline{\omega}^2}\right) \right\},$$

except in the immediate vicinity of a certain point $p = \Omega$. For this point we require the additional term which has already been discussed.

Now the real molecule or atom is more complicated than that which we selected. Instead of one small negative particle we have a great many. Each of these will contribute a term of the above form along with a term which is numerically insignificant except in the immediate vicinity of a certain point. If these particles are all similar the $\overline{\omega}^2$ will be the same for all, but it does not follow that the critical values at which they are thrown off from the parent atom is the same for all. I conclude that for the real atom we shall have the contribution

$$\mu^2 - 1 = \frac{\kappa\rho}{\overline{\omega}^2} \left\{ 1 - E\left(\frac{p^2}{\overline{\omega}^2}\right) \right\}$$

along with a number of terms each of which is insignificant except at a certain point. These points I identify with the spectral lines ; and this view seems preferable to the view that spectral lines correspond to free periods of vibration of the atom ; for if there are free periods in coordinates not capable of being affected by temperature, it is difficult to see how radiation could be kept up, or become appreciable as the temperature is raised. On the present view this difficulty is removed. We have motion in coordinates which can be affected by temperature; nevertheless the frequencies at which ionization occurs are independent of temperature, while the amount of ionization increases with the temperature ; and this is in agreement with the electrical and optical experiments.

We have already seen that merely as obstacles the molecules give refraction of the form $\mu^2 - 1 \propto \rho$. If then we combine this with the refraction produced by orientation in the molecule, we get as the general form

$$\mu^2 - 1 = k_1\rho + \frac{k_2\rho}{\theta}\left\{1 - \mathrm{E}\left(\frac{p^2}{\bar{\omega}^2}\right)\right\}$$

where $\bar{\omega}^2$ is proportional to θ, and k_1 and k_2 are constants depending on the molecule. I would again point out that at certain points additional terms must be introduced.

(7) *General Application of the Formula.*

We have now to examine whether the formula

$$\mu^2 - 1 = k_1\,\rho + k_2\,\rho/\theta\left\{1 - \mathrm{E}\left(\frac{p^2}{\omega^2}\right)\right\}$$

is capable of explaining the general features which were summarized earlier.

For $p = 0$ the formula becomes

$$\mu^2 - 1 = \mathrm{K} - 1 = k_1\,\rho + k_2\,\rho/\theta.$$

If the first term is the greater we shall have approximately $\mathrm{K} - 1 \propto \rho$, but if the second term is greater $\mathrm{K} - 1$ will vary more nearly proportional to ρ/θ. We thus have a means of explaining Baedecker's results for denser gases.

Referring to the curve for $\left\{1 - \mathrm{E}\left(\frac{p^2}{\bar{\omega}^2}\right)\right\}$ we see that refraction increasing with the frequency may be explained by the first portion of the curve. In order to meet the fact that $\mu^2 - 1$ is very nearly proportional to ρ we require that $\frac{k_2}{k_1\theta}$ should be small. Further, the values of $\mu^2 - 1$ must be

greater than the value of K -1. Again, since $\left(1-\mathrm{E}\dfrac{p^2}{\omega^2}\right)$ diminishes as ω^2 increases we require that $\dfrac{\mu^2-1}{\rho}$ should diminish with increasing temperature. We also require that $\dfrac{p^2}{\omega^2}$ should be less than one. The first portion of the curve has thus comparatively limited power of explaining the actual facts, and the conclusion that p^2 should be less than ω^2 is highly improbable.

Turning now to the portion of the curve beyond A, we see that refraction increasing with the frequency may be explained. There the values of $\dfrac{p}{\omega}$ are greater than unity, and the function $\left(1-\mathrm{E}\dfrac{p^2}{\omega^2}\right)$ is negative.

To explain a refractive index greater than unity we require that $k_1\rho$ should be greater than $\dfrac{k_2\rho}{\theta}\left\{1-\mathrm{E}\dfrac{p^2}{\omega^2}\right\}$ in the spectrum. Now this can be secured even when $\dfrac{k_2\rho}{\theta}$ is greater than $k_1\rho$ by taking $\dfrac{p}{\omega}$ sufficiently large. Thus we may have $\dfrac{k_2\rho}{\theta}$ as the important term in K -1, but $k_1\rho$ will be the important term in the visible spectrum.

Writing the formula

$$\mu^2-1=k_1\rho\left\{1+\dfrac{k_2}{k_1}\dfrac{\overline{\omega}^2}{\theta\overline{p}^2}\cdot\dfrac{p^2}{\omega^2}\left(1-\mathrm{E}\dfrac{p^2}{\omega^2}\right)\right\},$$

we see that the variation with temperature will be greater or less than that indicated by Gladstone and Dale's law, according as the coefficient of $k_1\rho$ diminishes or increases with temperature. That is according as $\dfrac{p^2}{\omega^2}\left(1-\mathrm{E}\dfrac{p^2}{\omega^2}\right)$ diminishes or increases in numerical value as $\dfrac{p^2}{\omega^2}$ increases. Referring to the curves we see that the minimum point marked B on the curve $\dfrac{p^2}{\omega^2}\left(1-\mathrm{E}\dfrac{p^2}{\omega^2}\right)$ occurs for a value of $\dfrac{p}{\omega}$ greater than that for the point A on the curve $\left(1-\mathrm{E}\dfrac{p^2}{\omega^2}\right)$. Hence, for values of $\dfrac{p}{\omega}$ between those corresponding to the

points A and B we get ordinary refraction with a temperature-coefficient less than, and for values of $\frac{p}{\omega}$ beyond B a temperature-coefficient greater than, Gladstone and Dale's law indicates. We note that at the point B the variation is zero, and again for large values of $\frac{p}{\omega}$ the variation becomes very small.

The manner in which $\frac{1}{\theta}\left\{1-\mathrm{E}\frac{p^2}{\omega^2}\right\}$ becomes largely independent of temperature calls for some explanation. It is due to the fact that for large values of $\frac{p}{\omega}$ the asymptotic expansion for $\left(1-\mathrm{E}\frac{p^2}{\omega^2}\right)$ is

$$\frac{1\,!\;\bar{\omega}^2}{p^2}+\frac{2\,!\;\bar{\omega}^4}{p^4}+\frac{3\,!\;\bar{\omega}^6}{p^6}+\;\;.\;\;.\;\;.$$

Since ω^2 is proportional to θ, we see that for large values of $\frac{p^2}{\omega^2}$ the expression $\frac{1}{\theta}\left\{1-\mathrm{E}\frac{p^2}{\omega^2}\right\}$ is to a great extent independent of temperature, while for small values of $\frac{p^2}{\omega^2}$ the expression varies nearly as $\frac{1}{\theta}$.

Let us now consider whether the dispersion indicated by the formula is dependent on temperature. The change of μ for a given wave-length was measured by counting the number of interference-bands displaced for a known difference of pressure, and the results of Mascart and Lorenz were that the ratio of the number of bands displaced for two given wave-lengths was the same at all temperatures.

Denoting the frequencies by p and p', the experiments, so far as they can be trusted, assert that $\frac{\mu_p-1}{\mu_{p'}-1}$ is the same at all temperatures between say $0°$ and $100°$ C.

Now, since $\mu-1$ is small, we have

$$\frac{\mu_p-1}{\mu_{p'}-1}=\frac{1+\dfrac{k_2}{k_1}\dfrac{1}{\theta}\left\{1-\mathrm{E}\dfrac{p^2}{\omega^2}\right\}}{1+\dfrac{k_2}{k_1}\dfrac{1}{\theta}\left\{1-\mathrm{E}\dfrac{p'^2}{\omega^2}\right\}}.$$

Now we have just proved that for moderately large values of $\frac{p^2}{\omega^2}$, $\frac{1}{\theta}\left\{1-\mathrm{E}\frac{p^2}{\omega^2}\right\}$ varies little with temperature, in

agreement with the experimental fact that $\dfrac{\mu^2 - 1}{\rho}$ varies by only a small amount when the temperature is increased from $0°$ to $100°$ C.

We must thus conclude that $\dfrac{\mu_p - 1}{\mu_{p'} - 1}$ will not alter by any measurable amount as the temperature changes within ordinary limits.

As an approximation, when $\dfrac{p^2}{\bar{\omega}^2}$ is large we get

$$\frac{\mu_p - 1}{\mu_{p'} - 1} = \frac{1 - \dfrac{k_2}{k_1 \theta} \dfrac{\bar{\omega}^2}{p^2}}{1 - \dfrac{k_2}{k_1 \theta} \dfrac{\bar{\omega}^2}{p'^2}},$$

and this is independent of temperature since $\bar{\omega}^2 \propto \theta$.

If, however, we were very near the point B the approximation would not be valid, but I doubt whether the present method of measuring dispersion in gases is sufficiently accurate to reveal the variation with temperature.

We have seen that values of μ greater than unity are required in the visible spectrum, but we may have values of μ less than unity in the infra-red. This must occur when $\dfrac{k_2}{k_1 \theta}$ is large, that is in cases where the value of $K - 1$ is much greater than the values of $\mu^2 - 1$ in the visible spectrum. This has an important bearing on Langley's experiments on the infra-red spectrum through the atmosphere. For suppose the sun to be low on the horizon but still visible, then if there is any constituent of the atmosphere for which $K - 1$ is much greater than $\mu^2 - 1$, infra-red rays in a certain range of the spectrum will be bent upwards and not downwards. They will suffer total reflexion at some point above the surface of the earth, and thus apparent absorption will occur. Water-vapour is one of those gases for which $K - 1$ is much greater than $\mu^2 - 1$, and it is always present in the atmosphere. The locality of the absorption in the infra-red must depend to a large extent on temperature.

The theory which has been proposed is thus capable of explaining qualitatively all the main features with which we are acquainted. It now remains to test whether the known facts can be explained quantitatively.

(8) *Experimental Verification.*

We shall suppose that the density of the gas is given by

$$\rho/\rho_0 = \frac{P}{(1+\alpha t)76} \left\{ 1 + \frac{\lambda_0 P}{76(1+\alpha t)} \right\},$$

where P is the pressure in cms. of mercury, t is the temperature centigrade, α the ordinary coefficient of expansion, and λ_0 a constant. The factor in brackets I shall call the compressibility term.

In my paper on refractive indices *, to which I must refer to avoid undue repetition, I reduced the observations by supposing that the compressibility term did not vary with temperature, being guided thereto by Mascart's conclusion that the temperature deviation was too great to be accounted for by the variation of the compressibility term. My experiments do not give such a large deviation as Mascart obtained, and in the case of SO_2 and NH_3 I find that it can be exactly accounted for in this way.

Sulphur Dioxide.

My experiments on the refractive index may be reduced as in the following table.

Temp.	Ratio.	p_1+p_2.	Ratio $\dfrac{\text{Ratio}}{1+\dfrac{\cdot0319(p_1+p_2)}{76(1+\cdot0039t)}}$.	$\dfrac{\text{Ratio} \times (1+\cdot000395t)}{1+\dfrac{\cdot0319(p_1+p_2)}{76(1+\cdot0039t)}}$.
81	11·535	137	11·050	14·585
37·4	13·305	130	12·700	14·575
14·9	14·465	126	13·775	14·585
14·2	14·490	121	13·825	14·600
14·2	14·270	81	13·825	14·600

Note.—Ratio means the number of bands displaced for 1 cm. difference of pressure.   Mean ... 14·585

The compressibility term was calculated from the last two observations, which give

$$\lambda_0 = \cdot0319 \text{ per } 76 \text{ cms. at } 0^\circ \text{ C.}$$

In order to get the value of μ the numbers in the last column have to be multiplied by 76, by the wave-length

(Na) 5890×10^{-10} metres and divided by $99 \cdot 9$ cms., the length of the tubes.

Thus we get for the Na line

$$\mu = 1 + \frac{\cdot 0006532\ P}{(1 + \cdot 00395t)76}\left\{ 1 + \frac{\cdot 0319\ P}{76(1 + \cdot 0039t)}\right\}.$$

The value of α for SO_2 is $\cdot 00390$, so that there is still a slight difference. The difference is, however, within possible experimental error.

At $P = 76$ and $t = 0°$ C. we get

$$\mu_{Na} = 1 + \cdot 000674.$$

The reduction of Baedecker's observations on the value of the dielectric constant is shown in the following table.

Temp.	Absolute Temp. θ.	$K-1$.	$(K-1)(1+\cdot 0039t)\theta \left\{1 + \frac{\cdot 0319}{1+\cdot 0039t}\right\}$	$(K-1)(1+\cdot 0039t)\theta \left\{1 + \frac{\cdot 0319}{1+\cdot 0039t}\right\}$	$(K-1)(1+\cdot 0039t)\theta(1+\cdot 0006\theta) \left\{1 + \frac{\cdot 0319}{1+\cdot 0039t}\right\}$
$10 \cdot 3$	$283 \cdot 3$	$\cdot 00928$	$\cdot 00936$	$2 \cdot 652$	$2 \cdot 266$
$16 \cdot 4$	$289 \cdot 4$	$\cdot 00893$	$\cdot 00922$	$2 \cdot 668$	$2 \cdot 273$
$21 \cdot 6$	$294 \cdot 6$	$\cdot 00869$	$\cdot 00915$	$2 \cdot 695$	$2 \cdot 290$
$37 \cdot 5$	$310 \cdot 5$	$\cdot 00794$	$\cdot 00881$	$2 \cdot 735$	$2 \cdot 305$
$42 \cdot 2$	$315 \cdot 2$	$\cdot 00765$	$\cdot 00867$	$2 \cdot 733$	$2 \cdot 298$
$49 \cdot 4$	$322 \cdot 4$	$\cdot 00734$	$\cdot 00852$	$2 \cdot 747$	$2 \cdot 301$
$64 \cdot 1$	$337 \cdot 1$	$\cdot 00669$	$\cdot 00815$	$2 \cdot 747$	$2 \cdot 284$
$76 \cdot 1$	$349 \cdot 1$	$\cdot 00634$	$\cdot 00802$	$2 \cdot 800$	$2 \cdot 315$
$103 \cdot 3$	$376 \cdot 3$	$\cdot 00551$	$\cdot 00755$	$2 \cdot 841$	$2 \cdot 317$
$\theta_0 = 273$				Mean ...	$2 \cdot 295$

Hence we get

$$K - 1 = \frac{2 \cdot 295\ P}{76(1 + \cdot 0039t)}\left\{ 1 + \frac{\cdot 0319\ P}{76(1 + \cdot 0039t)}\right\}\left\{ \frac{1 + \cdot 0006\ \theta}{\theta}\right\}$$

or

$$K - 1 = \frac{\cdot 001377\ P}{76(1 + \cdot 0039t)}\left\{ 1 + \frac{\cdot 0319\ P}{76(1 + \cdot 0039t)}\right\}\left\{ 1 + \frac{6 \cdot 10\ \theta_0}{\theta}\right\}.$$

The accuracy is only about 1 per cent.

According to the theory we deduce

$$\mu^2 - 1 = \frac{\cdot001377\,P}{76(1+\cdot0039t)}\left\{1 + \frac{\cdot0319\,P}{76(1+\cdot0039t)}\right\}\left\{1 + \frac{6\cdot10\,\theta_0}{\theta}\left(1 - E\frac{p^2}{\omega^2}\right)\right\}.$$

We can now determine $\dfrac{p^2}{\omega^2}$ for the Na line at $0°$ C. from the measurements of dispersion.

Ketteler gives

$$\frac{\mu_p - 1}{\mu_{p'} - 1} = 1\cdot0064 \text{ from Li red to Na.}$$

Hence if p refer to the Na line, $\dfrac{p^2}{p'^2} = 1\cdot297.$

If $\dfrac{p^2}{\omega^2}$ is large we have

$$\frac{1 - 6\cdot10\dfrac{\omega^2}{p^2}}{1 - 6\cdot10\dfrac{\omega^2}{p^2} \times 1\cdot297} = 1\cdot0064.$$

Therefore

$$6\cdot10\,\frac{\overline{\omega}^2}{p^2} = \cdot021,$$

or

$$\frac{\overline{\omega}^2}{p^2} = \cdot0034 \text{ for the Na line at } 0° \text{ C.}$$

The supposition that $\dfrac{\overline{\omega}^2}{p^2}$ is small is justified. We must note that this estimate of $\dfrac{\overline{\omega}^2}{p^2}$ cannot be considered accurate to more than 10 per cent. We may now apply a test by calculating the theoretical value of μ for the Na line at 76 cms. and $0°$ C. from Baedecker's measurements. We obtain $\mu_{Na} = 1\cdot000695.$

From my measurements on refraction we deduced $\mu_{Na} = 1\cdot000674.$

I think the agreement must be considered very satisfactory, especially when we remember the uncertainty of some of the measurements, and the fact that the value of $K - 1$ is about 7 times the value of $\mu^2 - 1$ in the visible spectrum.

Ammonia Gas.

In reducing my observations on the refractive index I have used Mascart's value for the compressibility term, viz.

$$\left\{1 + \frac{\cdot0135\,P}{76(1+\cdot0038\,t)}\right\},$$

which agrees well with the deviation from Boyle's law.

Temp.	Ratio.	$\dfrac{\text{Ratio}}{1+\dfrac{\cdot 0214}{1+\cdot 0038\,t}}$	$\dfrac{\text{Ratio} \times (1+\cdot 00380\,t)}{1+\dfrac{\cdot 0214}{(1+\cdot 0038\,t)}}$	p_1+p_2.
11·8	8·160	7·995	8·355	120
23·7	7·810	7·660	8·350	,,
32·0	7·605	7·460	8·370	,,
56·1	7·015	6·895	8·365	,,
76·4	6·585	6·475	8·355	,,
90·5	6·300	6·200	8·335	,,

Mean ...... 8·355

For NH_3 gas $\alpha = \cdot 00382$, so that the temperature-effects are accounted for within the limits of experimental error.

Hence we get

$$\mu = 1 + \frac{\cdot 0003742\,\mathrm{P}}{(1+\cdot 00380\,t)\,76}\left\{1 + \frac{\cdot 0135\,\mathrm{P}}{(1+\cdot 0038\,t)\,76}\right\}.$$

Hence for the Na line at 76 cms. and 0° C. we get

$$\mu_{\mathrm{Na}} = 1 + \cdot 000379.$$

Baedecker's observations on K are reduced in the following table.

Temp.	θ.	$K-1$.	$\dfrac{K-1}{1+\dfrac{\cdot 0135}{1+\cdot 0038\,t}}$	$\dfrac{(K-1)(1+\cdot 0038\,t)}{\left(+\dfrac{\cdot 0135}{1+\cdot 0038\,t}\right)}$	$\dfrac{(K-1)(1+\cdot 0038\,t)\,\theta}{1+\dfrac{\cdot 0135}{1+\cdot 0038\,t}}$	$\dfrac{(K-1)(1+\cdot 0038\,t)\,\theta(1+\cdot 0004\theta)}{\left(1+\dfrac{\cdot 0135}{1+\cdot 0038\,t}\right)}$
18·4	291·4	·00730	·00721	·00771	2·246	2·011
19·0	292·0	·00704	·00695	·00745	2·175	1·947
59·4	322·4	·00547	·00541	·00663	2·204	1·945
62·1	335·1	·00538	·00532	·00657	2·201	1·941
83·8	356·8	·00483	·00477	·00629	2·244	1·963
95·3	368·3	·00453	·00448	·00610	2·246	1·957
108·4	381·4	·00434	·00430	·00607	2·315	2·008

Mean...... 1·96

It is clear that the observations at $18°\cdot4$ and $19°$ cannot both be correct, and the same applies to the observations at $95°\cdot3$ and $108°\cdot4$. Excluding the first and last, the agreement is about 1 per cent.

Hence we get

$$K - 1 = \frac{1\cdot96\,P}{(1 + \cdot0038t)76}\left\{1 + \frac{\cdot0135\,P}{(1 + \cdot0038t)76}\right\}\left\{\frac{1 + \cdot00040\theta}{\theta}\right\}$$

or

$$K - 1 = \frac{\cdot000784\,P}{(1 + \cdot0038\,t)76}\left\{1 + \frac{\cdot0135\,P}{(1 + \cdot0038\,t)76}\right\}\left\{1 \div 9\cdot17\frac{\theta_0}{\theta}\right\}.$$

Hence we deduce

$$\mu^2 - 1 = \frac{\cdot000784\,P}{(1 + \cdot0038\,t)76}\left\{1 + \frac{\cdot0135\,P}{(1 + \cdot0038\,t)76}\right\}\left\{1 + 9\cdot17\frac{\theta_0}{\theta}\left(1 - E\frac{p^2}{\omega^2}\right)\right\}.$$

For the dispersion Lorenz gives $\dfrac{\mu_p - 1}{\mu_{p'} - 1} = 1\cdot0038$ between Li red and Na, $\therefore \dfrac{p^2}{p'^2} = 1\cdot297.$

Thus we have

$$\frac{1 - 9\cdot17\dfrac{\bar{\omega}^2}{p^2}}{1 - 9\cdot17\dfrac{\bar{\omega}^2}{p^2}\times 1\cdot297} = 1\cdot0048,$$

from which we obtain

$$9\cdot17\frac{\bar{\omega}^2}{p^2} = \cdot016,$$

or $\qquad \dfrac{\bar{\omega}^2}{p^2} = \cdot0017$ for the Na line and $0°$ C.

Using this value, we deduce from Baedecker's measurements the theoretical value

$\mu = 1\cdot000391$ for the Na line at 76 cms. and $0°$ C.

My experiments on refraction gave us

$$\mu = 1\cdot000379.$$

As in the case of SO_2, the agreement is very satisfactory. The values of $K - 1$ are about 10 times the value of $\mu^2 - 1$ in the visible spectrum.

Atmospheric Air.

In this case no experiments on the variation of K with temperature have been made, and the only datum we have is

$K = 1\cdot000558$ at 76 cms. and $16°$ C. (Boltzmann).

The method of verification must thus be changed, and we have to rely on the temperature-variation of μ and the dispersion for the determination of the constants in the formula. Considerable uncertainty is thus inevitable. A rough trial to explain the measured temperature-effect and dispersion showed that the visible spectrum must be rather close to the point A, so that I had to tabulate the numerical values of the function in the vicinity of the point A with great care. The results (which we also require for CO_2 and H_2) are given in the following tables.

TABLE I.

x.	$E(x)$.	$\dfrac{\partial E}{\partial x}$.	$\dfrac{\partial^2 E}{\partial x^2}$.	$\dfrac{\partial^3 E}{\partial x^3}$.	$\dfrac{\partial^4 E}{\partial x^4}$.
3	1·48372	+·01086	−·17209	+·22222	−·19828
4	1·43824	−·07868	−·03088	+·07794	−·09375
5	1·35383	−·08306	+·1229	+·01847	−·03324

Table I. was calculated by aid of the formulæ

$$\frac{\partial E}{\partial x} = 1 - \left(1 - \frac{1}{x}\right)E, \quad \frac{\partial^2 E}{\partial x^2} = -\left(1 - \frac{1}{x}\right) + \left(1 - \frac{2}{x}\right)E$$

$$\frac{\partial^3 E}{\partial x^3} = \left(1 - \frac{2}{x} - \frac{1}{x^2}\right) - \left(1 - \frac{3}{x}\right)E, \quad \frac{\partial^4 E}{\partial x^4} = -\left(1 - \frac{3}{x} - \frac{2}{x^2} - \frac{2}{x^3}\right) + \left(1 - \frac{4}{x}\right)E.$$

Table II. was computed by aid of Table I. and the formula

$$E(x + \zeta) = Ex + \frac{\zeta \partial E(x)}{\partial x} + \frac{\zeta^2}{2} \frac{\partial^2 E}{\partial x^2} + \frac{\zeta^3}{6} \frac{\partial^3 E}{\partial x^3} + \frac{\zeta^4}{24} \frac{\partial^4 E}{\partial x^4}.$$

TABLE II.

x.	Ex.	x.	Ex.	x.	Ex.
2·7	1·47165	3·7	1·46007	4·7	1·37920
2·8	1·47780	3·8	1·45325	4·8	1·37006
2·9	1·48175	3·9	1·44594	4·9	1·36219
3	1·48372	4	1·43824	5	1·35383
3·1	1·48398	4·1	1·43024	5·1	1·34559
3·2	1·48274	4·2	1·42199	5·2	1·33749
3·3	1·48017	4·3	1·41357	5·3	1·32954
3·4	1·47822				

TABLE III.

x.	Ex.	$\dfrac{x}{1\cdot366}$.	$\text{E}\dfrac{x}{1\cdot366}$.	$1-\text{E}(x)$ $-\dfrac{1}{1\cdot366}\left(1-\text{E}\dfrac{x}{1\cdot366}\right)$.	$\dfrac{x}{1\cdot297}$.	$\text{E}\dfrac{x}{1\cdot297}$.	$1-\text{E}(x)-$ $\left(1-\text{E}\dfrac{x}{1\cdot297}\right)$
3	1·48372	2·19	1·37639	−·2082	2·31	1·39880	−·0849
4	1·43824	2·92	1·48214	−·0853	3·08	1·48392	+·0457
4·7	1·37920	3·44	1·47744	−·0297	3·62	1·46392	+·0847
5	1·35383	3·66	1·46320	−·0140	3·85	1·44959	+·0957
6	1·27872	4·39	1·40600	+·0185	4·62	1·38651	+·1078
7	1·22242	5·12	1·34397	+·0296	5·39	1·32239	+·0999
8	1·18183	5·85	1·28961	+·0302	6·16	1·26972	+·0879
9	1·15280	6·59	1·24551	+·0269	6·93	1·22636	+·0735
10	1·13146	7·32	1·20943	+·0219	7·71	1·19279	+·0613
11	1·11542	8·05	1·17980	+·0162	8·48	1·16310	+·0477

The number 1·366 is the ratio in which $\bar{\omega}^2$ is increased as the temperature increases from 0° to 100° C., and 1·297 is the ratio of p^2 for the Na line to p'^2 for the Li red line.

If we call $\dfrac{p^2}{\bar{\omega}^2}$ for the Na line at 0° C. x, we have theoretically

$$\frac{1+\dfrac{k_2}{k_1\theta_0}(1-\text{E}x)}{1+\dfrac{k_2}{k_1\theta_0}\dfrac{1-\text{E}\dfrac{x}{1\cdot366}}{1\cdot366}} = \frac{\mu_{0°}-1}{\mu_{100°}-1}$$

and

$$\frac{1+\dfrac{k_2}{k_1\theta_0}(1-\text{E}x)}{1+\dfrac{k_2}{k_1\theta_0}\left(1-\text{E}\dfrac{x}{1\cdot297}\right)} = \frac{\mu_{\text{Na}}-1}{\mu_{\text{Li}}-1}.$$

These two equations are theoretically sufficient to determine $\dfrac{k_2}{k_1\theta_0}$ and x. $\mu_{0°}$ and $\mu_{100°}$ were calculated from the experiments, and then by trial and error the constants were determined.

Ketteler gives for the dispersion

$$\frac{\mu_p-1}{\mu_{p'}-1}=1\cdot0034.$$

If we take $\dfrac{p^2}{\bar{\omega}^2}=4\cdot7$ at 0° C. for the Na line, we get

$$\frac{1-\dfrac{k_2}{k_1\theta_0}\times\cdot3792}{1-\dfrac{k_2}{k_1\theta_0}\times\cdot4639} = 1\cdot0034.$$

Hence $$\frac{k_2}{k_1\theta_0} = \cdot039.$$

In the following table my measurements are first corrected for the compressibility term, which is

$$\left\{ 1 + \frac{\cdot00069}{(1+\cdot0036t)}\ \frac{P}{76} \right\}.$$

Temp.	Ratio.	Ratio. $\dfrac{\cdot0014}{\left(1+\dfrac{\cdot0014}{1+\cdot0036t}\right)}$	Ratio $(1+\cdot00365t)$ $\dfrac{\cdot0014}{\left(1+\dfrac{\cdot0014}{1+\cdot0036t}\right)}$	Ratio $(1+\cdot00365t)$ $\dfrac{\left\{1+\dfrac{\cdot039\theta_0}{\theta}\right\}}{\left(1+\dfrac{\cdot0014}{1+\cdot0036}\right)\left(1-\mathrm{E}\dfrac{p^2}{\bar{\omega}^2}\right)}$
$10\overset{\circ}{\cdot}6$	6·290	6·280	6·525	6·525
11	6·285	6·275	6·525	6·525
23·9	6·025	6·015	6·540	6·540
24·3	6·000	5·999	6·530	6·525
52	5·510	5·505	6·550	6·645
53	5·495	5·490	6·550	6·645
76·5	5·120	5·115	6·545	6·635
100·9	4·790	4·785	6·545	6·635
			Mean	6·635

The agreement of the numbers in the last column is as close as the experiments warrant, and shows that the temperature effect is accounted for.

We obtain

$$\mu = 1 + \frac{\cdot000297\,P}{(1+\cdot00365t)76} \left\{ 1 + \frac{\cdot00069\,P}{(1+\cdot0036t)76} \right\} \left\{ 1 + \cdot039\,\frac{\theta_0}{\theta}\left(1 - \mathrm{E}\frac{p^2}{\omega^2}\right) \right\}.$$

Hence

$$\mathrm{K} = 1 + \frac{\cdot000594\,P}{(1+\cdot00365t)76} \left\{ 1 + \frac{\cdot00069\,P}{(1+\cdot0036t)\,76} \right\} \left\{ 1 + \cdot039\,\frac{\theta_0}{\theta} \right\}.$$

Hence at 76 cms. and 16° C. we get

$$K = 1 + \cdot 000582.$$

Boltzmann gives

$$K = 1 + \cdot 000558 \text{ at 76 cms. and 16° C.}$$

When we remember that the basis of calculation has been the dispersion and the temperature deviation, the agreement is as close as we could expect.

Carbon Dioxide.

Ketteler gives for the dispersion

$$\frac{\mu_p - 1}{\mu_{p'} - 1} = 1 \cdot 0034 \text{ from Li red to Na.}$$

If we take $\dfrac{p^2}{\bar\omega^2} = 8$ for the Na line and 0° C., we get

$$\frac{1 - \dfrac{k_2}{k_1 \theta_0} \times \cdot 1818}{1 - \dfrac{k_2}{k_1 \theta_0} \times \cdot 2697} = 1 \cdot 0034.$$

Hence

$$\frac{k_2}{k_1 \theta_0} = \cdot 038.$$

My experiments may be reduced as follows.

Temp.	Ratio.	Ratio $\dfrac{\cdot}{1 + \dfrac{\cdot 0116}{1 + \cdot 0037t}}$	Multiplied by $(1 + \cdot 0037t)$.	Divided by $1 + \dfrac{\cdot 038\,\theta_0}{\theta}\left(1 - E\dfrac{p^2}{\bar\omega^2}\right)$.
10·5	9·960	9·585	9·955	10·025
21·7	9·290	9·190	9·930	10·000
31·5	8·985	8·895	9·930	10·005
74·9	7·845	7·775	9·930	10·010

Mean ...... 10·010

Hence

$$\mu = 1 + \frac{\cdot 0004483 \, P}{(1 + \cdot 0037t)76} \left\{ 1 + \frac{\cdot 0058 \, P}{(1 + \cdot 0037t)76} \right\} \left\{ 1 + \frac{\cdot 038\,\theta_0}{\theta}\left(1 - E\frac{p^2}{\omega^2}\right) \right\} \cdot$$

From this we deduce

$$K = 1 + \frac{\cdot 0008966\, P}{(1 + \cdot 0037t)76}\left\{1 + \frac{\cdot 0058\, P}{(1 + \cdot 0037t)\, 76}\right\}\left\{1 + \frac{\cdot 038\,\theta_0}{\theta}\right\}.$$

Hence at $16°$ C. and 76 cms. we get

$$K = 1 + \cdot 000886.$$

Boltzmann gives $K = 1 + \cdot 000892$ at 76 cms. and $16°$ C.

Hydrogen.

In this case the variation of μ with temperature was distinctly less than that given by the law $\mu^2 - 1 \propto N$.

Although the compressibility term for hydrogen is less than unity and gives a temperature variation of the same sign as that found experimentally, the variation is insufficient to account for the observed facts. Hence the only portion of the curve which will meet the case is that between the points A and B. Ketteler gives $\frac{\mu_p - 1}{\mu_{p'} - 1} = 1\cdot 00456$ from Li red to Na. If we take $\frac{p^2}{\bar{\omega}^2} = 4$ for the Na line and $0°$ C., we get

$$\frac{1 - \dfrac{k_2}{k_1\theta_0} \times \cdot 4382}{1 - \dfrac{k_2}{k_1\theta_0} \times \cdot 4839} = 1\cdot 00456.$$

Hence $$\frac{k_2}{k_1\theta_0} = \cdot 095.$$

My experiments may be reduced as follows :—

Temp.	Ratio.	Ratio $1 - \dfrac{\cdot 0014}{(1 + \cdot 00365t)}$.	Multiplied by $(1 + \cdot 00365t)$.	Divided by $1 + \dfrac{\cdot 095\,\theta_0}{\theta}\left(1 - E\dfrac{p^2}{\bar{\omega}^2}\right)$.
$10°65$	3·048	3·052	3·170	3·305
10·85	3·039	3·043	3·163	3·298
23·3	2·920	2·924	3·172	3·305
32·35	2·830	2·833	3·167	3·297
33	2·827	2·830	3·171	3·301
65·8	2·560	2·563	3·178	3·299
77·5	2·479	2·482	3·184	3·302
81·7	2·449	2·452	3·185	3·301
83·2	2·445	2·448	3·191	3·307

Mean ...... 3·302

Hence

$$\mu = 1 + \frac{\cdot 0001478\,\text{P}}{(1 + \cdot 00365 t)\,76} \left\{ 1 - \frac{\cdot 0007\,\text{P}}{(1 + \cdot 0036 t)\,76} \right\} \left\{ 1 + \frac{\cdot 095\,\theta_0}{\theta} \left(1 - \text{E}\,\frac{p^2}{\omega^2} \right) \right\}.$$

Hence

$$\text{K} = 1 + \frac{\cdot 0002956\,\text{P}}{(1 + \cdot 00365 t)\,76} \left\{ 1 - \frac{\cdot 0007\,\text{P}}{(1 + \cdot 0036 t)\,76} \right\} \left\{ 1 + \frac{\cdot 095\,\theta_0}{\theta} \right\}.$$

Hence at 76 cms. and 16° C. we get

$$\text{K} = 1 + \cdot 000304.$$

Boltzmann gives $\text{K} = 1 + \cdot 000250$ at 16° C.; but this is the mean of two series of experiments which gave respectively

$$\text{K} = 1 + \cdot 000272,$$

$$\text{K} = 1 + \cdot 000228.$$

Thus Boltzmann's experiments do not admit of great accuracy; and this is not surprising in view of the tremendous difficulty of measuring the value of $\text{K} - 1$ for hydrogen.

(9) *Conclusions.*

I think it must be admitted that the numerical test has been very severe, and the agreement most satisfactory when we remember the uncertainty of the experimental values of the dispersion and temperature deviations. The agreement is particularly good in the two cases where we have most data, SO_2 and NH_3; and in these cases the value of $\text{K} - 1$ is very much greater than the values of $\mu^2 - 1$ in the visible spectrum.

The theory gives a substantial explanation of all the essential facts connected with refraction and the dielectric constant which I summarized at the beginning of the paper. It shows that dispersion is controlled by the temperature and not by free periods of vibration; but notwithstanding this the dispersion does not vary much with temperature except near the point A. In the case of hydrogen we should expect that by raising the temperature a few hundred degrees the dispersion ought to be completely reversed.

If the theory is correct, it throws great doubt on estimates of molecular quantities based on theories which do not explain the temperature effects, and shows that until we have experiments over a much greater range than the visible spectrum it is unwise to trust much to an approximate formula which represents only some of the facts in a comparatively limited region.

I should like, in conclusion, to express how much I have been stimulated in the study of refraction by Professor Voigt, in whose laboratory at Göttingen I made my preliminary experiments on refraction in gases, and by Professor J. J. Thomson, under whom the experiments were completed. For their encouragement and advice in conversation and for their published works I am very grateful.

LX. *The Vapour-Densities of some Carbon Compounds; an Attempt to Determine their correct Molecular Weights.* *By* Sir WILLIAM RAMSAY, *K.C.B., F.R.S., and* BERTRAM D. STEELE, *D.Sc.*[*]

[Plate XX.]

THE accurate determination of the densities of gases has been for long an object to which chemists have paid attention. On the other hand, the density of vapours has only been roughly estimated, as a means of arriving at a conclusion regarding molecular weights; whilst accurate molecular weights have been deduced from the results of analysis, and from previous determinations of atomic weights.

Every method which brings additional evidence to bear on so important a class of constants as atomic weights must be welcome; and it was with great interest that the memoirs of M. Daniel Berthelot were perused, " Sur la détermination rigoureuse des poids moléculaires des gaz en partant de leurs densités et de l'écart que celles-ci présentent par rapport à la loi de Mariotte (*Comptes rendus*, 1898, xii. pp. 954, 1030, 1415, & 1501). In these papers M. Berthelot has brought Regnault's determinations of the compressibility of hydrogen, nitrogen, oxygen, and carbon monoxide between one and six atmospheres to bear on determinations of their density by M. Leduc, in such a manner that their relative weights can be compared when, if Avogadro's hypothesis be granted, equal volumes contain equal numbers of molecules. In the case of the elementary gases, oxygen, hydrogen, and nitrogen, inasmuch as the molecules are diatomic, the determination of the molecular weight is at the same time a determination of the atomic weight; and with carbon monoxide the atomic weight of carbon is arrived at by simple subtraction. To quote Berthelot's words: " Le volume moléculaire d'un gaz à 0° et sous la pression atmosphérique étant égal à 1 pour un gaz qui suivrait exactement la loi de Marriotte, ce volume a la valeur 1—*a* pour un gaz qui ne la suit pas." The definition

[*] Communicated by the Physical Society: read December 12, 1902.

of a is that given by Regnault, viz. $\dfrac{p_0 v_0}{pv} - 1 = a(p - p_0)$, and represents the deviation of the compressibility of the gas from Boyle's law for increase of pressure of one atmosphere. The error in the atomic weights of oxygen, hydrogen, nitrogen, and carbon thus determined is estimated not to exceed 1 part in 5000. M. Berthelot also discusses deviations from Boyle's law exhibited by easily liquefiable gases, namely, carbon dioxide, nitrous oxide, hydrogen chloride, acetylene, phosphoretted hydrogen, and sulphur dioxide, making use for this purpose of van der Waals' modification of the simple gas laws. From the previously determined densities of these gases he calculates their molecular weights, using as a basis the molecular weight of oxygen taken as 32; whence by subtraction the atomic weights of hydrogen, carbon, nitrogen, sulphur, and chlorine are deduced.

Another method of applying M. Berthelot's procedure is this :—Determine the density of the gas ; determine also its compressibility at low pressures, preferably below one atmosphere ; assuming the compressibility to be a linear function of pressure, calculate the value of the product pv at zero pressure : the ratio of the pv's will be the ratio of the densities when equal volumes contain equal numbers of molecules. The molecular weights, and consequently the atomic weights, follow as a matter of course. The method can also be applied graphically, by plotting as ordinates the pressure, and as abscissæ the values of pv/T: where the curve cuts the line of zero pressure the theoretical value of pv/T has been reached.

There appeared every prospect that the determination of the density of vapours should, if carried out with pure substances, lead to results as interesting and important as those deduced by Daniel Berthelot. And if the estimations were made with pure substances the atomic weights of the elements contained in the compounds might be determined with extreme accuracy. Moreover, the method of deducing atomic weights from density may be regarded as a "statical" method, in contrast with the usual analytical methods, which, on account of the transformations which the elements are made to undergo during the determination, may well be termed "dynamical."

After some preliminary attempts to apply a modification of Dumas' method it was discarded in favour of a modification of Gay-Lussac's ; this method has the advantage that while densities are being determined compressibilities may be simultaneously estimated with the same sample of material,

at least within certain limits.   But for low pressures separate
estimations of compressibility were made with a special
apparatus designed by one of us for the purpose (Trans.
Chem. Soc. 1902, vol. lxxxi. p. 1176).   We had intended to
investigate the molecular weights of a number of substances
containing various elements ; but the results which some
compounds of carbon and hydrogen, and of carbon, hydrogen,
and oxygen have led to are so remarkable that it appears
desirable not to delay publication.   We may here anticipate
our conclusions, in order that the importance of each point
may not be overlooked.   Our evidence goes to show that the
densities of certain compounds calculated for zero pressure
are not proportional to their molecular weights deduced from
the atomic weights of the elements which they contain.
This conclusion involves one, or it may be several, of the
following assumptions :—

1. The substances employed may not have been pure ;

2. The methods of experiment may not be capable of giving
sufficiently accurate results ;

3. Avogadro's hypothesis may not hold for vapours for one
of the reasons below.

    (*a*) The vapours may adhere to the glass and increase
        their apparent density ;

    (*b*) The vapours may contain some complex molecular
        groups, or, in other words, they may display partial
        molecular association at the temperature of experi-
        ment, even under very low pressure ;

    (*c*) The atomic weights of the elements may alter their
        values according to the ratios between the number
        of atoms in the compounds.

The first two suppositions, until disproved, are of course
by far the most likely, and we shall therefore commence
with a description of the sources and preparation of the com-
pounds under experiment, in order that our readers may be
in a position to judge of their probable purity.

Preparation of the Compounds.

These were : hexane, two octanes, benzene, toluene, ether,
and methyl alcohol.   The first three we owe to the kindness
of Prof. Sydney Young.   Beyond drying them with phos-
phoric anhydride and distilling them into our weighing-bulbs
they underwent no process of purification.

1. *Normal hexane* (Young, Trans. Chem. Soc. vol. lxxiii.
pp. 910–913).—This specimen was fractionated from petro-
leum ether.   The fractionation was carried out with a
combined regulated temperature still-head and six-column

dephlegmator. The separate fractions were heated with
fuming nitric acid for several days, and after treating with
potash, and drying, they were distilled. The comparison of
the constants is as follows :—

	Petroleum.	Normal hexane from Propyl iodide.
Boiling-point................	68·95°	68·95°
Density at 0°/4° (four fractions afterwards mixed together)..	{ 0·67693 0·67699 0·67697 0·67702	0·67697
Critical temperature..........	234·8°	234·8°
„ pressure	22540 mms.	22510 mms.

The concordance of these results affords a guarantee of the
almost absolute purity of the specimen.

(2) *Normal octane* (Young, *ibid.* xvi. p. 166).—This
sample was made from octyl iodide by Kahlbaum. Its
boiling-point at normal pressure after purification by Young
was 125·8°. It was collected in three fractions, which showed
the densities A, 0·71850; B, 0·71847; C, 71848. These
fractions were mixed. The boiling-points, determined at
different times, were 125·8°, 125·85°, and 125·75°.

(3) *Di-isobutyl* (Young, *ibid.* xvi. p. 165).—Made by
Young and Miss Fortey by treating isobutyl bromide with
sodium in ethereal solution. "No great difficulty was ex-
perienced in the preparation of di-isobutyl, and the yield was
fair. As the boiling-point of the paraffin (109·2°) is con-
siderably higher than that of isobutyl bromide (92·3°), it
could be separated fairly completely from the unaltered
bromide by fractional distillation. The final purification
was effected by treatment with a mixture of nitric and sul-
phuric acids, and subsequent fractional distillation through a
twelve-column Young and Thomas dephlegmator. Owing
to the partial conversion of the isobutyl bromide into the
tertiary bromide a small quantity of hexamethyl-ethane is
formed ; but this is completely removed during the fractional
distillation." The boiling-points of three fractions were
109·2°, 109·2°, and 109·25°; the density at 0° was 0·71021 :
Schiff found 0·7103.

The remaining substances were prepared by ourselves.

(4) *Benzene.* –A sample of Kahlbaum's purest thiophene-
free benzene was twice subjected to fractionation ; the boiling-
point was absolutely constant.

(5) *Toluene.*—Sample A was prepared from paratoluidine
by diazotization. Its boiling-point was constant. Sample B
was synthesized from bromobenzene and methyl iodide ; the
constant boiling fraction was crystallized. As the process of

crystallization at very low temperatures has, so far as we are aware, not been described, a short account of it may prove interesting.

Some ordinary crude toluene was cooled by pouring liquid air into it until it was partially solid. The purified toluene was introduced into a tube into which was sealed an inverted filter, plugged with cotton-wool. On dipping this tube into the cooled toluene the pure toluene slowly froze, forming large crystals; when about one-third had frozen the unfrozen portion, containing any possible impurities, was filtered off. The crystals when melted formed the sample termed B.

(6) *Ether.*—Prepared from absolute alcohol in the usual way. It was then fractionated from phosphorus pentoxide until the boiling-point was constant. The samples of ether were treated in three different ways; sample A was distilled from phosphoric anhydride into the bottle in which it was preserved; sample B was a portion of A which had been allowed to stand for three days over phosphoric anhydride, and then shaken with mercury to remove possible ethyl peroxide; it was then distilled into the weighing-bulb from pentoxide by a method to be described. As samples A and B both gave the same puzzling density, and as it was possible that the methods of purification left something to be desired, sample C was a portion of B which was frozen in a manner similar to that described for toluene. It was cooled in a tube immersed in frozen crude ether until the ether in it had crystallized. The large clear crystals, some of which were two inches long, were separated from twice their volume of mother-liquor by upward filtration. The crystals when melted formed sample C.

(7) *Methyl alcohol.*—Methyl oxalate was prepared from " pure " methyl alcohol. The crystallized oxalate was washed with water until the washings gave no iodoform reaction. The ester was decomposed with potash, and the alcohol dried with lime of very high quality, which reacted violently and at once with water. The sample was re-distilled with lime until the boiling-point became constant.

It is constancy of boiling-point, and not temperature, which we regard as a criterion of purity. To register a really accurate temperature is a difficult undertaking, and one which was unnecessary for our purpose.

Temperatures of Experiment.

These were approximately 100°, 115°, and 130°. They were obtained by surrounding the tubes to be heated with the vapour of pure chlorobenzene, which had been very

carefully fractionated so as to ensure a constant boiling-point, boiling under correspondingly reduced pressures. The arrangement did not materially differ from that described in the Trans. Chem. Soc. 1885; all the joints were sealed so that no leakage occurred, and temperature could be recovered from day to day within 1/100 of a degree.

As the standard of comparison for the densities of the substances mentioned above was oxygen, its density at these temperatures was required. Instead of a direct determination of the densities of oxygen at these temperatures, however, the coefficient of expansion of the gas between 0° and 130° was directly measured with an unknown weight; it was found to be $1/272\cdot52 = 0\cdot0036694$. The mean of the determinations of the weight of one litre of oxygen at 0° by Regnault, Jolly, Leduc, and Rayleigh was accepted as the basis of calculation, namely, $1\cdot42961$.

As the coefficient of expansion of oxygen was determined by a method practically identical with that employed for the measurements of densities, it will be more convenient to defer the description of these experiments until an account of the apparatus has been given.

Description of Density-apparatus.

The apparatus consists essentially of the volume-tube and pressure-gauge A and B (fig. 1, p. 498), which are connected by a short piece of thick-walled india-rubber tubing. The volume-tube A consists of a glass tube of about 20 mms. diameter and about 780 mms. in length, constricted at its lower end, near which is attached the side-tube a. The top is blown into a cap b of the form shown in the figure; this serves the double purpose of trapping the weighing-bulb after its insertion, and also of retaining a small globule of mercury. The bulb is thus prevented from floating on the surface of the mercury in A, and from breaking off the glass points 1, 2, 3, &c. The globule of mercury gives off vapour when the tube is heated, and the space rapidly becomes saturated with mercury-vapour; experiments have shown that if this precaution be omitted the pressure of mercury-vapour takes many hours to rise to its maximum, on account of its great density, and consequent slow rate of diffusion.

The whole tube is inclosed in a jacket D, which contains during the experiment the vapour of chlorobenzene; and D is attached to the apparatus for regulating the pressure under which the chlorobenzene boils. The whole system of tubes rests on its cork g in mercury, contained in a vessel E,

through which water is made to circulate. The temperature
of the volume-tube A, from its top to where it enters the
mercury c, required to protect the rubber cork d from being

Fig. 1.
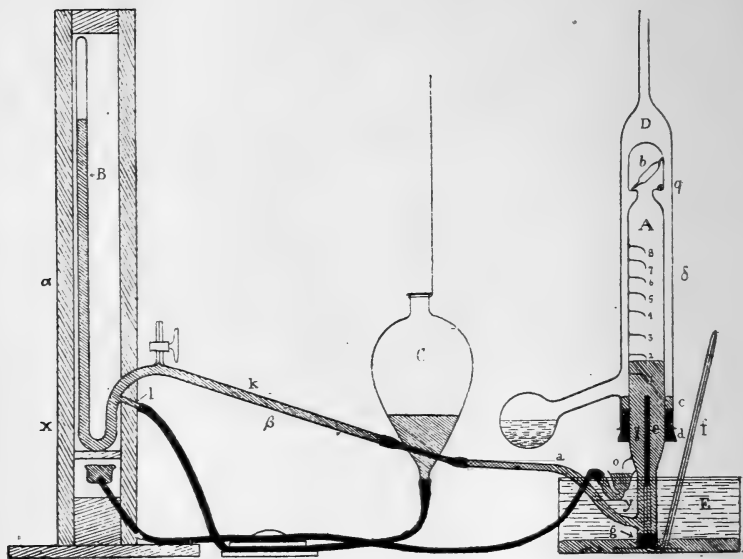
attacked by the hot chlorobenzene, is that of the chloro-
benzene vapour, that of the portion standing in water is also
known, and that of the intermediate portion is determined by
means of a thermometer with a long bulb e, of the same
length as the portion of tube of unknown temperature; the
temperature is thus integrated between the point c and
the surface of the water. The temperature is read on the
stem f.

The volumes of the vapour are determined by adjusting
the surface of the mercury to a set of points 1 to 8, of blue
enamelled glass, which should be ground on a whetstone
with oil in the manner recommended by Lord Rayleigh.   To
determine once for all the volumes indicated by the different
points stopcocks were sealed to the end of the tube at g and
to the end of a, as shown in fig. 2.   The tube was completely
exhausted with a Töpler pump, and filled with warm mercury;
by attaching a Fleuss pump to the exit of the weighing-flask
F the mercury flowed out through the capillary end of the

stopcock *h* into F. The mercury was drawn below the level
of the point 1 in question ; the Fleuss pump was disconnected

Fig. 2.

from F, and by turning the stopcock
rapidly the mercury rose until it nearly
touched the point. Adjustment was made
by help of a single lens. During calibra-
tion the whole tube was surrounded with
water of known temperature. The average
error was about 1/30,000 of the total
volume.
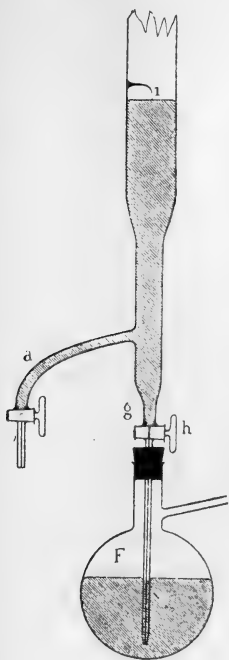
After calibration the next step was to
determine the vertical distance between
points 1 to 8 and a point 0 (fig. 1) attached
to the outside of the tube below the jacket.
The volume-tube was set up opposite the
scale B, and made vertical with a plumb-
line, and the distances read with a telescope
to within 0·02 mm. In order to correct
these differences the linear expansion of
the glass was required; as a sufficient
approximation the cubical expansion of
this sample of glass (0·000029) was deter-
mined, and one-third taken as the linear
expansion. The cubical expansion was, of
course, also required to correct the read
volumes.

The manometer consisted of a glass
U-tube B, 100 cms. long and 13 mms. in
diameter. It was filled with mercury
and boiled out *in vacuo*. This gives no trouble provided the
tube is carefully freed from dust by washing out with bichrome
and sulphuric acid, and subsequently with distilled water,
before drying it. The side-tube *k* is provided at the highest
point with a stopcock through which any bubbles of air
can be displaced. The short limb of the U-tube is connected
by means of the side-tube *l* and thick-walled rubber tubing
with the mercury reservoir C. Final adjustment of the
mercury surface to the points in the volume-tube A is made by
means of a squeezer, consisting of two pieces of wood and a
screw. The manometer was water-jacketed, a delicate ther-
mometer being inclosed in the jacket. The scale was obtained
from Zeiss. It was tested by means of a travelling microscope,
reading to 0·001 mm., and was found to be consistent to
0·03 ; a table of corrections was made by which readings
were corrected to within 0·01 mm.

In order to prevent the expansion of the scale the hot volume-tube stood at some distance, and the level was "carried" from the external point on the volume-tube to the scale by means of two mercury cups and a connecting tube. The cups were placed fairly level by eye, and mercury was poured in until the meniscus stood just a trace below the level of the point. By means of a squeezer, compressing the indiarubber tube which connected the two mercury cups, the level of the mercury was so adjusted as accurately to touch the point. The level of the mercury on the scale was the same, and readings could be made of the distance between the external point, and consequently of all the internal points, and the height of the mercury in the manometer.

The reading of the levels of mercury in the cup and in the manometer was made with a telescope standing at a distance of 10 feet. By a simple device due, we believe, to Prof. Poynting, readings to an accuracy of 0·01 mm. were easily made. It consists of a piece of truly plane parallel glass in front of the telescope which can be rotated through an angle by means of an attached pointer. The rotation of this plane glass displaces the object viewed; it is easy to count the divisions on an arc, ruled in sine divisions, which correspond to 1 mm. of the scale, and in this way hundredths of a millimetre may be directly and simply estimated. The device offers a simple and inexpensive substitute for a cathetometer, and is not behind one of the best in accuracy.

In order to prevent absorption of moisture during the filling of the bulbs this operation was carried out *in vacuo* by means of the apparatus shown in fig. 3. The small bulb x is about 2 cms. in diameter, there are sealed to it a number of small weighing-bulbs y; x is attached by the tube to a somewhat larger bulb z.

After this apparatus is made a small quantity of phosphoric anhydride is introduced into z, and the whole is exhausted and sealed at v. It is then allowed to stand for a length of time so that all moisture introduced in making and sealing the apparatus may be absorbed; the point v is then broken and the liquid with which the bulbs are to be filled is introduced; the apparatus is again exhausted and again sealed at v. Finally, after the liquid has stood in contact with the anhydride for a number of hours, a portion of it is distilled over into x, and so into the bulbs y; these are sealed off in such a manner by a fine blow-pipe flame that a long capillary stem remains attached to y, containing no liquid or foreign gas, but only the vapour of the contained liquid. The size which y should possess for a given substance is calculated

approximately from the molecular weight and density of the
liquid in question; after sealing off the point of the stem is

Fig. 3.

melted so as to form a knob about 1 or 2 mms. in diameter,
and a nick is made with a glass-cutter in the capillary about
1 cm. from the end. (The filled bulb is shown in fig. 3, *u.*)
In the case of methyl alcohol lime was substituted for phos-
phoric anhydride.

The Limits of Accuracy.

The accuracy of the final result depends on the measure-
ments of four quantities, namely, the pressure, the volume,
the temperature, and the weight of substance introduced
into the volume-tube. The smallest pressure that has been
measured with this apparatus is 220 mms. The divisions of
the scale employed, when corrected by the calibration table
previously referred to, are consistent to 0·01 mm. The
pressures given in the tables on subsequent pages are the
differences between two readings on the scale, each of which
is the mean of three independent readings for which the
surface of the mercury is re-set to the point under observa-
tion. In no case did the eight settings differ by more than
0·05 mm., whilst in the great majority of cases they lie
within 0·03 mm.; so that the extreme error due to reading
of pressure may be taken as about 0·05 mm., or about 1 in
4000. As will be seen from the curves and tables given
later it is only occasionally that a point is found to diverge
so widely from the curve.

The error arising from the determination of the volumes in the tube A is negligible ; this is seen from the following result of a calibration carried out in duplicate :—

	February 16th.	March 5th.
I.	228·947	228·962
II.	201·866	201·856
III.	179·473	179·463
IV.	156·399	156·379
V.	114·734	114·742
VI.	95·171	95·167

The temperature of the vapour-jacket can easily be maintained constant to within 0·02° by occasionally adjusting the pressure under which the liquid boils; this has been repeatedly tested and proved during the course of the research. The temperature at which the majority of the experiments were carried out was 130°; the absolute temperature was therefore about 403°, and a variation of 0·02° would therefore amount to only 1 part in 20,000.

The weighings were made with a long-beam Oertling's balance which, on reading by oscillations, recovered weighings correct to within 0·00002 gram; duplicate weighings of one of the toluene and of one of the methyl-alcohol bulbs will serve to illustrate the accuracy that could be attained by careful weighing :—

	Toluene.	Methyl-alcohol.
I.	0·77528 gram	0·39244 gram
II.	0·77527 „	0·39242 „

As, however, each experiment involved two weighings, one of the full and one of the empty bulb, the total error was probably about one twenty-fifth of a milligram, or 0·00004. Since the average weight of substance taken was about 0·15 gram the error should be about 1 in 4000; and this is about the degree of concordance which has been found in cases in which more than one experiment with the same substance has been carried out.

The Coefficient of Expansion of Oxygen.

As already explained it was necessary to determine this coefficient in order that the densities of the vapours might be compared directly with that of oxygen. What was done practically was to express the temperature of the experiment in terms of an oxygen thermometer.

The actual densities of the various vapours were determined at three temperatures, namely, the temperatures of chloro-benzene boiling under a pressure of 292·75 mms., 468·5 mms., and 719·0 mms. of mercury. In order to ascertain these temperatures the pressure exerted by a constant volume of oxygen was measured, first at the temperature of melting ice ; second at 99·82°, the temperature of water boiling under a pressure of 755·1 mms. ; third, at the temperature of chlorobenzene boiling under a pressure of 292·75 mms. ; and lastly with chlorobenzene under a pressure of 719·0 mms.

For this purpose the portion of the apparatus A and D (fig. 1) was replaced by that shown in fig. 4, where a' is

Fig. 4.
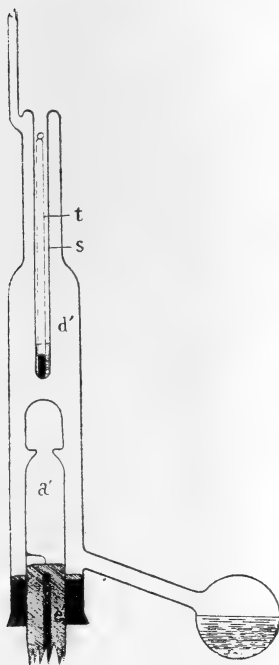
a smaller tube, the volume of which was about 92 c.cs. ; this tube was dried very carefully, filled with warm dry mercury, and the mercury was displaced with perfectly dry oxygen which had been prepared by the de-composition of potassium per-manganate. The tube a' was then connected with the mano-meter, after the thermometer e' had been inserted. It was next jacketed successively with chloro-benzene, with water, and with ice.

For each measurement a num-ber of readings were made, which are given later on. It is neces-sary first to explain the various corrections that have to be applied to the actual readings.

These will best be understood by aid of the Greek letters on fig. 1.

Section a, the manometer column, is surrounded with run-ning water, the temperature of which is observed ; a correction must here be applied to reduce the height of the mercury column to its height at 0°. This correction is given in the table under a.

Section β, forming the connecting tube between manometer and volume-tube, is not jacketed ; but its vertical height is small, and an error of even a few degrees in its temperature

is unimportant; the correction is given under heading *b* in the table.

The temperature of γ, the lower portion of the volume-tube, is ascertained by means of the long-bulbed thermometer previously described, and its correction is given under heading *c*; the temperature of δ, its upper part, is that of the vapour-jacket, and the correction is given under heading *d*. Corrections for the portions *x* and *y* are not so important since *x* is small, and since the two limbs of *y* are of the same length and at the same temperature. A constant correction of a few hundredths of a millimetre was made for section *x* when the temperature of α was much lower than that of the room.

Further corrections have to be applied for depression due to capillarity in the manometer-tube, and for vapour-pressure in the volume-tube; the former of these is already applied to the figure given under the heading "manometer." Hence, if the actual reading of the manometer is *m*, its true height is $m - a - b - e$, where *e* is the correction due to vapour-pressure of mercury in the volume-tube. From this has to be subtracted the height of the mercury in the volume-tube. The position of the external point 0 (fig. 1) is directly measured; to its height must be added the vertical distance between the points 0 and 1, plus the correction for linear expansion of the glass; point $(0-1) + f$ gives the actual height of the mercury which has to be corrected for temperature by subtracting the quantities *e* and *d*; hence

Pressure corrected to 0°

$$= (m - a - b - e) - (\text{point } 0 + \text{point } (0-1) + f - c - d),$$
$$= (m + c + d) - (\text{point } 0 + \text{point } (0-1) + f + a + b + e).$$

The pressures thus found must finally be corrected for the change in volume brought about by the cubical expansion of the glass; this correction is given under the heading *k*. For these experiments the correction applied to section *x* is 0·06 mm.

The jacket *d'* (fig. 4) was so constructed that it was possible to insert the delicate thermometer *t* into the inner tube *s*; and it was thus possible to obtain the values of its graduations in terms of the oxygen scale.

Table I.

The coefficient of the expansion of oxygen.

I. Temperature 0° C.

A...	length	=542 mms.	temperature =	8·9°
B...	,,	=172 ,,	,,	=15·8°
C...	,,	=118 ,,	,,	= 7·8°
D...	,,	= 16 ,,	,,	= 0·0°

Manometer (1) 719·33 (2) 719·34   Point 0   (1)   21·51 (2) 21·50
+c   ,,   0·17 ,,   0·17   +point(0−1)+f ,,   113·79   ,, 113·79
+d   ,,   0·00 ,,   0·00   +a   ,,   0·84   ,,   0·84
+0·06   ,,   0·06 ,,   0·06   +b   ,,   0·49   ,,   0·49
                                     +e   ,,   0·00   ,,   0·00
    X = 719·56   719·57
                                            Y   = 136·63   136·62

Hence P=(X−Y)=(1) 582·93 ; (2) 582·95.
Mean pressure = 582·94.

II. Temperature that of water boiling at 99·82° under a pressure
of 755·1 mms.

A... length=752 mms.   temperature= 8·5°
B...   ,,   =172 ,,   ,,   =17·4° to 17·7°
C...   ,,   =118 ,,   ,,   =48·0° to 48·3°
D...   ,,   = 16 ,,   ,,   =99·82°.

Manometer...   (1) 929·79   (2) 929·81   (3) 929·84
  +c   ,,   1·02 ,,   1·03 ,,   1·03
  +d   ,,   0·29 ,,   0·29 ,,   0·29
  +0·06   ,,   0·06 ,,   0·06 ,,   0·06

        X   =   931·16   931·19   931·22

Point 0   (1) 21·22   (2) 21·22   (3) 21·25
+point(0 −1)+f ,, 113·84   ,, 113·84   ,, 113·84
  +a   ,,   1·16 ,,   1·16 ,,   1·16
  +b   ,,   0·54 ,,   0·55 ,,   0·55
  +e   ,,   0·26 ,,   0·26 ,,   0·26

    Y   =   137·02   137·03   137·06

Hence P=(X−Y)=(1) 794·14 ; (2) 794·16 ; (3) 794·16
       +k ,,   2·30   ,,   2·30   ,,   2·30

True pressure   ,, 796·44 ,, 796·46 ,, 796·46
Mean pressure   =   796·453 mms.

III. Temperature that of chlorobenzene boiling at 99·71° under a pressure
of 292·75 mms.

A... length=752 mms.   temperature= 8·6°
B...   ,,   =172 ,,   ,,   =18·4° to 18·6°
C...   ,,   =118 ,,   ,,   =48·8° to 49°
D...   ,,   = 16 ,,   ,,   =99·71°.

Manometer...   (1) 929·59   (2) 929·60   (3) 929·62
  +c   ,,   1·04 ,,   1·04 ,,   1·04
  +d   ,,   0·29 ,,   0·29 ,,   0·29
  +0·06   ,,   0·06 ,,   0·06 ,,   0·06

        X   =   930·98   930·99   931·01

Point 0   (1) 21·21   (2) 21·23   (3) 21·23
+point(0−1)+f ,, 113·84   ,, 113·84   ,, 113·84
  +a   ,,   1·16 ,,   1·16 ,,   1·16
  +b   ,,   0·57 ,,   0·57 ,,   0·57
  +e   ,,   0·26 ,,   0·26 .,   0·26

    Y   =   137·04   137·06   137·06

Hence P=(X−Y)=(1) 793·94 ; (2) 793·93 ; (3) 793·95
       +k ,,   2·30   ,,   2·30   ,,   2·30

True pressure   ,, 796·24 ,, 796·23 ,, 796·25
Mean pressure   =   796·24 mms.

IV. Temperature that of chlorobenzene boiling under a pressure
of 719 mms.=129·60°.

A... length=816 mms.   temperature=   8·1°
B...   ,,   =172 ,,   ,,   = 17·2° to 17·4°
C...   ,,   =118 ,,   ,,   = 58·6°
D...   ,,   = 16 ,,   ,,   =129·60°.

Manometer	...	(1)	993·65	(2)	993·62	(3)	993·60	(4)	993·64
+c		,,	1·34	,,	1·34	,,	1·34	,,	1·34
+d		,,	0·37	,,	0·37	,,	0·37	,,	0·37
+0 06		,,	0·06	,,	0·06	,,	0·06	,,	0·06
	X =		995·42		995·39		995·37		995·41

Point 0	(1)	21·26	(2)	21·26	(3)	21·24	(4)	21·24
Point(0−1)+f	,,	113·86	,,	113·86	,,	113·86	,,	113·86
+a	,,	1·19	,,	1·19	,,	1·19	,,	1·19
+b	,,	0·54	,,	0·54	,,	0·53	,,	0·53
+e	,,	1·12	,,	1·12	,,	1·12	,,	1·12
Y =		137·97		137·97		137·94		137·94

P=(X−Y)	=	(1)	857·45	(2)	857·42	(3)	857·43	(4)	857·47
+k	=	,,	2·49	,,	2·49	,,	2·49	,,	2·49
True pressure		,,	859·94	,,	859·91	,,	859·92	,,	859·96

Mean pressure = 859·93.

From the above figures the coefficient of expansion of
oxygen between 0° and 100° is found to be

0·0036694,

whilst the temperature of chlorobenzene boiling

at 292·75 mms. is 99·71°;
and at 719·00 ,,   is 129·6°;

and from comparison with the thermometer in *s*, the temperature

at 468·5 mms. is 114·9°,

all of them being expressed on the oxygen scale of tempe-
rature.

Method of Experiment.

Before each experiment the volume-tube was carefully
washed with distilled water and with pure alcohol and dried;
it was then attached to a Töpler pump by means of an india-
rubber cork, a tube containing phosphoric anhydride being
interposed between the tube and the pump. The side tube *a*
(fig. 1) was attached by a short piece of thick-walled rubber

tubing to a capillary tube, dipping into clean dry mercury; this rubber tube was closed by a screw clip. The volume-tube was then placed in a long glass jacket having a boiling-bulb sealed to one side near the lower end ; it contained bromobenzene, and the vapour kept the tube at the temperature of 160° for a considerable time, during which it was from time to time exhausted by the pump. After cooling, mercury was allowed to enter through the capillary tube attached to *a,* until the volume-tube was completely filled. After this treatment no bubbles of air or gas are developed on heating to 130°, the temperature of experiment. The tube was then disconnected from the pump ; the rubber stopper with the attached drying-tube removed, and the volume-tube was placed mouth downwards in a trough of mercury.

The previously weighed bulb containing liquid was then broken at the nick under the mercury, and the two portions of the tube were immediately inserted into the mouth of the volume-tube, into which they rise, and are trapped in the cap at the top.

On account of the bulb not being completely full of liquid nothing escapes ; on the contrary, as soon as the tube is broken mercury rushes in and fills the capillary portion of the stem, which, as previously mentioned, contains only vapour at very low pressure.

The volume-tube is then removed from the trough, the mouth being closed by the finger, and placed in a nearly vertical position ; the thermometer *e* is next inserted, and the side tube attached to the manometer. The jacket is slipped over the volume-tube, and after adjusting to verticality with a plumb-line the condenser-tube at the top of the volume-tube is sealed to the gauge-tube, and the pressure under which the chlorobenzene boils is adjusted.

During the experiment the bulb remains at the top of the volume-tube, along with a small globule of mercury ; the latter gives off vapour quickly, and the heavy mercury-vapour falls and mixes with the vapour of the liquid ; it is only in this manner that it is practicable to saturate the space with mercury-vapour without protracted delay.

After the experiment is finished the rubber tube joining *a* to *k* is clipped, disconnected, and attached to a flask in which a partial vacuum is made with a Fleuss pump ; the greater part of the mercury is thus withdrawn from the volume-tube. The rubber tube is again clipped, disconnected from the flask, and attached to a capillary tube of very fine bore; on opening

the clip air gradually enters the volume-tube without splashing up mercury. When the volume-tube is full of air the cork and thermometer are removed ; any adhering globules of mercury are brushed away from its interior with a hair pencil, and the two pieces of the bulb and the globule of mercury at the top are shaken out on to a clean plate ; these are dried and weighed separately, and their volume deducted from the volume of the tube. To ensure accuracy the density of the sample of glass used to make the bulb was determined.

We have given a detailed account of all these operations because unless they are carried out in the manner described the experiment fails. It was only after many failures that the right method of manipulation was gradually evolved.

Experimental Results.

A complete record of one experiment is given, with all the necessary corrections, in order that it may be perfectly clear how these are applied. It is, however, unnecessary to quote more than the final figures for the other experiments, because any arithmetical mistakes that might have been made in applying the corrections or in making the calculations were automatically detected on plotting the curves ; many such cases have been found, but on interpreting the results graphically one point was found to lie off the curve, and on again going over the calculations it was always found to be due to a simple arithmetical slip.

In the annexed tables special sections are devoted to " Compressibility." It will be noticed that the pressures range, in these sections, from 40 mms. to about 200 mms. It would be impossible to weigh with sufficient accuracy, with the volume space practicable in such experiments, a quantity of substance which would yield the small amount of vapour necessary ; hence the quantity of substance was only roughly measured. The curve of compressibility representing rise of pv for decrease of pressure, therefore, is not continuous with the density curve ; but if its position be shifted it forms a continuation of that curve. The shift in position is easily effected since the two curves generally overlap. The factor by which the compressibility numbers must be multiplied in order to make the two curves continuous is given in each case.

TABLE II.

Methyl Alcohol. I.

Weight of substance = 0·08819 gram = m.
Volume corrections :— $a = 0·176$. $\beta = 0·099$. $(a+\beta) = 0·275$.
Temperature = 129·60° ; that of chlorobenzene boiling under 719·2 mms.
Length in mms. of A = (M−177) ; B = 168 ; C = 118.

Point	1	2	3	4	5
Length of D	21	86	138	195	246
Volume in c.cs.	228·962	201·856	179·463	156·379	134·656
$(a+\beta)$	0·275	0·275	0·275	0·275	0·275
	228·687	201·581	179·188	156·104	134·381
h	0·858	0·756	0·674	0·587	0·504
True volume	229·54	202·34	179·86	156·69	134·88
Temp. of A	16·6°	16·6°	16·2°	16·4°	16·4°
,, B	18·5°	19·0°	18·7°	19·0°	19·0°
,, C	69·0°	69·5°	69·5°	69·5°	69·3°
Manometer	450·89	555·25	650·03	760·17	881·30
+0·06	0·06	0 06	0·06	0·06	0·06
+c	1·47	1·48	1·48	1·48	1·48
+d	0·49	2·01	3·22	4·55	5·73
X=	452·91	558·80	654·79	766·26	888·57
Points	126·85	192·05	245·64	300·69	352·48
+f	0·11	0·18	0·23	0·29	0·34
+point 0	23·73	23·88	23·83	23·78	23·75
+a	0·82	1·13	1·38	1·72	2·08
+b	0·56	0·58	0·57	0·58	0·58
+e	1·10	1·10	1·10	1·10	1·10
Y=	153·17	218·92	272·75	328·16	380·33
P=(X−Y)	299·74	339·88	382·04	438·10	508·24
Pv/mT	1940·2	1939·2	1937·8	1935·7	1933·2

Methyl Alcohol. II.

Weight of substance = 0·10427 gram.   Temperature = 129·60°*.

Volume in c.cs.	229·61	202·40	179·92	156·75
Pressure in mms. ...	363·82	401·19	451·01	517·11
Pv/mT	1937·6	1936·6	1935·3	1933·2

Compressibility.

Volume	32·162	19·267	12·279
Pressure	75·893	126·59	198·50
Factor	0·79735	0·79735	0·79735
Pv/mT	1946·2	1944·9	1943·5

Pv/mT at zero pressure = 1947·3.

* In this experiment a sample of chlorobenzene was used as a jacket, which was afterwards found to be impure, for its boiling-point varied by some tenths of a degree; it also contained moisture.

Ethyl ether, Sample A. I.

Weight of substance=0·17196=*m*. Temperature=129·6°.

Volume in c.cs.	229·48	202·27	179·79	156·62	134·82	95·18
Pressure in mms. ...	256·06	287·08	322·68	370·00	429·24	605·48
P*v*/*m*T	839·82	839·66	838·99	838·04	836·90	833·40

Weight of substance the same. Temperature=99·71°.

Volume in c.cs.	229·28	202·09	179·64	156·47	134·70	95·096
Pressure in mms. ...	234·21	265·57	298·47	342·06	396·74	559·38
P*v*/*m*T	838·94	838·50	837·65	836·22	834·90	831·06

Sample B. II.

Weight of substance=0·18303 gram=*m*. Temperature=99·71°.

Volume in c.cs.	229·21	202·02	179·56	156·41	134·63	95·024
Pressure in mms. ...	249·37	282·64	317·68	364·14	422·38	595·35
P*v*/*m*T	838·97	838·10	837·32	835·99	834·66	830·37

Sample C. III.

Weight of substance=0·15806 gram=*m*. Temperature=129·6°.

Volume in c.cs.	279·76	134·77	95·136
Pressure in mms. ...	296·84	395·03	557·61
P*v*/*m*T	839·54	837·62	834·64

Compressibility at 129·6°.

Volume	57·394	30·274	18·136	11·551
Pressure	40·047	75·893	126·59	198·60
Factor............	0·36678	0·36678	0·36678	0·36678
P*v*/*m*T	843·03	842·71	842·08	841·00

P*v*/*m*T at zero pressure=843·45.

Toluene. Sample A. I.

Weight of substance=0·22343 gram=*m*. Temperature=99·71°.

Volume in c.cs.	229·28	202·10	179·64	156·49	134·70	
Pressure in mms. ...	242·03	273·88	307·70	352·02	407·33	
P*v*/*m*T	667·24	665·54	664·63	662·46	659·72	

Weight the same. Temperature=114·9°.

Volume in c.cs.	229·37	202·18	179·72	156·56	134·76	95·141
Pressure in mms. ...	252·13	285·53	320·63	367·20	425·09	596·63
P*v*/*m*T	668·09	666·89	665·70	664·14	661·80	655·76

Weight the same. Temperature=129·6°.

Volume in c.cs.	229·48	202·27	179·80	156·62	134·82	95·182
Pressure in mms. ...	262·05	296·89	233·48	381·97	442·36	620·99
P*v*/*m*T	669·32	668·39	667·36	665·85	663·79	657·87

Sample B. II.

Weight of substance=0·24181 gram. Temperature=129·6°.

Volume in c.cs.	229·50	202·29	179·81	95·079
Pressure in mms. ...	283·45	321·13	360·74	671·78
P*v*/*m*T	669·00	668·06	666·83	656·88

Compressibility at 129·6°.

Volume	32·388	19·386	12·337
Pressure	75·893	126·59	198·50
Factor............	0·27411	0·27411	0·27411
P*v*/*m*T	673·49	672·40	671·00

P*v*/*m*T from experiments I. and compressibility=674·40

,, ,, II. ,, =674·73

Mean...=674·57

Benzene. Sample A. I.

Weight of substance=0·26700 gram. Temperature=129·60°.

Volume in c.cs.	179·937	162·09	142·87	126·39	107·05
Pressure in mms.......	453·44	502·72	569·34	641·99	755·41
Pv/mT	789·48	788·49	787·09	785·17	782·52

Weight the same. Temperature=99·71°.

Volume in c.cs.	179·83	162·00	142·79	126·34
Pressure in mms.	418·86	464·19	525·46	592·44
Pv/mT.......................	787·39	786·10	784·30	782·40

Sample B. II.

Weight of substance=0·18606 gram. Temperature=129·60°.

Volume ...	229·24	202·08	179·62	156·48	134·71	114·69	95·075
Pressure ...	239·35	271·31	304·85	349·22	404·88	474·41	570·09
Pv/mT	792·24	791·61	790·64	789·03	787·52	785·64	782·61

Weight the same. Temperature=99·71°.

Volume ...	229·46	202·25	179·77	156·62	134·83	114·36	98·80
Pressure ...	258·84	293·32	329·62	377·82	438·12	513·50	617·98
Pv/mT	793·82	792·92	791·98	790·88	789·52	787·86	785·98

Compressibility at 129·6°.

Volume	55·505	29·276	17·532	11·162
Pressure.........	40·047	75·893	126·59	198·50
Factor............	0·35875	0·35875	0·35875	0·35875
Pv/mT............	797·44	797·09	796·20	794·88

Pv/mT at zero pressure=798·1.

Hexane. Sample A. I.

Weight of substance=0·29424 grams=m. Temperature=99·71°

Volume in c.cs.	218·37	198·47	153·86	132·93
Pressure in mms.	358·38	393·63	505·15	582·28
Pv/mT.......................	714·54	713·30	709·63	706·71

Weight the same. Temperature=114·90°.

Volume in c.cs.	218·47	198·56	153·93	132·99
Pressure in mms.	373·27	409·98	526·33	607·21
Pv/mT.......................	715·38	714·12	711·81	708·37

Weight the same. Temperature=129·6°.

Volume in c.cs.	239·11	218·56	198·64	175·73	153·99
Pressure in mms.	355·02	388·07	426·48	480·98	547·55
Pv/mT	717·44	716·83	715·99	714·33	712·63

Sample A. II.

Weight of substance=0·19298 gram. Temperature=129·60°.

Volume in c.cs.	229·43	202·22	156·58	134·77	114·78	95·15
Pressure in mms. ...	243·67	276·04	355·54	412·34	483·01	580·73
Pv/mT	720·42	719·34	717·37	716·12	714·43	711·93

Weight the same. Temperature=99·71°.

Volume	229·23	202·05	179·59	156·44	134·65	114·68	95·133
Pressure	225·37	255·21	286·84	328·59	380·76	445·88	535·54
Pv/mT............	719·21	717·85	716·97	715·62	713·77	711·67	708·65

Compressibility at 129·60°.

Volume	52·472	27·660	16·562	10·540
Pressure.........	40·047	75·893	126·59	198·50
Factor............	0·3446	0·3446	0·3446	0·3446
Pv/mT............	724·24	723·51	722·61	721·10

Normal Octane.

Weight of substance=0·36007 gram.   Temperature=125·6°.

Volume in c.cs.	239·13	198·62	154·04	133·17
Pressure in mms.	320·14	383·73	490·78	564·74
Pv/mT	534·04	531·68	527·16	524·08

Weight the same.   Temperature=131·4°.

Volume in c.cs.	198·90	175·07	153·42	132·55
Pressure in mms.	389·24	438·15	498·31	573·26
Pv/mT	531·66	529·44	527·66	524·84

Compressibility.

Volume in c.cs.	57·740	30·386	18·155	11·581
Pressure in mms.	40·047	75·893	126·59	198·50
Factor	0·23514	0·23514	0·23514	0·23514
Pv/mT	543·71	542·23	540·40	538·20

Pv/mT at zero pressure=544·4.

Di-isobutyl.

Weight of substance=0·26359 gram.   Temperature=129·60°.

Volume	229·27	202·06	179·58	156·40	134·61	94·97
Pressure	249·57	282·67	317·53	363·44	420·83	590·45
Pv/mT	539·83	538·86	537·97	536·27	534·43	529·03

Weight the same.   Temperature=114·90°.

Volume	229·18	201·99	179·53	134·57	94·95
Pressure	240·03	271·68	304·94	404·01	565·69
Pv/mT	538·66	537·37	536·06	532·39	525·96

Compressibility.

Volume	46·181	24·316	14·539	9·241
Pressure	40·047	75·893	126·59	198·50
Factor	0·29494	0·29494	0·29494	0·29494

Pv/mT at zero pressure=546·2.

Inasmuch as the value of this research is absolutely dependent on the purity of the liquids employed it appears necessary to give a rigorous proof of the fact. Mere confirmation of the concordance of any two sets of determinations with the same sample of any one substance would be convincing only as regards the accuracy of the method; it is necessary to prove that, at least in several cases, samples of the same substance, differently prepared or differently treated, yield practically identical results. With the substances kindly furnished by Prof. Young redistillation from phosphoric anhydride was the only means of purification which we could attempt, for we had only small amounts at our disposal; an example of the effect of thus treating normal hexane will be found in the following table. It did not appear obvious how we could improve on a sample of the purest thiophene-free benzene of absolutely constant boiling-point which had been purified by frequent recrystallization. Nor could we devise another and better way of obtaining a reliable sample of methyl-alcohol than by hydrolysing methyl oxalate, itself prepared from "pure" methyl-alcohol, and drying the resulting alcohol by help of the best lime obtainable. It is true it might have been recrystallized, but as the only likely

impurity was water, the handling of the sample would have reintroduced moisture, and the purification would have been an illusory one. We therefore rely on results obtained with ether and with toluene in order to show that the deviation from what may be conveniently termed " theory " is real, and not due to the presence of traces of impurity.

In order that the results of different series of measurements shall be comparable (for the pressures are not the same in any two observations) curves were constructed independently from the observations made with each sample, and the values of pv at certain definite pressures were read off from these curves, on which the values of pv/T were plotted on a very open scale. We give first comparative results with two samples of hexane, or rather with two tubes filled at the same time from the same sample; the observations have been made at two temperatures. This serves to show the accuracy of which the method is capable.

Hexane.

Temp.	Pressure.	$Pv/mT.$		Difference.
		A.	B.	
129·60°	540	712·90	712·93	0·03 = 1 part in 24000
	450	715·16	715·20	0·04 = 1 part in 18000
	390	716·65	716·70	0·05 = 1 part in 14500
	300	718·93	719·00	0·07 = 1 part in 10300
99·71°	480	710·42	710·40	0·02 = 1 part in 35000
	420	712·40	712·42	0·02 = 1 part in 35000
	360	714·45	714·50	0·05 = 1 part in 14000
	300	716·46	716·53	0·07 = 1 part in 10000

The greatest difference between any two measurements is thus 1 part in 10000. With benzene, the results were obtained from two different portions, distilled at different times from the same stock.

Benzene.

Temp.	Pressure.	$Pv/mT.$		Difference.
		A.	B.	
129·60°	680	784·34	784·42	0·08 = 1 part in 9800
	560	787·00	787·10	0·10 = 1 part in 7900
	440	789·65	789·77	0·12 = 1 part in 6600
	320	792·32	792·42	0·10 = 1 part in 8000
99·71°	680	779·58	779·82	0·24 = 1 part in 3200
	560	783·02	783·28	0·26 = 1 part in 3000
	440	786·50	786·76	0·26 = 1 part in 3000
	320	789·96	790·23	0·27 = 1 part in 3000

Here the limit of accuracy is 1 part in 3000.

The methods of preparation of the three samples of ether have already been mentioned on p. 496. Sample A was distilled from phosphoric anhydride; sample B was left in contact with phosphoric anhydride for some days, and had been shaken with metallic mercury before distillation; while sample C had been recrystallized, and the crystals alone used.

Ether.

Temp.	Pressure.	Pv/mT.			Difference.
		A.	B.	C.	
129·60°	500	833·70	...	833·78	0·08 = 1 part in 10000
	400	835·94	...	836·03	0·09 = 1 part in 9200
	300	838·20	...	838·28	0·08 = 1 part in 10000
	250	839·36	...	839·43	0·07 = 1 part in 12000
99·71°	500	830·03	830·25	...	0·22 = 1 part in 3800
	400	833·03	833·24	...	0·21 = 1 part in 4000
	300	836·00	836·19	...	0·19 = 1 part in 4300
	250	837·50	837·68	...	0·18 = 1 part in 4600

These results show that the ether of all the samples may be regarded as pure. The maximum difference is about 1 part in 4000.

Sample A of toluene was prepared from paratoluidine; sample B from bromobenzene and methyl iodide; sample B was subsequently recrystallized, as described on p. 495.

Toluene.

Temp.	Pressure.	Pv/mT.		Difference.
		A.	B.	
129·60°	600	658·61	658·93	0·32 = 1 part in 2000
	480	662·62	662·94	0·32 = 1 part in 2000
	360	666·43	666·78	0·35 = 1 part in 1900
	270	669·12	669·36	0·24 = 1 part in 2800

The greatest difference is observable with toluene; it is probable that the recrystallized sample was purer than the other. But after all, the difference is by no means large enough to affect the conclusions to be drawn.

The next step is to deduce from the measurements given in Table II. the values of *pv/mT* at zero pressure, when, if Daniel Berthelot's contention is correct, equal volumes may be expected to contain equal numbers of molecules; a comparison of the true molecular weights should then be possible.

The values of pv/mT were therefore transferred to curves, which are reproduced in Pl. XX. : and the curves representing compressibilities, having also been mapped, were shifted in position, so as to become continuous with those in which the true value, and not merely the variation of pv/mT with pressure, was known. From these two sets of data the complete curves were drawn. The lower limit of pressure lies approximately at from 40 to 60 millimetres.

These curves were extrapolated until they cut the axis of zero pressure. The justice of thus extrapolating the curves will be considered later.

From the weight of a litre of oxygen at 0° C., namely 1·42961, and from the found coefficient of expansion with temperature, 0·0036694, the molecular weights of the substances employed were calculated, as shown in Table IV.

Table IV.
Molecular Weights at 129·6°.

K = value of Pv/T for one gram-molecule of oxygen at 129·6° at zero pressure
= 62423.
K' = value of Pv/T for one gram of substance at 129·6°, and at zero pressure.
K/K' = found molecular weight = M'.
M = calculated molecular weight on the bases O = 32, C = 12, H = 1·007.

Substance.	K'.	M'=K/K'.	M.	Difference 1 part in
Methyl Alcohol	194·73	32·056	32·03	1111
Ethyl Ether	84·345	74·01	74·07	1250
Hexane	72·475	86·13	86·10	2500
Di-isobutyl	54·62	114·29	114·13	714
Normal Octane..........	54·44	114·66	114·13	213
Benzene..................	79·81	78·21	78·04	454
Toluene	67·44	92·56	92·06	182

The absolute value of the molecular weight M' in each case depends on the coefficient of expansion accepted for oxygen ; and this, as has been shown by researches (unpublished) made in the laboratory of University College, varies with the pressure. Were the number 0·003675 to be accepted instead of 0·36694, the value of M' for methyl-alcohol would be 32·105, and that for ether 74·12. But such an assumption would have no influence on the main question, viz. the variation in the discrepancies between the values of M and M'. From the last column these are seen to vary from 1 part in 1250 (minus) to 1 part in 182 (plus). The discrepancy is made more apparent by accepting for ether the "theoretical" molecular weight ; the discrepancy is then :—

Methyl alc.	Ether.	Hexane.	Di-isobutyl.	Norm. Octane.	Benzene.	Toluene.
1 part in 588...	∞	833	455	182	333	159

Now the greatest divergence between results with different samples occurs in the case of toluene; and it is only 1 part in 1900; it is clear, therefore, that the discrepancy cannot be attributed to impurity of material, nor to inaccuracy in experiment. To what cause, then, is it to be ascribed ? We have given on p. 494 several hypotheses which may explain the divergence between found and "theoretical" results ; these shall now be considered.

First, it is evident that Avogadro's law does not hold for vapours, even under the ideal condition of zero pressure. There would appear to be only three possible causes which would explain the anomaly.

We regard it as unlikely that the volume of the vapour is decreased by the adherence of a film to the glass walls of the volume-tube. It is to be expected that this phenomenon would become more observable at high, than at low, pressures ; and one of the authors, in conjunction with Prof. Young, has investigated the behaviour of ether and methyl-alcohol at very high pressures, and Prof. Young has himself examined the hydrocarbons without finding any ground for entertaining this supposition. On the other hand, evidence that it does take place in the case of water has been obtained by them. It is manifested by an apparently continuous, instead of an abrupt, change on passing from the gaseous into the liquid condition. It has an effect in modifying what is often termed the "Andrews" diagram in the same manner as if a permanent gas were present along with the substance under experiment. To quote from the paper mentioned (Phil. Trans. 1892, A, p. 113):—"As the vapour-pressure is approached, the curves, instead of cutting the vapour-pressure line so as to form an angle, as is the case with the other liquids which we have examined, gradually turn and run nearly parallel to the vapour-pressure line at a somewhat lower pressure. When a considerable amount of liquid has condensed the true vapour-pressure is reached." Now this phenomenon increases with rise of pressure ; but in the experiments of which an account has been given in this paper, the curvature is such as to show that the rate of increase of pv/T is less at low than at high pressures. Not merely, then, is the phenomenon absent in the class of liquids under experiment, but even if it should be conceived to be present its variation with pressure is in the wrong direction. We think, therefore, that it may be dismissed from further consideration.

That the formation of complex molecular groups is the cause of the high density of these vapours is certainly a conceivable supposition, but hardly a tenable one. For methyl-alcohol, which is known to consist of complex groupings in

the liquid state, exhibits, if ether is taken to possess the theoretical molecular weight, a divergence of only 1 part in 588; whereas the hydrocarbons, which show no such tendency, are characterized by a greater discrepancy, all except hexane.

Another supposition which deserves consideration is that although there may be no molecular groupings, yet the molecules attract one another, and thus lower the value of the product Pv. And it is conceivable that the nearer the boiling-point of the substance to the temperature of experiment, 129·6°, the more this attraction should manifest itself. This possibility finds support to some extent by the results. The order of divergence does present a general correspondence with the order of boiling-points; but the numbers are reversed between three pairs of substances, as may be seen from the table which follows :—

	Ether.	Hexane.	Methyl alc.	Di-iso-butyl.	Benzene.	Octane.	Toluene.
1 part in	...	833	588	455	333	182	159
Boiling-pt.	35°	69°	66°	109°	80°	126°	110°

The boiling-points of methyl-alcohol and hexane, of benzene and di-isobutyl, and of toluene and octane follow the wrong order. Still, if substances belonging to the same class be compared, hexane, di-isobutyl, and octane show correspondence; and so, too, do benzene and toluene.

Another plan of attacking the problem is to compare the values of the expression $d(\mathrm{P}v)/d\mathrm{P}$ for all the substances at certain definite pressures and temperatures, the same for all; in other words, to compare the slopes of the curves with the divergences from the " theoretical " vapour-densities. This is done in the following table :—

	Discrepancy 1 part in	Temp.	Relative values of $d(\mathrm{P}v)/d\mathrm{P}$ at the pressures.		
			500 mms.	300 mms.	100 mms.
Me. Alc....	588	130°	0·0000197	0·0000160	0·0000114
Ether ...	...	130	0·0000282	0·0000271	0·0000184
		100	365	347	
Benzene ...	333	130	286	278	208
		100	376	348	
Hexane ...	833	130	379	321	277
		115	422	402	
		100	480	434	
Toluene...	159	130	466	420	282
		115	548	512	
		100	664		
Di-isob. ...	455	130	623	595	480
		115	740	710	
Octane ...	182	130	720	656	593

It is clear that the order of discrepancy bears no relation to the order of slope at any of the temperatures or pressures.

With a theoretically "perfect" gas, in the ideal state in which its molecules occupy no space and exercise no attraction on one another, the curve representing Pv/T should run parallel to the pressure-axis. This condition is nearly satisfied by the gases examined by M. Daniel Berthelot, all of which were weighed at temperatures much above their critical points. One would have imagined that at sufficiently low pressures the distance between the molecules of vapours such as those examined would be so great that neither molecular attraction nor the size of the molecules would influence the result; and it might have been expected that the curves showing variation of the value of Pv/T with pressure should become parallel to the pressure-axis at sufficiently low pressure. It is, of course, possible that at pressures lower than 40 millimetres—the lowest measured—the curves might become parallel. A glance at figure 5 (Pl. XX.) will show this. But it is to be noticed that no such change in the slope of the curve in extrapolating to zero pressure would have had any important influence on the discrepancy. We have for all these reasons been unable to establish any connexion between the volatility of the substances examined and their divergence from the "theoretical" molecular weights.

The last supposition, viz. that it is possible that the atomic weights of the elements may depend on the proportion in which they are present in the compounds which contain them, is added only for the sake of completeness. Even if it be considered, there is no regularity in the cases examined which would lend probability to the hypothesis.

It must therefore be concluded that the determination of the density of a vapour does not serve as a means of arriving at a conclusion regarding the accurate atomic weights of the elements present in the compound.

LXI. *On a Modification of the Plug Experiment.* By EDGAR BUCKINGHAM*.

IN spite of various advances in our knowledge of the other quantities concerned in the computation of the relation of gas scales to the absolute scale of temperature, we are, as regards the Joule-Thomson effect, almost exactly in the same position as when Joule and Thomson finished their plug experiments. No mathematical ingenuity can tell us anything

* Communicated by the Author.

certain about the desired corrections, outside the range which they covered. The gas scale is now in use over so vastly wider an interval than forty years ago, that it is very desirable indeed that the plug experiment should be repeated over the original range and extended as far as possible towards both low and high temperatures. At the same time, the greatest attainable accuracy should be sought. The weak point of the original method seems to me to lie in the fact that the passage through the plug was not strictly adiabatic. By a slight modification, suggested by the ease and accuracy with which electrical measurements may now be carried out, this defect of the original method may be avoided.

Instead of making the passage through the plug adiabatic, let us make it isothermal. Instead of the fall of temperature, we then have to measure the energy which must be supplied to keep the temperature from falling. Let a platinum heating-coil be embedded in the plug, and by adjusting the current in this coil let us establish a steady state such that there is no fall of temperature as the gas passes through the plug.

This method has several advantages. It reduces the measurement of temperature to a null method—we place two platinum thermometer coils in the stream of gas, one before and one after the passage through the plug; we balance these as two arms of a Wheatstone's bridge, and we regulate the heating current so as to preserve the balance. Beside thus avoiding the correction for thermal leakage into the plug, we furthermore avoid *all* differences of temperature which may, in the original form of the experiment, cause a thermal leakage against which it is difficult to guard.

The quantities to be measured, in addition to the fall of pressure, the rate of flow of gas, and the *uniform* temperature of the whole apparatus, are now merely the heating current needed for the steady state and the fall of potential in the heating coil. The originally difficult part of the measurement is thus performed electrically, and such measurements are easy and accurate if our standards are really correct. As regards the accuracy of the absolute values obtainable, we substitute two electrical units for a specific heat and Joule's equivalent; and this is not, I think, to be regarded as a disadvantage, in the present state of our knowledge.

As a further precaution, we may adopt the guard-ring principle and use two independent concentric plugs; if, in the steady state, both plugs are giving similar results, we are reasonably sure that the inner plug is not subject to any thermal leakage from outside. It is easy to distribute

thermometer coils in the various places where different temperatures are to be expected, and so test the uniformity of temperature throughout the apparatus in some detail; and such differential measurements, with the platinum coils arranged after the manner of the bolometer, may be given a very high degree of sensitiveness.

The fundamental equation of the plug experiment, performed in the manner outlined above, assumes a particularly simple form: Let v, p, and ϵ be the specific volume, the pressure, and the specific internal energy of the gas before the passage through the plug; and let v', p', and ϵ' be the corresponding values after the passage. Let $\lambda\delta p$ be the energy supplied per unit mass of gas to keep the process isothermal when the fall of pressure is δp. For a finite fall of pressure we shall then have

$$\epsilon' = \epsilon + pv - p'v' + \int_{p'}^{p} \lambda dp,$$

or for an infinitesimal fall of pressure

$$\delta\epsilon = -\delta(pv) + \lambda\delta p.$$

Letting θ represent the absolute temperature, and η the entropy of unit mass, we have, for any reversible change of state,

$$\delta\epsilon = \theta\delta\eta - p\delta v.$$

The passage through the plug is not reversible, but since ϵ and η are functions only of the coordinates which define the instantaneous state of the system, the changes of entropy and energy *of the mass of gas* are the same as if the process had been reversible, and we may apply the formulæ for reversible processes to the case in hand. Equating the two values of $\delta\epsilon$, we have

$$\delta(pv) - \lambda\delta p = -\theta\delta\eta + p\delta v,$$

whence

$$-\theta\delta\eta = v\delta p - \lambda\delta p,$$

or

$$-\theta\left(\frac{\partial\eta}{\partial p}\right)_{\theta} = v - \lambda.$$

But we have the familiar thermodynamic relation

$$\left(\frac{\partial\eta}{\partial p}\right)_{\theta} = -\left(\frac{\partial v}{\partial\theta}\right)_{p},$$

whence, by comparison with the last equation, we obtain

$$\theta\left(\frac{\partial v}{\partial\theta}\right)_{p} = v - \lambda,$$

or

$$\log \frac{\theta_2}{\theta_1} = \int_{v_1}^{v_2} \frac{\delta v}{v - \lambda},$$

where v_1 and v_2 are the specific volumes of the gas at the constant pressure p and at the absolute temperatures θ_1 and θ_2.

We have thus a relation between the absolute temperature and the temperature measured by a constant-pressure thermometer filled with the gas under investigation. It will be noticed that, as was stated above, neither J nor C_p enters into this equation, their place being taken by λ, the measured value of which depends upon two electrical units.

The method thus briefly sketched is, of course, not applicable to hydrogen at temperatures above $-80°$ C., which Olszewski has found to be the inversion temperature of the Joule-Thomson effect for that gas. But for other gases, below their as yet unknown inversion temperatures, the method appears to have certain advantages.

U.S. Department of Agriculture,
Washington, D.C., June 5th, 1903.

LXII. *Single-Piece Lenses.* *By* Thomas H. Blakesley [*].
[Plate XXI.]

IF the definition of a lens is taken to be a mass of refracting material contained between two spherical surfaces, then any lens is completely determined in shape and size when the radii of curvature of its two end surfaces and the distance between them are given. The dimensions which are measured at right angles to the axis do not affect the focussing properties of the lens except in the second order of small quantities, and such dimensions are not brought into consideration in what follows.

If the kind of material in reference to any kind of light, as defined by the index of refraction of the material for that light (μ) is supposed given, there remain only the three linear matters, viz. the two radii of curvature of the surfaces and the distance between the surfaces.

Let these be symbolized under the letters r_1, r_2, and d.

The third may be considered as always positive, and as to the radii, let them be considered as having a positive value when the light encounters the surfaces upon the concave side; in other words, if the light as it passes generally along the

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axis passes the centre of a surface *before* it encounters the surface itself.

The variables r_1, r_2, and d being three in number may then be looked upon as sufficient to determine all the properties relating to focussing for small central pencils of light.

If we contemplate the relations $\frac{r_1}{d}$, $\frac{r_2}{d}$, and symbolize them under the letters x and y, or, in other words, if we consider d as unity, we reduce the variables to two in number, and can then represent all the ordinary properties of lenses under a simple system of coordinates, where x implies the relation of the radius of the first surface to the thickness or length of the lens, and y a similar magnitude applying to the second surface.

The first diagram is based upon this principle. If we take any point upon the paper its position will indicate some particular lens, and all lenses having some one the same property will lie upon a line drawn upon the diagram. If two such lines meet the point of intersection will correspond to a lens having the properties appertaining to both the lines. The general characters of a lens depend upon its shape and not upon its scale. But if the general characters of a lens are known, and the point on the diagram determined, the strength of a lens is then simply dependent upon the scale, and can be raised as desired.

The diagram presented is based upon the supposition that $\mu = 1\cdot5$, a supposition which, though it was long employed for glass as sufficiently exact for academical approximation in the casual text-books of Cambridge University, when it was certainly not so, has, under the laborious care of German experiment, become not inapplicable to the glasses of low dispersion produced by Schott.

It may make the use of such a diagram simpler to take an example or two.

Suppose the condition which it is desired to obtain to be that the second principal focus shall lie upon the second surface.

Then from any table of optical properties, *e. g.* that given of lens quantities in my 'Geometrical Optics,' this condition implies that the following equation must hold good :—

$$\overline{\mu - 1}\, d + \mu r_1 = 0.$$

Hence

$$\frac{r_1}{d} = -\frac{\mu - 1}{\mu},$$

or if $\mu = 1\cdot5$,

$$x = -\frac{1}{3}.$$

The line is therefore drawn parallel to the axis of y and at a distance $-\frac{1}{3}$ from it.

The condition that the focal length shall be infinite is

$$\frac{\mu-1}{\mu} d = r_2 - r_1.$$

Hence $y - x = \overline{\dfrac{\mu-1}{\mu}}$, or $\dfrac{1}{3}$ if $\mu = 1.5$.

This straight line is shown upon the diagram and is called the "Telescope Line."

The condition that the focal length shall not change for small variations in the value of the index is of course found by forming the equation $\dfrac{df}{d\mu} = 0.$

This condition is given by the equation

$$\frac{\mu^2-1}{\mu^2} \cdot d = r_2 - r_1.$$

Hence
$$y - x = \frac{\mu^2-1}{\mu^2}, \text{ or } \frac{5}{9} \text{ if } \mu = 1.5.$$

This line is marked "Focal Length a minimum," and is of course parallel to the Telescope Line. All lenses upon this line possess a high degree of achromatism.

A ray may so pass through a lens that it encounters the two surfaces in such a way as to receive equal deviation in the same direction at each surface, and therefore on the whole minimum deviation, as in the symmetrical passage of a ray through a prism. I have shown elsewhere how this can take place for all the rays of a pencil emanating from any point on the axis of a lens.

If the point is at an infinite distance the condition is

$$\overline{2-\mu}\, r_2 + \mu r_1 + \overline{\mu-1}\, d = 0.$$

Hence
$$\overline{2-\mu}\, y + \mu x + \overline{\mu-1} = 0,$$

and if $\mu = 1.5$ this becomes

$$y + 3x + 1 = 0.$$

The line is marked upon the diagram "minimum deviation for rays from infinity."

It passes through the sphere point $x = -\frac{1}{2}, y = \frac{1}{2}$ as all such lines must.

The curve which results from making the position of the

second principal focus invariable for a small change in the index is marked $\dfrac{dv_0}{d\mu} = 0$.

It is an hyperbola whose asymptotes make an angle of 45° one with the other. One of these asymptotes is the straight line already considered marked "Focal Length a minimum."

The curves obtainable from such conditions as $f = n.d.$ *i. e.* for lenses in which the focal length has a specified relation to d, are rectangular hyperbolas whose major axis is the line

$$y + x = 0.$$

The numbers corresponding to n are marked along this line. The points are the centres of the corresponding hyperbolas.

The semiaxes of these hyperbolas are given by the expression

$$\sqrt{\frac{2n + 3n^2}{6}}.$$

The second diagram shows the shapes and the general optical properties of certain lenses which have been constructed on these principles.

The conditions selected are described above, and the relations of the radii of curvature and the focal length to the thickness below, each lens.

The uniform scales indicated along, above, and below the axes give the magnification values for any point, the upper scale for light before, the lower for light after, passage through the lens.

It should also be mentioned that the value of μ selected for the computation of the actual quantities was not always 1·5, but depended upon the glass selected.

The first, third, and fourth make very good eyepieces, especially the first, which has all the advantages of a Huyghens' eyepiece, to which it bears a close analogy in its ruling conditions.

LXIII. *On Astigmatic Aberration.*
By R. J. Sowter, *B.Sc., A.R.C.S.*[*]

THE paper affords a simple explanation for some of the shadow phenomena observed by Prof. S. P. Thompson in his experimental researches on the aberration of lenses[†],

namely, in those experiments in which the aberration is wholly or in part astigmatic. In investigating the properties of non-homocentric pencils or beams with one or more kinds of aberration present, it is necessary to determine what part is played by each of the several kinds of aberration. It is of interest to know what is the effect of, or part played by, astigmatic aberration in the production of shadows by objects placed in, or moved about in, non-homocentric pencils or beams. It will be found that the presence of astigmatic aberration in a non-homocentric beam is accountable merely for the twist or rotation shown in the shadow or image formed by an object placed in or moved about in the beam.

To examine mathematically the twists produced in a pure astigmatic non-homocentric pencil or beam, the symmetrical standard pencil, such as is investigated by Heath (Geometrical Optics, p. 142), is assumed.

If the axis of the beam is the axis of z, and the origin is at the intersection of this axis with a selected orthogonal surface or plane cutting the beam in an ellipse of which the axes are 2α, 2β, respectively, the equation to the bounding surface of the beam is

$$\frac{x^2}{\alpha^2\left(1-\dfrac{z}{f_1}\right)^2} + \frac{y^2}{\beta^2\left(1-\dfrac{z}{f_2}\right)^2} = 1*, \quad . \quad . \quad . \quad (1)$$

where f_1 and f_2 are the distances of the focal lines from the selected plane.

If ξ, η, are the coordinates of a point in the selected ellipse it is easily shown that

$$\xi = \frac{x}{1-\dfrac{z}{f_1}} \quad \text{and} \quad \eta = \frac{y}{1-\dfrac{z}{f_2}},$$

and if the eccentric angle of the point $(\xi\,\eta\,0)$ in the ellipse is ϕ then

$$\xi = \alpha \cos \phi,$$
$$\eta = \beta \sin \phi,$$

and

$$x = \alpha\left(1-\frac{z}{f_1}\right)\cos \phi,$$
$$y = \beta\left(1-\frac{z}{f_2}\right)\sin \phi.$$

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* Heath, Geom. Optics, § 142, p. 161.

So that if θ is the inclination of the radius to the point $x\,y\,z$

$$\tan \theta = \frac{\beta}{\alpha} \cdot \frac{\left(1 - \frac{z}{f_2}\right)}{\left(1 - \frac{z}{f_1}\right)} \cdot \tan \phi.$$

One section of the beam being assumed elliptical it follows from the equation (1) of the bounding surface that all sections of the beam are elliptical, the circular sections being particular forms.

Let the lengths of the primary and secondary focal lines be $2\,a$ and $2\,b$ respectively, and let the focal interval be δ.

The equation to the bounding surface of the beam, the origin being transferred to the middle point of the primary focal line, can be expressed as

$$a^2(\delta - z)^2(\delta^2\,y^2 - b^2\,z^2) + b^2\,\delta^2\,x^2\,z^2 = 0.$$

This equation is the equation of the skew surface generated by the movement of a variable ellipse between the focal lines, the axes of the generating ellipses remaining parallel to the focal lines.

Through any point $(x\,y\,z)$ on this skew surface there is one generator or ray, viz. :—

$$x = \frac{a \cos \phi}{\delta}(\delta - z)$$

$$y = \frac{b \sin \phi}{\delta}(z).$$

This ray intercepts the primary and secondary focal lines at distances from the axis of the beam of $a \cos \phi$ and $b \sin \phi$ respectively.

If θ is the inclination of the radius to the point $x\,y\,z$

$$\tan \theta = \frac{b\,z}{a(\delta - z)} \cdot \tan \phi.$$

Any ray or generator of the astigmatic surface has for its equation

$$\phi = \text{constant} = \tan^{-1} \frac{a\,\epsilon}{b\,\gamma},$$

where γ and ϵ are the intercepts of the ray or generator on the focal lines, the intercepts being measured from the axis of the beam.

This simple formula $\phi = \text{constant}$ expresses in a most convenient form the general equation for any ray of a symmetrical astigmatic beam, and exemplifies for such a ray the property which causes the twisted effects observed in connexion with astigmatic beams.

It is obvious since $\gamma = a \cos \phi$ and $\epsilon = b \sin \phi$ that in constructing a string model of the astigmatic surface, points in the focal lines that correspond and that can be connected by a string are points with a phase-difference of $\frac{\pi}{2}$ in harmonic oscillations along the focal lines.

To investigate the shadow produced by an object placed in an astigmatic beam, such as has been dealt with above, let the receiving screen SS′ be beyond the secondary focal
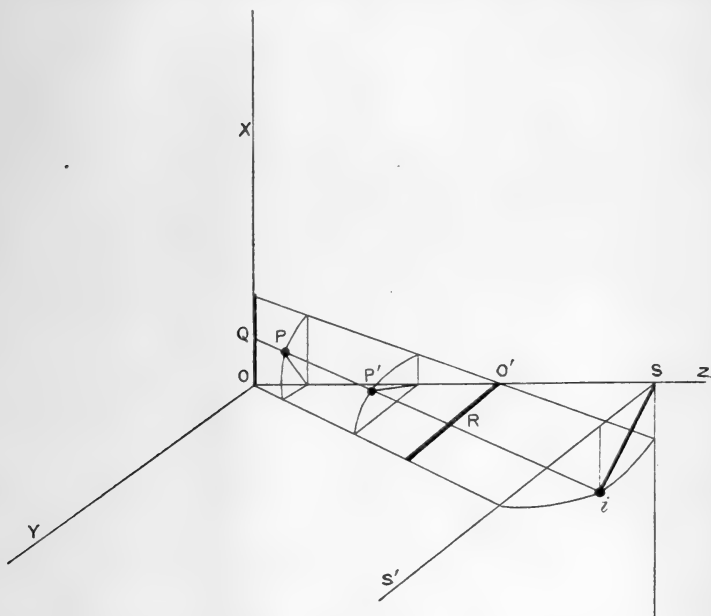
line at a distance s from the plane O X Y. Then the ray Q P P′ R, or ϕ_1, will intersect the screen SS′ in a point i, the coordinates of which are

$$\begin{cases} x = -\left(\dfrac{s-\delta}{\delta}\right)\gamma, \\[2mm] y = \dfrac{s\,\epsilon}{\delta}, \\[2mm] z = s, \end{cases}$$

since $\phi_1 = \tan^{-1}\dfrac{a\,\epsilon}{b\,\gamma}$, and ϕ_1 is the eccentric angle of the point i in the ellipse on the receiving screen.

A small point-like object placed in the beam at P will cast a small point-like shadow on the screen, and the shadow with the positions taken will be inverted in respect to top and bottom, and will be unchanged in respect to side and side.

A straight wire placed say at 45° across the beam and cutting the axis of the beam will depict on the screen a straight-line shadow in the ellipse on the screen, and this shadow is at a definite inclination. The inclination depends on the relative distances and positions.

If the wire is considered to intercept the bounding surface of the beam in a point, the eccentric angle of which is ϕ, the line shadow in the ellipse on the screen is at an inclination defined by the eccentric angle ϕ.

Generally, if the wire is inclined at θ and is at a distance z from the primary focal line, the value of ϕ is obtained from the equation

$$\tan\theta = \frac{b\,z}{a(\delta - z)}\tan\phi.$$

If an inclined straight wire is placed across the pencil or beam and is moved, without varying its inclination, from the primary focal line to the secondary focal line, the shadow rotates through an angle of $\dfrac{\pi}{2}$, for ϕ changes by $\dfrac{\pi}{2}$ in passing from one focal line to the other.

In non-homocentric pencils or beams that are not purely astigmatic but are compounded of astigmatic and other aberrations, the presence of astigmatism shows itself in or is accountable for the twists or rotations that are discernible in the shadows or images formed by objects placed in or moved about in such pencils or beams.

For instance, if in the experimental pencil or beam there are astigmatic and spherical aberrations present, then a straight wire placed across the beam and moved about does not cast a straight-line shadow, but casts a distorted shadow, such as a figure of S, which is displaced through an angle or is rotated, the displacement or rotation being due to the astigmatic aberration present in the beam.

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LXIV. *Note on "Minimum Deviation through a Prism."* *By* R. Chartres[*].
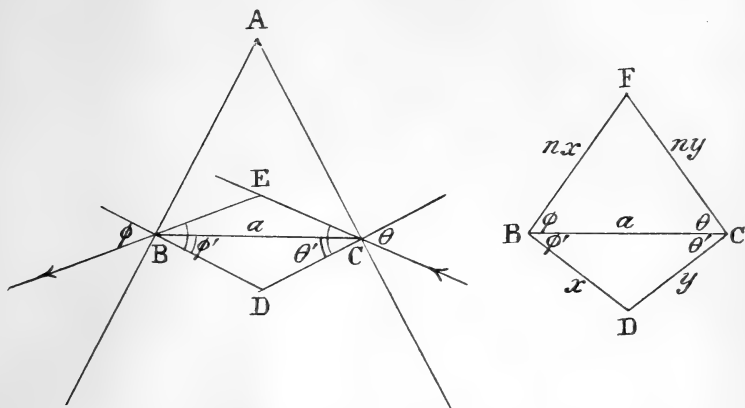
$\theta + \phi =$ minimum. $\quad \theta' + \phi' =$ constant. $\quad \operatorname{Sin} \theta = \mu \sin \theta'. \quad \operatorname{Sin} \phi = \mu \sin \phi'.$

Make F C B $= \theta, \quad$ F B C $= \phi,$

then F is to be a maximum, while D is constant.

$$\therefore \ \cos F - \cos D = \frac{a^2(n^2 - 1)}{2n^2 xy} \ \text{is a minimum.}$$

$$\therefore \ xy = \text{maximum.} \qquad \therefore \ x = y, \ \text{ or } \ \theta' = \phi', \ \text{ or } \ \theta = \phi.$$

LXV. *Mr. J. H. Jeans' Theory of Gases.* (*Note on his Paper in Phil. Mag. June* 1903.) *By* S. H. Burbury, *F.R.S.*[†]

Mr. JEANS proposes in this paper to put the Theory of Gases on a new basis.

The orthodox theory rests, he says, on the *molekular ungeordnet* hypothesis of Boltzmann. But this has never been explained either by its author or by anybody else. And Jeans concludes from Boltzmann's practice that the orthodox theory is effectively based on what in my Kinetic Theory of Gases I called assumption A. That assumption is that the chance of any molecule having its velocities within assigned limits is everywhere and at every instant independent of the

velocities and positions of the other molecules for the time being. We may express this also in the form that, whatever the velocities may be, the molecules are distributed in space at haphazard with respect to them. If this assumption be regarded as the interpretation of Boltzmann's hypothesis, that hypothesis comes before us, like the Ghost in Hamlet, in a questionable shape.

2. Jeans proceeds to show, rightly as I think, that this assumption A, the basis of the orthodox theory, is mathematically impossible if the motion is continuous, that is, if the state of the system at any instant is a necessary consequence of its past history. I think I shall not be misinterpreting Jeans if I put the argument thus:—If the $6N$ coordinates and velocities of N molecules are given at any instant, and if the motion be, as above defined, continuous, the state of the system at any other instant is completely determined in terms of these $6N$ variables and the time t. There are then, including t, only $6N + 1$ independent variables. But if assumption A is made at every instant, we are using in effect an infinite number of independent variables. Therefore assumption A is a mathematical impossibility.

The coordinates and velocities of every pair of molecules are necessarily correlated, because they are functions of the same $6N + 1$ quantities, and of no others. It is probably true, if N be very great, that the correlation is a negligible quantity, except for those pairs of molecules which happen to be near each other. But it cannot be always negligible for these. And it is for such pairs, and for such pairs only, that the assumption A has to be made. It is untrue in the only case for which we require to make it.

3. Assumption A cannot then be accurate except in discontinuous motion, a motion, that is, which is continually receiving disturbances at haphazard, which in fact takes a fresh start from chaos at every instant. It is fair to say that in nature disturbances are very frequently taking place. The isolated system, with its $6N$ variables left to its own forces, hardly exists in practice.

4. Having thus proved that assumption A is not *a priori* consistent, Jeans goes on to show (art. 3) that it is not justified *a posteriori* by its results, instancing the well-known difficulty of Boltzmann's H theorem and the supposed reversed motion. I agree with him in his view that the assumption is not justified by its consequences, though I should not give precisely the same explanation that he gives of the H theorem difficulty.

5. But Jeans having proved in arts. 2 and 3 that assumption

A is untrue, in fact impossible, goes on to prove that Maxwell's law of distribution, given in complete form in art. 34, is true, that is, that the chance of the system of N molecules having velocities $u \ldots u + du$ &c., and coordinates $x \ldots x + dx$ &c. is

$$A e^{-h(\Sigma m(u^2 + v^2 + w^2) + 2\chi)} du_1\, dv_1 \ldots dz_N,$$

in which χ is the potential of external forces (intermolecular forces are not here considered). But from this result it immediately follows that assumption A, which we have just proved to be false, is true. The above formula (34) is the concise mathematical expression of it. Maxwell's law and assumption A can no more exist one without the other than head and tail of the same living animal.

6. The explanation of the contradiction is Jeans' assumption, art. 37 (d), "that at any instant that part of the total energy which is accounted for by the intermolecular forces forms an infinitesimal fraction of the whole." For that implies that the gas considered is an infinitely rare gas. Now Maxwell's law is, as I maintain, never exactly true ; but we can make it approximate to the truth as nearly as we please by diminishing the density. If the law be not as Maxwell says $A e^{-h\Sigma m u^2}$, it will be $A e^{-hQ}$, in which

$$Q = a_1 u_1^2 + b_{12} u_1 u_2 + a_2 u_2^2 + \&c.,$$

a complete quadratic function of the velocities. But the coefficient b of products of the velocities in this quadratic function diminishes as the density diminishes, and as we approach the infinitely rare gas becomes negligible. Therefore in the only case contemplated in Jeans' (37) Maxwell's law, though not accurately, is approximately true. By consequence its satellite the law of equipartition of energy is approximately true ; and so is assumption A. That is the explanation of Jeans' apparent self-contradiction.

7. Jeans' case (37) is a limiting case. The study of limiting cases is always instructive, and may be important. It is exceptionally important in the theory of gases, because air under ordinary conditions is supposed to approximate to the state of the infinitely rare gas. If, however, we are always to assume the infinite rarity of our gas, we are like the early navigators who never went out of sight of land. They also with their imperfect means obtained important results, but they did not obtain a complete science of navigation.

8. The difference between Boltzmann and Jeans is one of method only. Boltzmann uses the ordinary Cartesian coordinates, x, y, z denoting the coordinates of the centre of a

2 M 2

sphere m, u, v, w its component velocities.   In Jeans' notation the 6N coordinates and velocities of the N spheres (which if given define the state of the system) are the " coordinates " (in a wider sense) of a " point " in generalized space of 6N dimensions.   That way of regarding the problem may be more advantageous or less so than Boltzmann's, probably in some respects more, in others less.   But the difference is only in mathematical method, not in the physical conditions.

Every physical assumption which is necessary in the one method is equally necessary in the other.

Of course the conservation of energy is assumed in both cases.   But Boltzmann concludes that conservation of energy in such a system is not alone sufficient to insure the existence of Maxwell's law.   We require, he says, in addition a special assumption.   And he, as Jeans and I think, in effect makes assumption A.   If that assumption is really necessary to Boltzmann, it is equally necessary to Jeans.   But Jeans is at liberty to show that some other assumption is available instead of A.   He does not, however, show this, does not in fact make any assumption except (37d).   And that by bringing in the infinite rarity of the medium, renders assumption A legitimate in the limiting case.

9. Notwithstanding that Jeans' consistency is thus preserved by (37d), I think he has made assumption A unawares, as great writers have done before him.   I will explain my reasons, freely admitting that I may be mistaken.

Jeans, like Boltzmann, has 6N (in a sense) independent variables, that is, 6N independent equations are required to define the state of the system.   It does not, however, necessarily follow that all combinations of the 3N coordinates, given the 3N velocities, are equally probable, or occur with equal frequency (as is asserted in assumption A).   Out of a number of such combinations, each of which is possible and sometimes occurs, some may occur more frequently than others.   But if you make no special assumption at all, you do, *ipso facto*, make all such combinations equally probable.   You make, therefore, assumption A.   And that Jeans has done by implication.   If he is not against assumption A, he is for it.

10. Again, in art. (27) the number of molecules whose velocities are between u and $u+du$, v and $v+dv$, w and $w+dw$ is $f(u, v, w)\, du\, dv\, dw$, or $fdu\, dv\, dw$.   Then Jeans says the most probable, and therefore the actual, distribution is found by making $\Sigma f \log f$ minimum subject to the conditions

$$\Sigma f = \mathrm{N}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and

$$\Sigma fm(u^2 + v^2 + w^2) = 2\mathrm{E}, . \quad . \quad . \quad . \quad . \quad (b)$$

E being the total kinetic energy, and that gives
$$f = A e^{-h \Sigma m (u^2 + v^2 + w^2)},$$
Maxwell's law of distribution of velocities, afterwards given in its complete form in (34).

Now either there exists or there does not exist some other probable relation between the velocities and the coordinates besides (*a*) and (*b*). If you assume that no such exists, you make assumption A. If any such does exist, it is not true that $\Sigma f \log f$ being minimum characterizes the motion. Maxwell's law does not follow.

11. In art. 38 Jeans treats of a gas with "mass velocity" $U = \iint \ldots f . u . dx \, dy \, dz \, du \, dv \, dw$. The integration is intended to include the whole gas under consideration. Now we may conceive a state of things in which every molecule has, in addition to its velocity in the "normal state," the velocity U. As an example U might be the earth's velocity in space. In the equipartition of energy we do not include the energy of the motion U. If assumption A be made, the only mass motion that the system can possibly have is that simple one formed from the normal state as above defined. For any irregularities, such as some groups of contiguous molecules having on average greater, others less, momentum in direction denoted by U, if they exist at one instant, will under assumption A cease to exist the next instant. They are assumed to be non-existent. Jeans' mass motion is then exactly what we should have if we make assumption A. If we do not make assumption A we cannot, except in the above-mentioned case of infinite rarity, obtain Maxwell's law. We must come to the law e^{-hQ}, where Q is the quadratic function before defined. And from that it follows at once, if the *b* coefficients be negative, that the velocities of neighbouring molecules have, on average, the same sign. That is, we have *stream motion* among the molecules of the gas, a different thing from Jeans' mass motion of the whole gas. Why should stream motion not exist?

12. Such a motion is conceivable. If I mistake not it, or a motion of the same kind, exists in the case of wind, which, as we are told, is not mass motion of the simple kind allowed by Jeans and by assumption A, but one in which the velocities in the given direction vary very rapidly from instant to instant, and from point to point at measurable distances in space. For some, but not for all, purposes, this may be represented by "mass motion" U. Whether such streams exist or not in fact is a proper subject for investigation. The orthodox theory rejects them without investigation, because they might not agree with the equipartition of energy.

On the Definition of Stream at a Point.

13. Evidently when we speak of the stream at a point P we have in our mind many molecules near P. And we do not regard molecules very distant from P. The way to express the idea is to weight the molecules (so to speak) according to their nearness to P. If m be the mass, u the x velocity of a molecule, let $\xi = \Sigma f . mu$ be the stream in x at P, the summation including all molecules, and f being a function of r, the distance from P which, when r exceeds a certain minimum, diminishes very rapidly with increasing r. For instance, when $r > c$

$$f = \frac{A}{r^2} e^{-\kappa r},$$

where A and κ are constants. Let η, ζ have corresponding meaning for the component velocities v and w.

Suppose, then, there is a molecule at P having velocities u, v, w. I should then define $\frac{1}{2}(u\xi + v\eta + w\zeta)$ to be the energy of the stream motion for P, and $\phi = \frac{1}{2}\Sigma(u\xi + v\eta + w\zeta)$ the whole stream energy. ϕ is of course zero under assumption A. But if assumption A be not made, it is on average positive for repulsive, negative for attractive, forces. That indicates that the molecules having finite dimensions choose in either case the least resisted motion. When the forces are repulsive the energy of the relative motion of neighbouring molecules is less than, according to assumption A, it should be—and the difference is the stream energy. For attractive forces the energy of the relative motion of a molecule and its immediate neighbours is greater than according to assumption A it should be. In steady motion ϕ must be constant or $\frac{d\phi}{dt} = 0$. That is

$$\tfrac{1}{2}\Sigma\left(u\frac{d\xi}{dt} + v\frac{d\eta}{dt} + w\frac{d\zeta}{dt}\right) = 0,$$

which can be reduced on average to

$$\tfrac{1}{2}\Sigma\left\{\begin{array}{l} u\left(u\dfrac{d\xi}{dx} + v\dfrac{d\xi}{dy} + w\dfrac{d\xi}{dz}\right) \\[2mm] + v\left(u\dfrac{d\eta}{dx} + v\dfrac{d\eta}{dy} + w\dfrac{d\eta}{dz}\right) \\[2mm] + w\left(u\dfrac{d\zeta}{dx} + v\dfrac{d\zeta}{dy} + w\dfrac{d\zeta}{dz}\right) \end{array}\right\} = 0.$$

For steady motion we should, according to Professor Bryan's canon, make $\frac{d^2\phi}{dt^2} = 0$.

Now the above expression for $\frac{d\phi}{dt}$ is what I denoted by M in my Kinetic Theory of Gases, chapters v., vi.; and I did solve the problem there stated by making $\frac{dM}{dt} = 0$, that is, $\frac{d^2\phi}{dt^2} = 0$. I did not, however, define the energy of the stream motion.

The Finite Dimensions of a Molecule.

14. Jeans' molecules may in the beginning of his paper be conceived as elastic spheres having diameter 2R. This he expresses by equation (3), namely

$$(x-x')^2 + (y-y')^2 + (z-z')^2 \geqq 4R^2,$$

and his limits of integration are taken accordingly. His view seems to be that by stating the fact, and remembering it in the limits of integration, he has given full account of all the consequences which need be deduced from the fact stated. It may be put on the shelf. Undoubtedly that view is true for the discontinuous motion characterized by assumption A; but not for continuous motion. For in continuous motion disturbances may exist having greater or less permanence, such, for instance, as the streams assumed to exist in the theory of viscosity. At the point (x, z) a positive stream κz parallel to x, at $(x, -z)$ a stream $-\kappa z$ parallel to x. Now, as I have said elsewhere, in presence of such continuing disturbances a molecule which has finite dimensions, or as a centre of force has finite sphere of action, is exposed to different influences on different sides or parts of its surface, and its motion is affected by these differences, and that may affect the mean potential, the mean virial, and the question of stability. It is true of course that molecular dimensions are exceedingly small, but so is the scale on which the disturbances take place.

Now unless assumption A be made, effects of this kind are an essential part of the phenomena. We do not allow for them by simply stating the fact that (for instance) a spherical molecule has radius R, or two molecules have a potential χ of their mutual action, and then putting the question on the shelf. If, on the other hand, we make assumption A, we assume the non-existence of permanent or continuing disturbances; and in that case the simple statement of fact, and attention to it in the limits of integration, is all that we want or can have. And that is what Jeans has given us. For which reason I think the theory developed in his paper is impliedly based, like the orthodox theory, on assumption A. But as above stated I admit that I may be mistaken.

LXVI. *On Unsymmetrical Broadening of Spectral Lines. By* GEORGE W. WALKER, *M.A., A.R.C.Sc., Fellow of Trinity College, Cambridge* *.

A PERUSAL of chapters 5 and 6, vol. ii. of Kayser's ' *Handbuch der Spectroscopie* ' readily convinces one that the conditions which determine the broadening of a spectral line are extremely complex, and that comparatively little progress has been made towards the disentanglement of the various factors which determine the appearance of a line.

Nevertheless, some general features appear to have been established beyond doubt. One of the most important of these is that when a line is broadened, by whatever means, the broadening is in general unsymmetrical.

The vast majority of lines show a broadening towards the red end of the spectrum ; but there are cases in which there is no measurable effect, and some in which the broadening is distinctly towards the violet.

The theories which have been proposed explain broadening towards the red, but are unable to explain a broadening towards the violet. The Doppler Theory applied to moving molecules leads to the result that the breadth of the line should be proportional to the wave-length. But although in general the breadth is found to be greater for greater wave-lengths, the law is by no means so simple as that just mentioned. Again, Godfrey † has shown that owing to a numerical error in Michelson's work ‡, the Doppler Theory is not sufficient to account for the whole breadth, but only for about one half of it.

My object in this paper is to show that unsymmetrical broadening may be accounted for in a manner quite different from any theories that have been hitherto proposed, and that the effect may in some measure contribute to an explanation of the observed facts.

Whether the luminous radiations are produced in a flame at high temperatures, or by the electric discharge in any form, there can be no doubt of the presence of free negatively-charged particles (corpuscles) in the immediate vicinity of the source from which the radiations proceed. I propose to show that these negative particles modify the light which they receive and again scatter, in a manner

quite different from either the Doppler effect or the effect of damping of the motion.

Several years ago I proved * that a charged particle under the influence of plane waves would vibrate in a period which differed from the period of the incident waves; but as the calculation has not been published, and is quite simple, I reproduce it here.

Let m be the mass of the particle, x, y, z the coordinates of its centre of mass, e its electrical charge; and let plane waves given by

electrical force $X = X_0 \cos (pt - \kappa z)$
magnetic force $M = \dfrac{\kappa}{p} X_0 \cos (pt - \kappa z)$ $\Big\}$ where $\dfrac{p^2}{\kappa^2} = V_0{}^2$

be incident on the particle.

Then the equations of motion of the particle are

$$m\ddot{x} = e(X - M\dot{z}) = \frac{eX_0}{p} \frac{d}{dt} \sin (pt - \kappa z),$$

$$m\ddot{y} = 0,$$

$$m\ddot{z} = eM\dot{x} \qquad = \frac{\kappa}{p} eX_0 \dot{x} \cos (pt - \kappa z).$$

Let us take a new variable ζ, so that

$$z = \frac{p}{\kappa} t + \zeta.$$

Then the equations become

$$\ddot{x} = -\frac{\kappa}{p} \frac{eX_0}{m} \dot{\zeta} \cos \kappa\zeta,$$

$$\ddot{y} = 0,$$

$$\ddot{\zeta} = \frac{\kappa}{p} \frac{eX_0}{m} \dot{x} \cos \kappa\zeta.$$

As integrals of these equations we get

$$\dot{x} = a - \frac{eX_0}{mp} \sin \kappa\zeta,$$

$$\dot{y} = b,$$

$$\dot{\zeta}^2 = c^2 + \frac{2aeX_0}{mp} \sin \kappa\zeta - \left(\frac{eX_0}{mp}\right)^2 \sin^2 \kappa\zeta,$$

where a, b, and c are constants of integration.

* Proc. Royal Society, London, vol. lxix. p. 394.

The complete solution in the usual notation for elliptic functions is

$$\sin \kappa\zeta = \frac{\mu \sin (\lambda t + \epsilon) + 1}{\mu + \sin (\lambda t + \epsilon)},$$

where ϵ is an arbitrary constant.

μ is a root of the equation

$$\frac{ea\mathrm{X}_0}{mp} \mu^2 + \left(c^2 - \frac{e^2\mathrm{X}_0^2}{m^2p^2}\right) \mu + \frac{ea\mathrm{X}_0}{mp} = 0,$$

$$\frac{\lambda^2}{k^2} = \frac{c^2\mu^2 + 2a\dfrac{e\mathrm{X}_0}{mp} \mu - \dfrac{e^2\mathrm{X}_0^2}{m^2p^2}}{\mu^2 - 1},$$

and the modulus of the elliptic functions is given by

$$k^2 = \frac{\mu^2 \dfrac{e^2\mathrm{X}_0^2}{m^2p^2} - 2a\mu \dfrac{e\mathrm{X}_0}{mp} - c^2}{c^2\mu^2 + 2a\mu \dfrac{e\mathrm{X}_0}{mp} - \dfrac{e^2\mathrm{X}_0^2}{m^2p^2}},$$

In general the motion will consist of a linear part along with a periodic motion. As a first approximation the periodic part will be simply harmonic, but the period will differ from the period of the incident waves.

The period will be $\dfrac{4\mathrm{K}}{\lambda}$, where K is the quarter-period of the elliptic functions.

If we take the larger root of the equation in μ, and proceed as far as squares of X_0, we find that

$$\lambda^2 = k^2c^2\left(1 - \frac{e^2\mathrm{X}_0^2}{m^2p^2} \frac{a^2}{c^4}\right),$$

$$k^2 = \frac{e^2\mathrm{X}_0^2}{m^2p^2} \frac{a^2 + c^2}{c^4},$$

$$\mathrm{K} = \frac{\pi}{2}\left(1 + \frac{1}{4} k^2\right).$$

Hence the periodic time is

$$\frac{2\pi}{\kappa c}\left\{1 + \frac{1}{4} \frac{c^2 + 3a^2}{c^4} \frac{e^2\mathrm{X}_0^2}{m^2p^2}\right\}.$$

We note that this is independent of the sign of e, but as the negative particles have a much smaller mass than the positive particles, the effect of the former will be of much greater importance than the effect of the latter.

Let $\dot{z} = d$ when ζ is zero. Then we get

$$c = V_0 - d.$$

Now a, b, and d are the linear velocities of the particle before the waves start, and they will be in general small compared with V_0.

We should get the usual Doppler effect by taking account of all velocities, and this we need not discuss. Independently of this, we see that the period of the scattered light will be increased by

$$\frac{2\pi}{p} \cdot \frac{1}{4} \frac{e^2 X_0^2}{V_0^2 m^2 p^2},$$

or the wave-length will be increased by

$$\frac{\pi e^2 X_0^2}{2 V_0 m^2 p^3}.$$

Thus, instead of homogeneous light, there will be a portion of light scattered by the free-charged particles, and the wave-length of this portion will be longer than the wave-length of the original light. The intensity of the scattered light will be proportional to the number of free-charged particles which are present in the vicinity of the luminous source. Probably this will not show as a new line, but by complication with the Doppler effect we shall get only unsymmetrical broadening to the red.

X_0^2 is a measure of the intensity of the original light, so that the broadening will increase with increasing intensity of the original radiations. We may thus expect very different amounts of broadening, not only in different lines, but also in different forms and intensity of the electric discharge.

Other things being equal, the increased breadth will be proportional to the cube of the wave-length.

It thus seems to me that we have a reasonable explanation of at least part of the observed broadening, and the formula possesses considerable power of adaptability to the different circumstances which determine the particular line in question.

So far, however, the theory leaves the question of broadening towards the violet unexplained. Now I think the charged negative particles may contribute to this in the following way. The continual stream of charged particles which goes on in the electrical discharge means that a local magnetic field of perhaps considerable strength will be set up in the immediate vicinity of the luminous source.

The phenomena will thus be complicated by the Zeeman effect. Now Zeeman has observed a lack of symmetry of the

components towards the violet. The theory given by Voigt [*] and that by myself [†] both lead to the conclusion that the lateral component towards the violet should be at a greater distance from the undisturbed line than the lateral component towards the red. It may well be that in some cases this cause may be more effective than those which produce unsymmetrical broadening towards the red, and consequently broadening towards the violet would be the net result.

LXVII. *Notices respecting New Books.*

Aide-Mémoire de Photographie pour 1903. *Publié sous les auspices de la Société Photographique de Toulouse par* C. FABRE. Paris : Gauthier-Villars. 1903. Pp. 240.

THIS useful little annual is divided into three parts. The first part contains a review of the progress made in photography during the preceding year, and a complete list of photographic societies and periodicals (both French and foreign). The second part is a concise treatise on the fundamental principles of photography, and contains many useful hints. The concluding part forms a tradedirectory.

Thermodynamik. Von Dr. W. VOIGT, *Professor der theoretischen Physik an der Universität Göttingen.* 1 Band. Mit 43 Figuren [Sammlung Schubert XXXIX]. Leipzig : G. J. Göschensche Verlagshandlung. 1903. Pp. xv+360.

THE subject of thermodynamics is probably one of the most difficult branches of mathematical physics, and abounds in pitfalls to the unwary. Several excellent textbooks on it have appeared recently, and most of them are distinguished from the older books by the care bestowed on the foundations of the subject, and the detailed manner in which the various assumptions on which the theory is based are explained.

The book under review is the first volume of a self-contained and comprehensive treatise on the subject. The plan adopted by the author consists in proving all the dynamical propositions required in thermodynamic theory, instead of referring the reader to special treatises on dynamics. The book opens with an introduction in which the subjects of thermometry, calorimetry, and heat conduction are dealt with. Then comes Chapter I., on the energy equation and the dynamical equivalent of heat. Chapter II. contains the thermodynamics of an ideal gas. Chapter III. is devoted to the thermodynamics of bodies whose state is completely defined by two independent variables (especially bodies under a uniform pressure), and the concluding Chapter IV. deals with the

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[*] *Annalen der Physik,* vol. i. p. 376 (1900).
[†] Phil. Mag. Feb. 1902.

case of bodies whose thermal state requires more than two variables for its complete specification.

The style is admirably clear, and the book will prove particularly useful to advanced students both on account of its comprehensiveness and its self-contained character, which renders unnecessary frequent and troublesome references to textbooks on allied subjects.

LXVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 384.]

February 4th, 1903.—Prof. Charles Lapworth, LL.D., F.R.S.,
President, in the Chair.

THE following communications were read:—
    1. 'The Granite and Greisen of Cligga Head (West Cornwall).'
By John Brooke Scrivenor, Esq., M.A., F.G.S.

The small granite-mass between St. Agnes and Perranporth has been described by Conybeare, Carne, Sedgwick, Foster, and others. It is a remnant of a much larger mass which has been partly denuded by marine action and partly hidden by a north-and-south fault. It is possible to distinguish two divisions of it: the main mass and the 'tongue,' throughout both of which 'bedding' is well developed. The granite bordering the bedding-planes has been altered into greisen, which, owing to the abundance of quartz, appears in the cliff-section as dark bands. Each greisen-band contains a quartz-vein, marking the original fissure along which metasomatism took place; the veins contain tourmaline, cassiterite, wolfram, mispickel, and chalcopyrite. Two main reactions appear to have taken place in the formation of the greisen: the felspars affording topaz, muscovite, and secondary quartz; the biotite brown tourmaline, magnetite, and secondary quartz. The fact that no tourmaline has been formed from the felspar, owing to the presence of abundant fluorine, distinguishes this greisen from luxullianite and trowlesworthite. The blue tourmaline-prisms included in original quartz appear to have been original constituents of the granite. Secondary quartz, deposited in optical continuity with the original grains, has also caused them to appear to have a crystal-outline. The fluorine and boron had not so great an effect on the extremity of the tongue as on the main mass, as shown by the poor development of greisen and the freshness of the biotite. Mica, topaz, and microcline-perthite have been re-deposited there by percolating water or vapour. The greisen is an example of Prof. Vogt's 'pneumatolytic' action in thoroughly acid rocks, resulting in the formation of tinstone-lodes, as contrasted with the similar action in syenitic rocks with the production of zircon, etc., and in basic rocks with the production of chlor-apatite and the scapolitization of the felspar.

2. 'Notes on the Geology of Patagonia.' By John Brooke Scrivenor, Esq., M.A., F.G.S.

The author was travelling in Patagonia from September 1900 until March 1901. The sedimentary strata consist of Tertiary, Cretaceous, and Jurassic formations, which, with the exception of the Jurassic, yield interesting and varied faunas, both vertebrate and invertebrate. The latest classification is that drawn up by Mr. J. B. Hatcher, who conducted the expeditions sent from Princeton University. Mr. Hatcher, aided by Dr. Stauton and Dr. Ortmann, has arrived at the following correlation :—

Shingle Formation (Téhuelche Pebble-Bed).	PLEISTOCENE.
Cape Fairweather Beds.	PLIOCENE.
Santa Cruz Beds.	UPPER MIOCENE.
Patagonian Beds.	LOWER MIOCENE and UPPER OLIGOCENE.
Upper Lignites.	MIDDLE OLIGOCENE.
Magellanian Beds.	LOWER OLIGOCENE and UPPER EOCENE.

Guaranitic Beds.	
Lower Lignites.	
Variegated Sandstones.	
Upper Conglomerates.	CRETACEOUS.
Belgrano Beds.	
Lower Conglomerates.	
Gio Beds.	
Mayer Shales.	JURASSIC.

Except in the north, where intrusions of an acid type have disturbed the sediments, the southerly dip is so gentle as only to be appreciable where sections can be followed for some distance. Mr. Hatcher considers that an unconformity separates the Magellanian and Guaranitic Series, also the Cretaceous and Jurassic.

Excellent sections of the Patagonian Beds were seen on the Santa Cruz River and in the coast-section at Monte Leon. They are littoral deposits, consisting of sandstones and mudstones. Calcareous nodules are frequently arranged along the bedding-planes. Petrologically the sandstone is remarkable for containing fresh hypersthene and plagioclase. At Monte Leon the top of the Patagonian Beds is marked by gypseous mudstones and a shell-bed. These are succeeded by estuarine beds, some of which yield impressions of *Fagus*. Conformable on the estuarine beds are the famous Santa Cruz Beds, which have yielded a rich vertebrate fauna. They consist chiefly of pumiceous mudstones, with a little hypersthene ; but a blue clay alternates with the mudstones, and there are also two bands of *Ostrea ingens,* and one or two of ferruginous sandstone. The Téhuelche Pebble-Bed passes down into the Cape Fairweather Beds imperceptibly : otherwise it overlies everything unconformably.

Very little is known of the igneous rocks. Apart from those of the Cordillera, there are vast plateaux of basalt and intrusions of quartz-porphyry. A good example of the latter occurs at Port St. Helena. The specimens of igneous rocks collected from the moraines of the Cordillera comprise biotite-granite, hornblende-granite, quartz-mica-diorite, gabbro, hornblende-picrite, quartz-

porphyry, rhyolite, obsidian, ophitic olivine-dolerites, olivine-basalts, and acid tuffs.

The basalt-flows cover an enormous area. They slope gently towards the Atlantic, and are cut off from the Cordillera by a longitudinal depression. In the neighbourhood of Lago Colhuapé there seems to have been a distinct centre of eruption, apart from that which commences nearer the Cordillera. All that can be said of their age is that they are older than the transverse depressions of the Cordillera, and older than the glaciation of the eastern slopes of that chain.

The Téhuelche Pebble-Bed, which covers nearly the whole of Patagonia, has been ascribed to marine action by some authors, by others to glacial action. A third suggestion is the agency of big rivers. No one of these agents alone could have produced the observed phenomena : the origin was complex. The bulk of the material was brought by glaciers from the Cordilleras to the sea, which then covered the greater part of the pampas. As the sea receded, it distributed the pebbles over the bottom, so forming a continuous layer, such as now exists between the eastern coast and the Falkland Islands. Torrents resulting from the melting of the glaciers assisted in distributing the material from the Cordillera. Part of the material on the present eastern coast was derived from islets of quartz-porphyry in the Pleistocene sea. A great difficulty is that no basalt-pebbles are found at Santa Cruz east of the flows.

The drainage-system includes several eastward-flowing rivers and numerous lakes, some of which occupy transverse valleys cutting through the Cordillera. An example of the latter is Lago Buenos Aires. The history of this lake can be gathered from the evidence observed on its shores. Lagos Musters and Colhuapé are two other interesting lakes near the eastern coast. The width and depth of the river-valleys are disproportionate to the present streams : this can be explained by a decreasing rainfall, and also by the diversion of many tributaries to the Pacific. Some valleys are dry, as, for example, the Great Cañadon Salado.

3. ' On a Fossiliferous Band at the Top of the Lower Greensand, near Leighton Buzzard (Bedfordshire).' By George William Lamplugh, Esq., F.G.S., and John Francis Walker, Esq., M.A., F.L.S., F.G.S.

This paper describes a newly-discovered fossiliferous band at the top of the Lower Greensand, overlain by the Gault, in the sand-pits at Shenley Hill near Leighton Buzzard, in Bedfordshire. The fossils of this band present a different facies from that of any other previously-known fossiliferous horizon of the Lower Greensand, and show closer affinities with the fauna of the Upper Greensand than have hitherto been recognized in any deposit below the Gault. The brachiopoda are closely allied to those contained in the Tourtia Beds of Belgium. The fossiliferous bed is rather sharply marked off from the underlying unfossiliferous ' silver-sands,' but is still more sharply marked off from the overlying Gault. Stratigraphically it forms part of the Lower Greensand, and cannot (without violence to

the accepted classification of the deposits) be considered to belong to the Gault. The fossils constitute the newest Lower Cretaceous fauna as yet recognized in England. Several species, hitherto supposed to be confined to the Selbornian, are now shown to have been in existence before the deposition of the Gault. The lithological characters of the bed indicate a sea-bottom of moderate depth, swept by powerful currents, and the conditions were thus similar to those which persisted in the neighbourhood throughout Lower Greensand times. The overlying Gault shows a change to more tranquil waters, probably of greater depth.

February 25th.—Prof. Charles Lapworth, LL.D., F.R.S., President, in the Chair.

The following communications were read :—

1. ' On the Occurrence of *Dictyozamites* in England, with Remarks on European and Eastern Floras.' By Albert Charles Seward, Esq., M.A., F.R.S.

2. ' The Amounts of Nitrogen and Organic Carbon in some Clays and Marls.' By Dr. N. H. J. Miller, F.C.S.

Analyses of soils are given to show that, under most conditions, decaying vegetable matter in soil tends to become more nitrogenous, on account of the greater ease with which gaseous compounds are formed with carbon than with nitrogen. Hilgard's experiments throw light on the effects of extreme conditions of climate, the amount of soluble humus being much greater in soils in humid than in arid climates ; thus, although the total amount of soluble nitrogen does not vary much, the percentage of it in the humus varied very considerably in the two cases. The large areas of peat-land known as ' Hochmoor' contain larger proportions of carbon and nitrogen at depths of 7 and 14 feet than at the surface. The organic matter of soils is of two kinds—the humous portion and the bituminous : the latter being regarded as belonging to the original deposit from which the soil is derived. Analyses of soils and subsoils are given to illustrate this point. Further light on this subject is derived from the analysis of a series of specimens from the following deposits, obtained through the kindness of Sir Archibald Geikie from borings in the possession of the Geological Survey :—Lower Lias, Oxford Clay, Kimmeridge Shale, Purbeck and Wealden strata, Gault, Chalk-Marl, and London Clay. Apart from the interest due to the great depths from which the samples were obtained, and the evidence which they afford of the enormous accumulations of combined nitrogen, they possess the further and greater value of representing the materials out of which large areas of soils have been derived. Calcium-carbonate varies from 82·1 to 0 per cent., organic carbon from 1·229 to 0·229, and nitrogen from 0·068 to 0·021 ; the highest proportion of carbon to nitrogen is 40·3 to 1, and the lowest 8·9 to 1. It would be important to determine, in the case of these older deposits, whether any of the organic matter at all is in the form of humus.

March 11th.—Prof. Charles Lapworth, LL.D., F.R.S.,
President, in the Chair.

The following communications were read :—

1. ' Petrological Notes on Rocks from Southern Abyssinia,
collected by Dr. Reginald Kœttlitz.' By Catherine A. Raisin, D.Sc.

The specimens described in this paper were collected by
Dr. Kœttlitz on an expedition (in 1898–99), starting from Berbera,
westward through Somaliland and Southern Abyssinia, and turning
northward to the Blue Nile. The paper gives petrological notes on
the different classes of rocks represented. The crystalline rocks
include granite, gneiss, and hornblende-schist or foliated diorite,
together with more basic types. They occur where the plateau
rises from the coastal plain, farther west underlying volcanic
rocks and sedimentary strata, in the south-west of Abyssinia,
and towards the Sudan. Some of the gneisses exhibit pressure-
effects, as if these older masses had been thrust up. The more basic
types include diabase, hornblende-gabbro, and one lustre-mottled
hornblende-pyroxenite, resembling a picrite.

The sandstones (which are chiefly from Somaliland and the south-
east of Abyssinia) are sometimes compacted into quartzites, and
are often ferruginous. Some of the limestones are concretionary,
others are dolomitic, and several from different localities are
fossiliferous, containing foraminifera, calcareous algæ, and, at
Jigjiga Pass (which leads into Abyssinia), *Turritella* in great
numbers.

The numerous specimens of volcanic rocks include one which is
practically a limburgite, many basalts (a few with olivine, and
some glassy), various less basic volcanic rocks, and several
pumiceous tuffs. But the most interesting are the phonolites
and allied rocks, containing nepheline, riebeckite, or other alkaline
minerals. They occur at several places, one being a volcanic hill
with a summit-crater. The authoress distinguishes several types
among these soda-bearing rocks, and compares two of them with
rocks of Central Abyssinia and of British East Africa respectively.
Thus the specimens here described may form a connecting-link
between the volcanic rocks of other East African localities.

2. ' The Overthrust Torridonian Rocks of the Isle of Rum and
the Associated Gneisses.' By Alfred Harker, Esq., M.A., F.R.S.,
F.G.S.

The Torridonian strata of Rum occupy all the northern part of
the island, together with a strip extending along the eastern coast, the
high ground in the south being made by plutonic rocks of Tertiary
age. The northern tract consists in general of sandstones having a
moderate dip to the north-west or west-north-west, and below these
there emerges on the east side a lower group composed of dark
shales. There are, however, two districts in which the strata are
highly disturbed and overthrust. One is a small area to the north-
west, on Monadh Dubh, where a cake of thoroughly brecciated and

mylonitized rocks rests on the relatively unmoved sandstones. Besides sandstone, this crushed mass contains abundant débris of Cambrian limestone, chiefly towards the base, and resting immediately upon the surface of overthrust. The limestone does not occur in place on the island.

The other and more extensive area of overthrust rocks forms a belt along the north-eastern and eastern border of the mountain-tract. The effect of the displacement has been to bring the shales of the lower group to rest on the sandstones of the upper. Above the main surface of movement the shales are violently contorted, and the sandstones, where these occur, brecciated. There is also considerable thermal metamorphism, due to the Tertiary intrusions. At numerous places along the disturbed belt are patches and lenticles of gneiss. These are intrusive in the Torridonian rocks, and the evidence points to their being of Tertiary age. They have arisen in great part from a granitic magma modified in varying degree by dissolving basic, and often ultrabasic, rock-débris. The heterogeneous composition thus imparted has, with flowing movement, resulted in well-marked gneissic banding. In a minor degree basic rocks, probably gabbros originally, have contributed more directly to the composition of the complex, namely, as bands or lenticles of rocks, now hornblendic, representing distinct intrusions enveloped and modified by the later and more voluminous invasion of acid magma.

The chief conclusions which the author wishes to establish are :—

(i) That the highly disturbed region of the North-West Highlands, already known to extend into the south-eastern part of Skye, is further prolonged into the Isle of Rum.

(ii) That at numerous places along the disturbed belt which borders the principal mountain-group of the island, the Tertiary plutonic intrusions assume the character of well-banded gneisses, comprising alternations of different lithological types.

(iii) That these complex gneisses were formed mainly by fluxion in a heterogeneous mass, the heterogeneity being due to the inclusion and incorporation in a granitic magma of relics of ultrabasic and basic rocks.

March 25th, 1903.—Prof. Charles Lapworth, LL.D., F.R.S., President, in the Chair.

The following communications were read :—

1. 'On a New Species of *Solenopsis* from the Pendleside Series of Hodder Place, Stonyhurst (Lancashire).' By Wheelton Hind, M.D., F.R.C.S., F.G.S.

2. ' Note on some *Dictyonema*-like Organisms from the Pendleside Series of Pendle Hill and Poolvash.' By Wheelton Hind, M.D., F.R.C.S., F.G.S.

3. 'The Geology of the Tintagel and Davidstow District (Northern Cornwall).' By John Parkinson, Esq., F.G.S.

The country described and mapped consists of some 22 square

miles in Northern Cornwall, extending from the coast eastward towards Camelford Station and St. Clether. In the eastern part it extends to the neighbourhood of the Brown Willy mass of granite, while on the north it approaches the boundary between the Lower Culm and the Upper Devonian. The rocks described are of the latter age, and contain *Spirifera disjuncta.*

Except in the southern coast-region (Tintagel and Trebarwith Strand) the strike is fairly uniform in an east-south-easterly and west-north-westerly direction, the beds having a northerly dip ; but north and south of Tintagel Head the higher members appear, greatly faulted, being brought in out of their true position partly by a change of strike, partly by dip-faults. The most distinctive rocks, utilized as a datum for mapping, are a group of ashes and lavas. The latter are often amygdaloidal, and possess original characters which are still recognizable ; but the whole group is frequently much altered or entirely reconstructed, with the formation of epidote (sometimes enclosing allanite), sphene, biotite, chlorite, etc. The rocks are associated in many instances with calcite, at least partly due to contemporaneous deposition, but frequently forming a corporate part of the renovated rock, and the mineral is found with quartz and translucent felspar.

Bluish-black slates and fine laminated quartzose beds overlie and underlie this volcanic series.

The remaining rocks are phyllites, closely resembling those from the Ardennes. The author divides them into four groups. The highest of these (Tredorn Beds) overlies the uppermost division of the Blue-Black Slates, and in the western part of the district contains a mineral forming small white spots, not yet determined. The beds underlying the Lower Blue-Black Slates (Hallwell Cottage Beds) are banded phyllites, with quartzose laminæ, typically containing abundant crystals of clinochlore with a habit resembling that of ottrelite. The underlying phyllites (Penpethy Beds and Slaughterbridge Beds) contain no distinctive mineral. Taken as a whole, the phyllites consist of a sericitic and chloritic groundmass containing unorientated crystals of white mica, micaceous ilmenite, hæmatite, and minor quantities of tourmaline and rutile. Northeast of Camelford (Grigg's Down) they furnish clear evidence of contact-metamorphism.

April 8th.—J. J. H. Teall, Esq., M.A., FR.S., Vice-President, in the Chair.

The following communications were read :—

1. ' On the Probable Source of some of the Pebbles of the Triassic Pebble-Beds of South Devon and of the Midland Counties.' By Octavius Albert Shrubsole, Esq., F.G.S.

After an account of previous researches on this subject, the author proceeds to describe the Budleigh-Salterton Pebble-Beds. Judging from lithological evidence, the bulk of the pebbles must have come

from a definite region of a comparatively simple geological character; and this is confirmed by the palæontological evidence. The supposition is natural that Devonian rocks were once represented either in the Calvados district or in some region in the same drainage-area as that which has supplied the Ordovician element. The Grès de May of Normandy and its associated rocks are next described, a massif which, according to Prof. Bonney, must have exceeded the Alps in breadth. When regard is had to the extent and original thickness of the Grès de May, it appears capable of furnishing abundant material, not only for the Ordovician pebbles of the Budleigh-Salterton Pebble-Bed, but also for a great deal more. A list of species common to the Grès de May, of May itself, and the Budleigh-Salterton deposit is given; and it is pointed out that in the Department of the Manche the former deposit varies in palæontological facies. In addition to the identity of the quartzites and felspathic grit in the two areas, it is noted that the so-called lydianstone (tourmaline-grit) of Budleigh and the Midlands may be paralleled with one referred to by MM. de Tromelin and Lebesconte in the Department of Maine-et-Loire. The author is struck with the resemblance of the Midland Bunter to that of Devon, and he gives the percentage of rock-types in the larger pebbles at Repton and in the smaller material of Drift derived from the Bunter at two localities in the Lickey Hills. Strong family likenesses subsist between certain specimens in the northern and southern Bunter and some of the undisturbed rocks of Normandy. A list of fossils from the Midland Bunter contains three southern forms; and a further table is given comparing fossils from Drift-pebbles, from Budleigh Salterton, and from Normandy. Fourteen out of twenty of the Drift and Bunter fossils are found at Budleigh Salterton and in Normandy. The hypothesis which presents the least difficulty appears to be that which regards the two pebbly deposits, north and south, as having had approximately a common origin. It does not necessarily follow that both deposits are due to the same river.

2. 'Note on the Occurrence of Keisley-Limestone Pebbles in the Red Sandstone-Rocks of Peel (Isle of Man).' By E. Leonard Gill, Esq., B.Sc.

Pebbles of a coarsely-crystalline, greyish-white, mottled limestone, collected by Prof. W. Boyd Dawkins from the conglomerates at Whitestrand, contain the following fossils:—*Illænus Bowmanni,* var. *brevicapitatus, Primitia Maccoyi, Orthis calligramma, O. testudinaria, O. biforata, Rafinesquina deltoidea, Plectambonites quinquecostata, Atrypa expansa, Hyatella Portlockiana, Dayia pentagonalis, Platyceras verisimile, Stenopora fibrosa,* and crinoid-stems. This assemblage of fossils corresponds strikingly with that of the Keisley Limestone; and it is therefore concluded that the pebbles have been derived from that rock. It seems hardly likely that they have come from so distant a locality as the Lake District; more probably there has been a local source, which would form a link between the limestone of Keisley and that of the Chair of Kildare.

balt.

599° C

746°

319°
818°

800 H

168°

8°

-786°

00
800   1000   1200° C

FIG. 1.

FIG. 2.

FIG. 3.

FIG. 4.

FIG. 5.

FIG. 6.

FIG. 7.

FIG. 8.

FIG. 9.

FIG. 10.

FIG. 11.

FIG. 12.

FIG. 13.

FIG. 14.

FIG. 15.

FIG. 16.

Fig. 1.

14 Sec Waves.
B.P. = 16.5^s.
14. Sec. Vib ^{n}.

13 Sec waves.
B.P. = 16.5^s.
13 Sec Vib ^{n}.

12 Sec. waves.
B.P. = 16.5^s.
12 Sec Vib ^{s}.

Fig. 2.

B.P. = 16.5^s. 10   15 Sec Waves   15 Sec. Vib ^{n}.
10 to 11.

Fig. 3.

16 Sec Waves

B.P.=16.5⁵

16 Sec Vib⁵

17 Sec waves

B.P.=16.6⁵

17 Sec Vib⁵

18 Sec waves

B.P.=16.5⁵

18. Sec Vib⁵⁵

Fig. 4.

19 Sec Waves.

B.P.=16.5⁵

19 Sec Vib⁵

FIG. 1.

FIG. 4.

FIG. 6.

FIG. 2.

FIG. 3.

FIG. 5.

FIG. 7.

Fig. 1

Fig. 2.

Fig. 3

Fig. 4.

Fig 5

Fig. 6.

Fig. 7

Fig. 8.

Fig. 9.

Fig. 10.

Fig. 11.

Fig. 12.

ABSCISSÆ VALUES OF $\frac{d}{\omega}$.

curve $\left(1 - E\,\frac{p^2}{\omega^2}\right)$

curve $\frac{p^2}{\omega^2}\left(1 - E\,\frac{p^2}{\omega^2}\right)$

Curve $\frac{p^4}{\omega^4}\left(1 - E\,\frac{p^2}{\omega^2}\right) + \frac{p^2}{\omega^2}\,E\left(\frac{p^2}{\omega^2}\right)$

Curve $\frac{p^2}{\omega^2}\,\epsilon^{-\frac{p^2}{\omega^2}}\,E_i\left(\frac{p^2 - \Omega^2}{\omega^2}\right)$ *for* $\frac{\Omega^2}{\omega^2} = 5$

FIG. 5.

FIG. 7.

Fig. 6.

Fig. 8

SINGLE-PIECE LENSES.
T.H. BLAKESLEY.
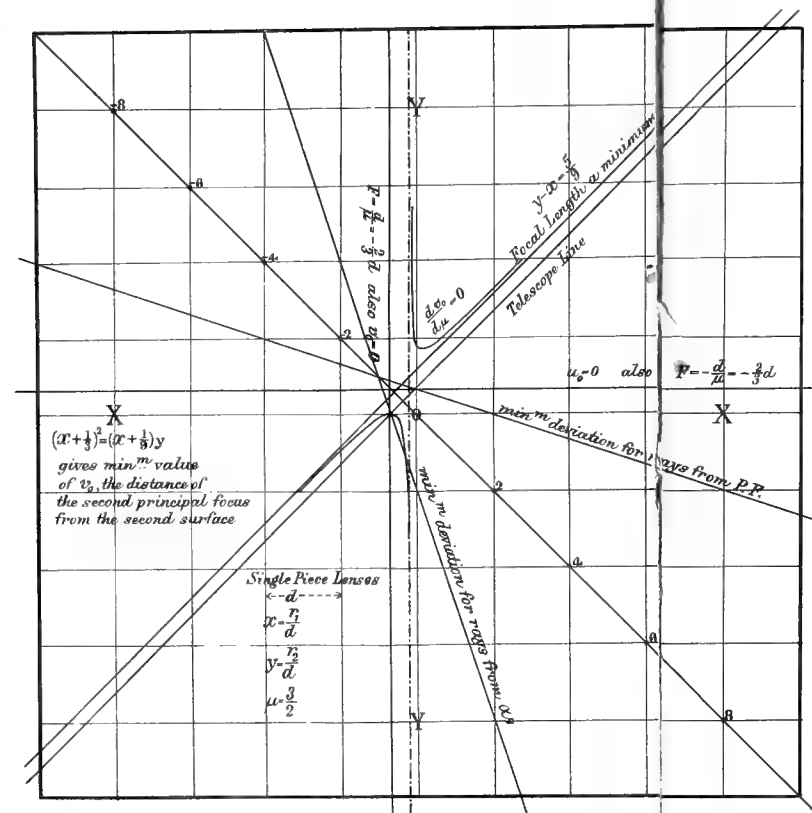
SINGLE-PIECE LENSES.
T.H. BLAKESLEY.
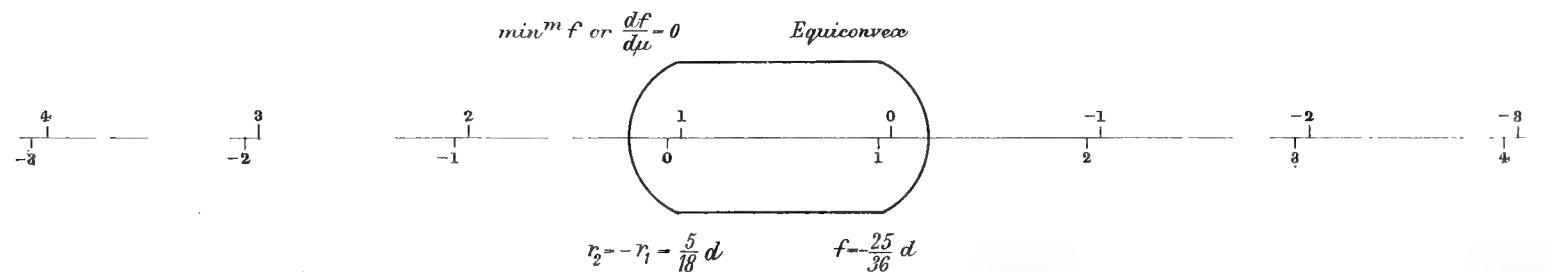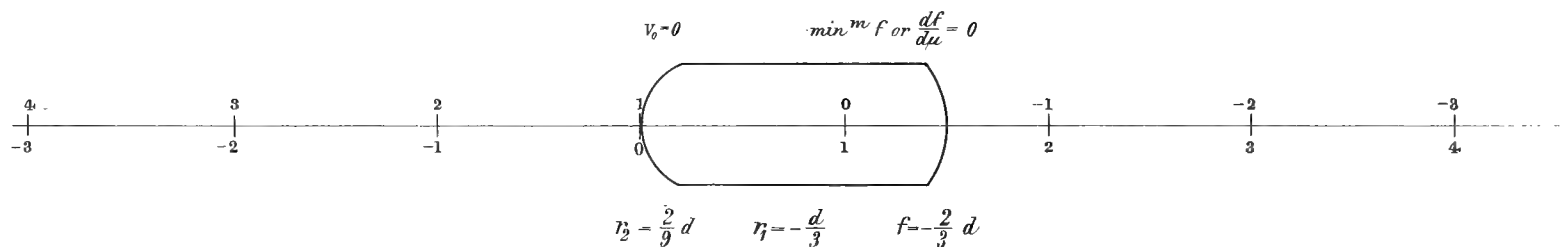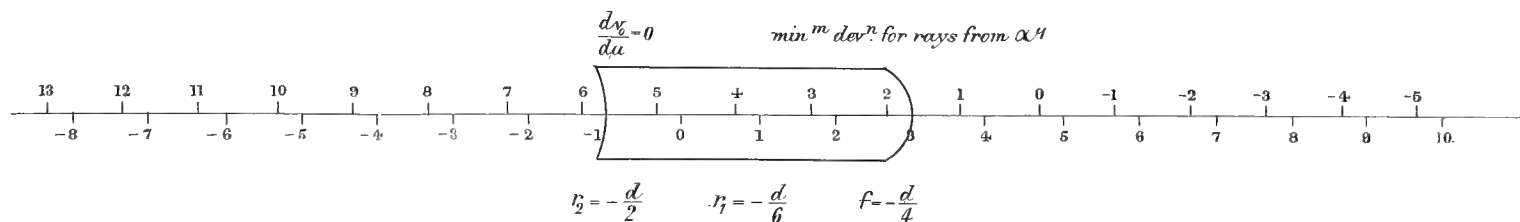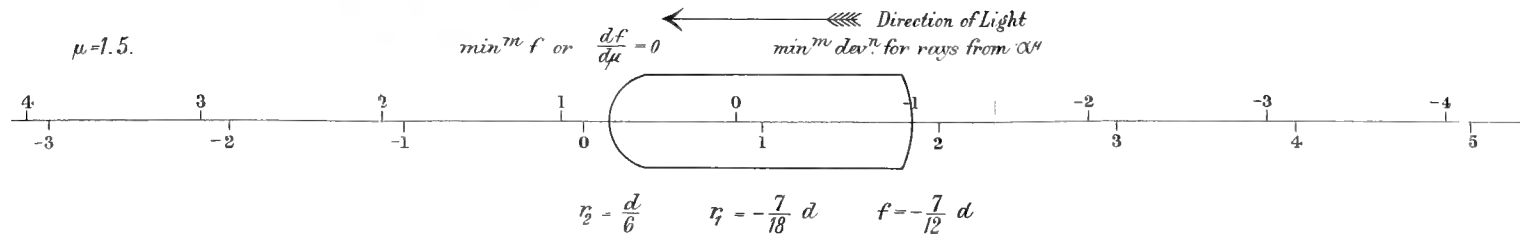
$\longleftarrow$ Direction of Light

$\mu = 1.5.$ $\min^m f$ or $\frac{df}{d\mu} = 0$ $\min^m dev^n$ for rays from α^q

$$r_2 = \frac{d}{6} \qquad r_1 = -\frac{7}{18}\,d \qquad f = -\frac{7}{12}\,d$$

$\frac{dv_0}{d\mu} = 0$ $\min^m dev^n$ for rays from α^q

$$r_2 = -\frac{d}{2} \qquad r_1 = -\frac{d}{6} \qquad f = -\frac{d}{4}$$

$v_0 = 0$ $\min^m f$ or $\frac{df}{d\mu} = 0$

$$r_2 = \frac{2}{9}\,d \qquad r_1 = -\frac{d}{3} \qquad f = -\frac{2}{3}\,d$$

$\min^m f$ or $\frac{df}{d\mu} = 0$ Equiconvex

$$r_2 = -r_1 = \frac{5}{18}\,d \qquad f = -\frac{25}{36}\,d$$

$(x+\tfrac{1}{3})^2 = (x+\tfrac{1}{9})y$

gives $\min^m$ value
of v_0, the distance of
the second principal focus
from the second surface

Focal Length a minimum

Telescope Line

$v_0 = 0$ also $F = -\frac{d}{\mu} = -\frac{2}{3}d$

$\min^m$ deviation for rays from P.P.

$\min^m$ deviation for rays from α^q

Single Piece Lenses

$x = \frac{r_1}{d}$

$y = \frac{r_2}{d}$

$\mu = \frac{3}{2}$

LXIX. *Preliminary Account of a Wave-Tracer and Analyser.*
By Prof. THOMAS R. LYLE [*].

A RESEARCH was lately undertaken by me in order to determine directly the magnetic properties of iron when subjected to alternating magnetizing forces, and to investigate its behaviour under the actual working conditions to which it is exposed in alternate-current working.

For this purpose it was necessary to obtain simultaneous values of magnetizing force and induction at any instant, for any wave-form of exciting current; and I have succeeded in doing so by means of a wave-tracer and analyser which I think will be of sufficient interest to electricians to merit a preliminary description. Some examples of the work it can do will be given in the present paper; and I hope to publish at as early a date as great pressure of teaching-work will allow a detailed account of the research for which it was originally devised.

Not only, however, does my wave-tracer and analyser determine accurately by means of a steady galvanometer-reading instantaneous values of an oscillating E.M.F., current, or magnetic flux at any point of their wave-form; and hence by a number of readings, or by photography, the wave-forms themselves of these quantities, but it also analyses these wave-forms, obtaining what I call their components. By the nth component of a periodic function I mean the periodic function

* Communicated by the Author.

which is the sum of the nth, $3n$th, $5n$th, &c., harmonics contained in the original function.

1. In order to either plot or photograph the wave-form of an alternate current, I send the latter through the primary of an air-circuit transformer whose secondary has few turns.

In the secondary circuit is a commutator (attached to the spindle of the generating alternator or of a synchronous motor) which commutates twice per period, and from the commutator the circuit is completed through a high resistance and a delicate Ayrton-Mather dead-beat galvanometer. Two of the four brushes of the commutator are attached to a divided circle which can be rotated by means of a tangent-screw, so that commutation can be effected at any point or instant of the phase of the primary alternate current-wave. For each position of the movable brushes the galvanometer will show a steady deflexion which is proportional to the instantaneous value of the primary or original current at the instants of commutation.

2. For if C be the current through the primary of the air-transformer, M the mutual inductance of its coils, and r the large resistance in its secondary circuit, then the secondary current will be

$$-\frac{M}{r}\frac{dC}{dt},$$

This is legitimate, as the time constant of the secondary is made so small as to be inappreciable not only relative to the time constant of the primary, but in itself.

If now commutation be effected at times t, $t+\frac{T}{2}$, $t+T$, &c., where T is the period, the quantity Q of electricity that flows through the galvanometer during one complete period will be given by

$$-\frac{r}{M}Q = \int_{t}^{t+\frac{T}{2}}dC - \int_{t+\frac{T}{2}}^{t+T}dC$$

$$= C_{t+\frac{T}{2}} - C_t - C_{t+T} + C_{t+\frac{T}{2}}$$

$$= -4C_t;$$

for, as only odd harmonics appear in alternate current-waves (see § 12), $C_{t+\frac{T}{2}} = -C_t$; therefore

$$Q = 4\frac{M}{r}C_t;$$

hence the deflexion of the galvanometer will be the same as that produced by a steady current

$$= \frac{Q}{T} \text{ or } \frac{4M}{r'T} C_t.$$

If C_t contains both even and odd harmonics, it is easy to show that $C_t - C_{t+\frac{T}{2}}$ is equal to twice the sum of the odd harmonics in C_t, in other words is equal to twice its first component. In such a case the deflexion of the galvanometer would be the same as that produced by a steady current

$$\frac{4M}{r'T} P_{1t},$$

where P_{1t} is the value of the first component of C at the instants of commutation.

3. The reducing factor λ of the galvanometer can be determined by means of a Clark's cell and a megohm, or by placing the galvanometer with a large resistance in its circuit as a shunt across a standard resistance through which a continuous current determined by a Kelvin balance flows. The latter is the preferable method, as the balance can be used to measure the alternating currents dealt with, and they can also be determined, as I shall show, from the square root of mean square of the galvanometer deflexions, thus giving a valuable check on the accuracy of the method.

λ having been determined, if γ be the steady galvanometer deflexion in § 2,

$$\lambda\gamma = \frac{4M}{r'T} C_t,$$

or

$$C_t = \frac{\lambda r T}{4M} \gamma,$$

by which the instantaneous value of the primary current at time t is given absolutely in terms of the galvanometer deflexion.

Also if $\bar{\gamma}$ be the square root of mean square of the galvanometer deflexions taken at suitable equal intervals over half a wave, then the primary current in virtual units is given by

$$\frac{\lambda r T}{4M} \bar{\gamma}.$$

4. The type of commutator used is the well-known one illustrated in fig. 1 (Pl. XXII.). This one is suitable for a

four-pole alternator if attached direct to it, or for a four-pole synchronous motor.

A and B are called the fixed brushes, though exigencies of construction render it necessary to attach one of them at least to the movable divided circle. Each has permanent contact, however, always with the same section of the commutator. C and D are the movable brushes whose points of contact are 90° (equivalent to 180° in phase) apart, and are attached to a divided circle which can be rotated by means of a tangent-screw.

For any definite position of C and D, given by the reading of the circle against a fixed index, commutation takes place at the corresponding point of the wave, and the steady galvanometer deflexion multiplied by the constant in § 3 gives the instantaneous value of the primary current at that point of the wave.

Readings of the galvanometer having been taken at suitable intervals over one half turn of the divided circle, these can be plotted against the circle readings multiplied by 2, and a complete wave-form is obtained.

It is essential that the large resistance *r* should be on the galvanometer side of the commutator. The latter will nearly always act on its own account as a feeble generator of statical electricity, and if the circuit be broken on the M side, the galvanometer will give small deflexions which vary in magnitude and sign with the position of the movable brushes. These deflexions are independent of the magnitude of *r*, and are greatly increased by approaching an excited rod of ebonite to the commutator. When the connexions are as described, these statical discharges are short-circuited through the very low resistance of M and cause no trouble.

5. In order to photograph wave-forms the galvanometer is inclosed at one end of a long camera-box, at the other end of which a dark slide containing a photographic plate moves in vertical grooves behind a metal screen, across the middle of which is cut a narrow horizontal slit. The slide is suspended by a vertical cord or fine wire, which is led through a hole in the box, thence over a pulley to a groove cut round the divided circle that carries the movable brushes. Light from an incandescent lamp enters through a vertical slit placed alongside the dark slide, and is reflected by the galvanometer-mirror to the horizontal slit behind which the photographic plate moves. Each plate can be standardized by means of a Clark's cell and megohm, or by the Kelvin-balance method in § 3.

6. In order to either plot or photograph the wave-form of a magnetic flux F oscillating in any specimen of iron, a coil of a few turns (v say) of wire is coiled round it, and the ends led to one pair of brushes of the synchronous commutator. From the other pair of brushes the circuit is completed through a high resistance r and a dead-beat galvanometer as before. In this case also for each position of the movable brushes the galvanometer will show a steady deflexion, which is proportional to the instantaneous value of the flux at the instants of commutation.

As the time-constant of the secondary is inappreciable, the current in it will be

$$-\frac{v}{r}\frac{d\mathrm{F}}{dt},$$

and it can be shown, as in § 2, that if commutation be effected at times t, $t+\dfrac{\mathrm{T}}{2}$, $t+\mathrm{T}$, &c., the galvanometer-deflexion obtained is the same as would be produced by a steady current

$$=\frac{4v}{r\mathrm{T}}\mathrm{F}_t,$$

that is, a constant into the instantaneous value of the flux at the time t.

If β be the galvanometer-deflexion obtained corresponding to commutation at t, &c., and if λ be the reducing factor of the galvanometer as before, then

$$\mathrm{F}_t=\frac{\lambda r \mathrm{T}}{4v}\beta,$$

by which the instantaneous value of the flux at time t is given absolutely in terms of the corresponding galvanometer-deflexion.

7. The following table gives the actual galvanometer-readings taken in an experiment with a ring made of 1358 turns of varnished iron wire of ·0375 cm. diameter.

The magnetizing-current was supplied by a rotary transformer (which also drove the commutator) supplied with direct current from storage-cells. It flowed through a coil of 187 turns wound uniformly in one layer all round the ring. The M of the air-transformer was ·000610 henries; the number of turns v on the secondary of the ring was 10, and the mean length of the iron circuit 31·89 cms.

Iron Wire Ring. No. II.

r in C circuit = 30,000 ω.   r in F circuit 60,000 ω.
C by centiampere balance at start ·491 amp.
"   "   "   at finish ·490 amp.
Speed at start 1000 ∽ in 30·6 sec. $\lambda = \dfrac{5\cdot46}{10^8}$ amp.
"   at finish 1000 ∽ in 30·8 sec.

Divided Circle Readings (Degrees).	Galvanometer Readings. Zero = 100.		Divided Circle Readings (Degrees).	Galvanometer Readings. Zero = 100.	
	Current.	Flux.		Current.	Flux.
0	115·8	138·0	90	84·4	62·0
5	110·2	136·8	95	89·9	63·2
10	104·7	135·1	100	95·5	64·8
15	99·0	133·0	105	101·1	66·9
20	93·3	129·8	110	106·8	70·0
25	88·0	124·7	115	112·1	75·2
30	83·3	114·6	120	116·8	85·3
35	79·8	98·0	125	120·2	102·0
40	76·2	80·7	130	123·8	119·3
45	72·4	67·9	135	127·6	132·1
50	68·6	62·1	140	131·4	138·0
55	65·9	59·6	145	134·0	140·5
60	64·8	58·7	150	135·1	141·5
65	65·2	58·7	155	134·7	141·4
70	67·3	59·1	160	132·6	141·1
75	70·3	59·5	165	129·4	140·8
80	74·4	60·1	170	125·4	140·1
85	79·1	60·9	175	120·6	139·3
90	84·4	62·0	180	115·2	138·2

In taking the above readings the galvanometer-key was kept down all the time. This is not the best method, as most galvanometer suspensions will show elastic fatigue. When greater accuracy is desired, deflexions on either side of the zero can be taken for each position of the movable brushes. It is not necessary, as was done in the above set, to take readings over the whole wave ; readings over half a wave are sufficient [see § 12]. Telescope and mm. scale, placed about 40 cms. from the galvanometer, were used ; estimation to 0·1 mm.

8. From any 18 consecutive current-deflexions got from the above table, their square root of mean square will be found to be 23·76, and the galvanometer-factor being $5\cdot46 \times 10^{-8}$, the current in virtual amperes is by § 3

$$= \frac{5\cdot46}{10^8} \frac{r\mathrm{T}}{4\mathrm{M}} 23\cdot76$$

$$= \frac{5\cdot46}{10^8} \cdot \frac{30000 \dfrac{30\cdot7}{1000}}{4 \times 0\cdot00061} \cdot 23\cdot76 = \cdot4897 \text{ amp.},$$

which agrees well with the Kelvin-balance reading of ·4905. This close agreement may be to some extent an accident, as the time of 1000 periods was taken by a stop-watch, and could not be accepted as being very accurate. The value of M is not 2 parts in 600 in error. [In the above calculation the resistance of the galvanometer, 260 ohms, was neglected.]

9. Fig. 2 (Plate XXII.) shows the wave-forms of the current and flux plotted from the data in the above table, these are typical of such associated wave-forms when the iron (of low permeability in this sample) is fairly saturated at the peak; and fig. 3 shows the curve got by plotting B against H from the same experiment. The area of this closed curve is proportional to the energy dissipated per cycle by both hysteresis and Foucault currents in the iron. Research bearing on these areas is at present in hand, and I hope soon to be able to publish the results in a more complete paper.

A number of satisfactory curves for different wave-forms were obtained, but cannot be reproduced here for want of space.

The Wave Analyser.

10. The apparatus having been arranged as before, but with a different commutator (see fig. 4) which effects commutation $2n$ times per period, there will be, for each position of the movable brushes, a steady galvanometer deflexion which, multiplied by the same factor as before ($\lambda \frac{rT}{4M}$ for current or $\lambda \frac{rT}{4v}$ for flux), is equal to n times the instantaneous value of the nth component of the original wave at the alternate instants of commutation (and with its sign changed at the intermediate times of commutation).

For if the same conditions as in § 2 with regard to time constants be fulfilled the current through the galvanometer will be

$$-\frac{M}{r} \frac{dC}{dt},$$

and it is commutated at times

$$t, \, t+\frac{T}{2n}, \, t+2\frac{T}{2n}, \, \&c., \, \&c. \quad t+(2n-1)\frac{T}{2n}, \, t+T, \, \&c.,$$

hence the quantity Q of electricity that flows through the

galvanometer in one complete period is given by

$$-\frac{r\mathrm{Q}}{\mathrm{M}} = \int_{t}^{t+\frac{\mathrm{T}}{2n}} d\mathrm{C} - \int_{t+\frac{\mathrm{T}}{2n}}^{t+2\frac{\mathrm{T}}{2n}} d\mathrm{C} + \&\mathrm{c.} \quad -\int_{t+(2n-1)\frac{\mathrm{T}}{2n}}^{t+\mathrm{T}} d\mathrm{C}$$

$$= -2\left\{\mathrm{C}_t - \mathrm{C}_{t+\frac{\mathrm{T}}{2n}} + \mathrm{C}_{t+2\frac{\mathrm{T}}{2n}} - \&\mathrm{c.} - \mathrm{C}_{t+(2n-1)\frac{\mathrm{T}}{2n}}\right\}.$$

If $\phi(t)$ be the expression in the brackets we see that

$$\phi(t) = -\phi\left(t + \frac{\mathrm{T}}{2n}\right) = \phi\left(t + \frac{\mathrm{T}}{n}\right),$$

so that $\phi(t)$ is periodic, of period $\frac{\mathrm{T}}{n}$, and contains only odd harmonics.

Hence, if $\mathrm{C}_t \equiv \Sigma_p \, c^p \sin(p\omega t - \theta_p)$ where $\omega\mathrm{T} = 2\pi$, only those terms whose arguments are $n\omega t, 3n\omega t, 5n\omega t, \&\mathrm{c.}$ need be considered in determining the value of $\phi(t)$. Neglecting the other terms we find that the remainders of

$$\mathrm{C}_t, \;\; -\mathrm{C}_{t+\frac{\mathrm{T}}{2n}}, \; \mathrm{C}_{t+2\frac{\mathrm{T}}{2n}}, \;\; \&\mathrm{c.} \;\; -\mathrm{C}_{t+(2n-1)\frac{\mathrm{T}}{2n}},$$

are all equal to one another, and to the nth component of C_t, and as there are $2n$ of them

$$\phi(t) = 2n \; (n\text{th component})$$
$$= 2n\mathrm{N} \text{ say}$$
$$\therefore \; \frac{r\mathrm{Q}}{4\mathrm{M}} = n\mathrm{N}_t,$$

and the steady galvanometer deflexion γ_n will be the same as that produced by a continuous current

$$\frac{\mathrm{Q}}{\mathrm{T}} \text{ or } \frac{4\mathrm{M}}{r\mathrm{T}} n\mathrm{N}_t.$$

λ being the galvanometer factor, we find that

$$n\mathrm{N}_t = \lambda \frac{r\mathrm{T}}{4\mathrm{M}} \gamma_n, \text{ hence, } \&\mathrm{c.}$$

Similarly, when magnetic flux is under examination, if β_n be the galvanometer deflexion when commutation occurs $2n$ times per period, then n times the instantaneous value of its nth component at alternate times of commutation is

$$= \lambda \frac{r\mathrm{T}}{4v} \beta_n.$$

11. On the formula established in § 10, that $2n$ times the nth component of any periodic function $f(t)$ is equal to

$$f(t) - f\left(t + \frac{T}{2n}\right) + f\left(t + 2\frac{T}{2n}\right) - \&\text{c.} \quad -f\left(t + \overline{2n-1}\,\frac{T}{2n}\right),$$

a simple method of analysing periodic curves (specially simple for those containing only odd harmonics) can be based which is very analogous to that proposed by Wedmore[*].

The formula on which his method depends is

$$f(t) + f\left(t + \frac{T}{n}\right) + f\left(t + 2\frac{T}{n}\right) + \&\text{c.} \quad + f\left\{t + (n-1)\frac{T}{n}\right\}$$
$$= n\{c_n \sin(n\omega t - \theta_n) + c_{2n} \sin(2n\omega t - \theta_{2n}) + \&\text{c.}\}.$$

Thus the function he gets from *adding* n equal spaced ordinates of a full wave is periodic, of periodicity n times that of the original wave, and contains only those of its harmonics that are any multiples of n, while the method here proposed will give components which are odd periodic functions, in fact the components as I have defined them in the early part of this paper.

12. It is obvious that the wave-analyser can be applied to wave-forms that contain both even and odd harmonics, and that the operation in § 2 is the simplest example of its use.

With a commutator that acts four times per period we have a very accurate means of testing whether the wave-forms of E.M.F., current, or flux that originate from a given generator contain only odd harmonics.

A careful test by this method was made of the E.M.F. curve of a Parson's generator, both when on practically open circuit and when carrying a heavy inductive load. In neither case was there the slightest indication that a second component was present. A similar test was made of the wave-form of current from a rotary transformer supplied with direct current from storage-cells, and also of the wave-form of the magnetic flux produced by this current in a ring of iron wire. In these cases again the second component was absent, the galvanometer remaining steady at zero independently of the position of the movable brushes. This of course does not prove that in wave-forms produced by other machines no second component is present.

13. Fig. 4 (Pl. XXIII.) shows a wave-analyser made to obtain the 1st, 3rd, 5th, and 7th components of E.M.F., current, or flux-waves when driven direct by the generator if it is a four-pole

* See Journal Inst. Elect. Engineers, vol. xxv. p. 224 (1896).

one, or when driven by a four-pole synchronous motor. If driven direct by a two-pole generator it would obtain the 2nd, 6th, 10th, and 14th components if such existed.

It is built up of five castings of hard gun-metal, turned inside and outside, insulated from the spindle by ebonite tubes and from each other by mica. Two nuts screwed on the spindle bolt the parts firmly together, and the hollow space inside is filled with an insulating compound that is poured in hot.

Fig. 4 is a photograph of the complete commutator, and shows how the brushes are mounted on a kind of squirrel-cage with four bars, which rotates on tubular extensions of the bearings.

The divided circle and tangent screw, by means of which the brushes are rotated and their different positions determined, can also be seen. Carbon brushes are used, and they are double-banked, two connected brushes being at opposite ends of a diameter of the drum. This greatly lessens the risk of loss of contact due to vibration. To the free end of the spindle is attached a speed-counter, which is so arranged that if desired it closes an electric circuit through a chronograph every 50 revolutions, and so a continuous record of the speed during an experiment can be kept. If great accuracy is not desired, the speed-counter can be used with a stop-watch. The smallest of the insulation-resistances measured between brushes bearing on contiguous sections of the drum was 250 megohms, and it runs without perceptible vibration at 2000 revolutions per minute.

14. *Examples of Wave-Analysis by Photographic Method.*— Fig. 5 is a wave-form of current from rotary transformer with 3 times its 3rd, 5 times its 5th, and 7 times its 7th component, the latter just perceptible.

The components in all photographs are similarly multiplied by numbers equal to the order of the component, and are in their correct positions as to phase relative to the first and to each other.

The fundamental harmonic of the wave can obviously be determined very approximately by subtracting from the full wave-form its different components.

As an illustration of the power of the method and the working of the instrument I may refer to fig. 6, obtained from a 2000 volt-150 KW. Parsons generator at 45 ampere load, partly inductive and partly a water-resistance.

With the wave-tracer we can obviously obtain all the periodic quantities involved in the action of a transformer, both in magnitude and in correct relative position as to

phase. I have obtained an interesting series of these, but as the examples already given are sufficient for the purposes of this paper, I will hold it over for publication at an early date.

15. It is interesting to note that in the device I have described the quantity whose wave-form is to be determined and analysed is first differentiated with respect to the time, then integrated by the galvanometer, the limits to the integrals being assigned by the commutator. It is possibly that the same device might be applied to determine the wave-forms of periodic quantities other than those that arise in alternate-current working. All that would be necessary for this would be to obtain an E.M.F. or a current proportional to the time-rate of variation of the quantity whose wave-form is required.

In conclusion I wish to thank my friend Mr. W. Stone, Electrical Engineer to the State Railway Department, for his invaluable assistance. Not only was the commutator which was used in getting the preceding results built under his supervision, but he, at great personal trouble, put at my disposal one of his Parsons turbine-driven generators, and assisted in the experiments whereby a number of the wave-forms given in this paper were obtained.

LXX. *The Effect of Errors in Ruling on the Appearance of a Diffraction-Grating. By* H. S. ALLEN, *M.A., B.Sc.**

[Plate XXIV.]

PART II.

IN a former paper † under this title I drew attention to the bands which can generally be seen when a diffraction-grating is viewed by the light going to form a single spectral line, and explained the formation of these bands at points of the grating where a change is taking place in the value of the ruling. This method has proved of service in locating the errors in Lord Blythswood's dividing-engine ; and has been employed by him in determining the correction required to eliminate the residual periodic error arising from the ratchet-wheel and thrust-block. I have therefore thought fit to give an account of a modification of the method formerly described, which renders it possible to determine directly the parts of the grating where the ruling is wider (or narrower) than the average, instead of the places at which the value of

the ruling is changing.   This new method resembles in principle, and was indeed suggested by, Toepler's Method of Striæ (*Schlieren Methode*), which I have had occasion to employ in another connexion *.

In the diagram (fig. 1, Pl. XXIV.) ABC represents a section at right angles to the ruling of a plane grating, the ruling in the portion AB being wider than that in the portion BC. The grating is supposed to be illuminated by a parallel beam of light from the left-hand side of the diagram.   The light from the broader ruling which goes to form a single line of the spectrum is brought to a focus at F_1 by the objective of the observing-telescope, the light from the narrower ruling is brought to a focus at F_2.   At A'B'C' an image of the grating would be formed by the objective.   In the focus of this lens is placed a slit S_1S_2 with its aperture parallel to the collimating-slit and to the lines of the grating.   The slit is capable of adjustment in a direction at right angles to its own length and to the axis of the telescope.   It is evident that a displacement of the slit towards S_1 will have the effect of obstructing the light coming to a focus at F_2; so that the light coming from the narrower ruling will be cut off and B'C' will not be illuminated.   On the other hand, a displacement of the slit in the opposite direction allows the light from the narrower rulings to pass, but cuts off that from the wider rulings.   By placing a photographic plate at A'B'C' it is possible to obtain a photograph of the grating, showing the narrower rulings either as dark or as light bands.   This method of determining the errors in a grating is more difficult in practice than the former one, as the adjustments have to be made with much greater care, and a longer exposure is required; but it has an advantage in that the negative is placed at the optical image A'B'C' instead of in a position such as A'''B'''C'''.

In using the photographs obtained by these methods to determine the position of the periodic error existing in the dividing-engine the negatives were measured under the Blythswood micrometer (Hilger) with a low-power microscope.   The position of each band was thus given by its distance from the commencement of the ruling.   Knowing the number of lines cut on the strip examined and the number (720) corresponding to one revolution of the ratchet-wheel, it was easy to determine the breadth representing a single " period."   The figures giving the positions of the bands were divided into sets, each set corresponding to one period, and the number fixing the position of any particular

band was reduced so as to give the distance of the band from the commencement of the period. It was then not difficult to pick out the bands that recurred from period to period, discarding occasional accidental errors. In this way the position of the correction necessary to eliminate such periodic errors was found. The magnitude of the correction was subsequently determined by a process of trial and error.

The illustrations accompanying this paper are reproduced from photographs of gratings taken by the methods described. They have been enlarged from the original negatives by about eight diameters, so as to make the reproductions of the same breadth as the ruled surface of the grating. Figs. 2 and 3 (Pl. XXIV.) are photographs of a strip cut on a flat speculum by Lord Blythswood's dividing-engine before any correction had been applied. This strip was selected for purposes of illustration, as the bands are more pronounced in consequence of the greater errors in ruling. The rate of ruling is about 14,400 lines per inch. The photographs were taken by the light from the blue line of helium (wavelength 4471·5) in an order to the left of the central image.

Fig. 2 was obtained by placing the photographic plate in a position such as $A'''B'''C'''$, the slit S_1S_2 being removed. In this case dark bands on the photograph correspond to places where the rate of ruling is diminishing in passing from left to right over the grating (right to left over the reproduction).

Fig. 3 was obtained by the method described in the present paper, the photographic plate being in the position $A'B'C'$ (in this case so arranged as to be perpendicular to the axis of the telescope) and the slit being displaced towards S_1. Here dark bands correspond to places where the ruling is narrower than the average.

In a sense, then, we may regard fig. 2 as a pictorial representation of the first differential of the ruling represented by fig. 3.

A Curious Case of Interference.

In the previous paper (*loc. cit.*) it was pointed out that the points F_1, F_2, regarded as two sources of light might produce interference-effects as in Fresnel's experiments ; but in general the effect could be disregarded.

I have, however, met with one interesting case in which interference-bands are produced by a grating. This is a grating ruled on a speculum-metal plate with Rutherfurd's dividing-engine, July 10, 1880. It is marked as containing 15,840 spaces at the rate of 17,296 per inch. Each line of the spectrum given by the grating is accompanied by a pair of

" ghosts," one on either side of the main line at an angular distance from it of about 4′ 30″ (that is, about three times the distance between the D lines in the first order). If the grating is examined with the eyepiece removed, it is seen to be crossed by a series of strongly marked dark bands, spaced at regular intervals, parallel to the lines ruled on the surface. At first it was assumed that these bands were coincident with periodic errors in ruling, being formed in the manner explained in the former paper. But when the grating was tested by the method just described, it was found that the bands in question entirely disappeared when the slit was adjusted so as to allow no light but that from the main line to pass. To render the bands visible it was necessary to allow the light from at least one of the ghosts to pass the slit together with that from the main line. So that the effect must be due to interference between the light from the ghosts and that from the main line.

The reproductions (figs. 4 & 5, Pl. XXIV.) are enlargements from photographs showing the interference-bands due, in the first case to the main line and two ghosts, in the second case to the main line and one ghost. The latter is the well-known case of Fresnel's experiment, and the reproduction (fig. 5) shows the series of alternately light and dark bands spaced at approximately equal intervals, similar to that which would be obtained with the bi-prism. The former case presents more novelty, as the interference from *three* light-sources has not, so far as I am aware, been observed previously. The main line is naturally much brighter than the ghosts; so that the interference-bands may be regarded as arising from two equal sources with a source of double the intensity midway between them.

It is evident that the system of bands obtained in a plane parallel to the plane of the sources will be symmetrical with regard to a central line. It can be shown by elementary methods that at some distance from this central line the bands will be practically identical with those obtained from a pair of sources. The centre line itself may be either bright or dark, according to its distance from the plane of the sources. A careful examination of fig. 4 will show a fine dark line in the centre of the symmetrical system of bands.

If it were desirable, it would be possible to produce cases of interference of still greater complexity by ruling gratings with definite periodic errors so as to give a number of 'ghosts.'

In conclusion, I must express my great indebtedness to Lord Blythswood for his continued interest and encouragement in this work.

Blythswood Laboratory,
    Renfrew, N.B.

LXXI. *Water Radioactivity.* By E. P. Adams [*].

THE following is an account of some experiments made to determine the nature of the radioactive gas found in Cambridge tap-water by Prof. J. J. Thomson. It will be seen that this gas possesses properties remarkably similar to the emanation of radium.

Rutherford and Curie have shown that the radioactivity of the radium emanation decays according to the law

$$I = I_0 \epsilon^{-\lambda t},$$

where λ is a constant over a wide range of conditions. It is therefore important to determine the rate of decay of the water radioactivity and to see whether a similar law holds.

A gold-leaf electroscope of the type devised by C. T. R. Wilson has been used in most of these experiments, but instead of using sulphur insulation, a quartz rod, 10 cm. long and 2·5 mm. in diameter, was employed. The volume of the electroscope was about 200 c.c. The electroscope was charged to 200 volts, and the movement of the gold-leaf read by means of a microscope containing a scale in the eyepiece. The gases to be studied were admitted directly into the electroscope, which was connected to a Toepler pump, through a tightly-packed plug of glass-wool and a phosphorus-pentoxide drying-bulb.

Rate of Decay of the Radioactivity.

Water direct from a tap was passed through a copper tube which was placed in a combustion furnace, so that the water was thoroughly boiled on going through the tube. The gas thus driven out of the water was collected in a bottle which stood in a trough through which a stream of cold water continually circulated. In this way all the gas from a large volume of water could be quickly collected, and the radioactivity obtained in a very concentrated form. Sufficient gas was collected to fill the electroscope to atmospheric pressure. Readings of the rate of movement of the gold-leaf were taken from time to time in order to get the rate at which the radioactivity decayed. Immediately after the gas was introduced into the electroscope, the leak was at the rate of 17 divisions per minute. This gradually increased for about two hours, after which it began to decrease slowly. The time required for the gold-leaf to move over the same 30 divisions of the scale was determined each time, so that

the sensitiveness was the same at each reading. The following table gives the results, beginning at the time when the maximum conductivity is reached:—

Hours.	Time to leak 30 divisions in seconds.	Reciprocals.	$1/\lambda$.
0	53·2	188	
16·7	62·7	160	366,000
40·4	77·6	129	401,000
64·8	92·7	108	494,000
88·9	116·3	86	381,000
139·6	190·0	53	372,000
160·8	217·9	46	573,000

The accompanying figure is plotted from the above table. The ordinates are the numbers in the third column, which are proportional to the conductivity of the gas, and the abscissæ are the times in hours.
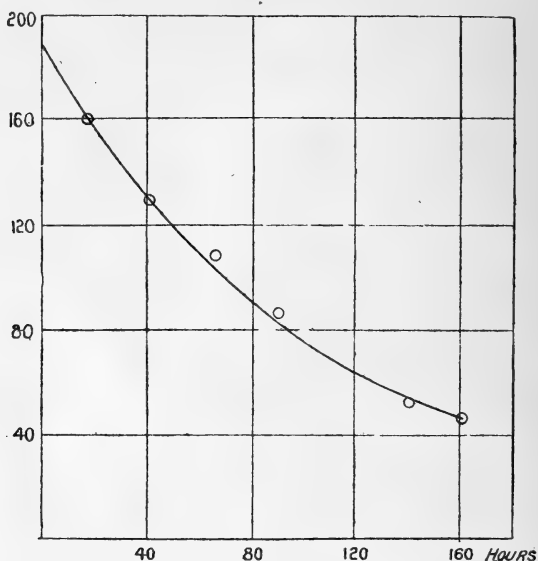
Assuming, as in the case of radium emanation, that the decay of the radioactivity can be represented by the formula

$$I = I_0 \epsilon^{-\lambda t},$$

the values of $1/\lambda$ corresponding to the successive intervals are given in the fourth column of the above table. The time in this formula is expressed in seconds. Giving these

values of $1/\lambda$ weights proportional to the corresponding intervals, we get for the mean value

$$1/\lambda = 425,000.$$

For radium emanation Rutherford gets

$$1/\lambda = 463,000,$$

while Curie gets for the same

$$1/\lambda = 497,000.$$

The value of λ obtained thus for tap-water is in sufficiently good agreement with that obtained for radium emanation to make it very probable that the former is actually due to radium emanation.

Similar results have been obtained for air which has been thoroughly bubbled through tap-water, so that it seems that the conductivity produced in air by bubbling it through tap-water is due to the same cause as the conductivity of the dissolved gases in water.

Absorption of Radium Emanation by Water.

According to Rutherford radium emanation passes unchanged through water. But experiments were made to see whether some of the emanation, if only a very small proportion of it, was not absorbed by the water. This was done as follows:—

A weak solution of a radium compound was prepared, using distilled water which had been thoroughly boiled to expel all air. Air was then bubbled through the radium solution, having first passed through a tight plug of glass-wool. From the radium solution the air passed through another tight plug of glass-wool, and then through a large bottle of thoroughly boiled distilled water. This was allowed to go on for three hours. At the end of this time air was blown into the bottle to sweep out thoroughly all of the radium emanation which remained above the surface of the water. Some of the water was then poured into a flask of about a litre capacity, provided with a rubber stopper with two holes, through one of which a glass tube passed to the bottom of the flask, while through the other a glass tube passed just through the stopper. The water was then boiled, and at the same time air from the room bubbled through it into a gas-holder. In this way all the dissolved gas and emanation, if any, was collected in the gas-holder. From the gas-holder the air was introduced into the electroscope. Before this air was put in, when the electroscope was filled

with dry air from the room, the leak in the electroscope was at the rate of 0·5 division per minute. But with the air boiled out of water through which radium emanation had passed, the rate increased to 10 divisions per minute, and after about two hours it increased to 18 divisions per minute. It then began to decrease. On allowing the same sample of water to stand for several days, and again testing the air drawn through it as above, it was found that the rate of leak was the same as for air taken directly from the room. This shows conclusively that a small amount of radium emanation is absorbed by water, and that the effect observed is not due to radium carried over in spray from the radium solution. The glass-wool plug stopped all of that.

When the same volume of water direct from the tap is tested in the above way, the maximum rate of leak is about 2 divisions per minute. But this varies considerably at different times. Hence a very small quantity of radium emanation dissolved in water would be sufficient to account for this.

Equal volumes of water containing the emanation in solution were tested from day to day to see whether the radioactivity decayed at the same rate in solution. It was found that the decay was considerably greater than for emanation kept in a closed vessel. The reason for this is that the decay is due to two causes: first, the decay in radioactivity of the emanation itself; and second, the escape of emanation from the solution. But a sample of water containing emanation in solution, when kept closed so that none of the emanation can escape, is found to decay at the same rate as emanation which is kept in a closed vessel.

But an important difference has been found between distilled water containing radium emanation in solution and tap-water. After the former has been thoroughly boiled so as to drive out all the emanation, it entirely loses its radioactivity, and on letting it stand for any length of time, and again testing the air drawn through the boiling water, this air is found to have no greater conductivity than air taken direct from the room. But with tap-water it is different. It is impossible to get rid of the radioactivity entirely by boiling the water so as to drive out all the dissolved gases. A minute trace remains. In fact, tap-water behaves as if it contained, in addition to dissolved emanation, an extremely minute quantity of a radium salt in solution. All attempts to evaporate a large quantity of water to dryness, and to find any radioactivity in the solid residue, have so far failed. But the evidence for the presence of the radioactive salt is

indirect. If a flask of tap-water is closed up air-tight, and allowed to stand for several days, it is found that the rate of decay of its radioactivity is considerably less than that of the gas driven out of it; in fact, on one or two occasions the radioactivity has actually increased on standing. This can be explained only by assuming that there is a continuous production of a radioactive emanation in the water. Further, tap-water is found to partially recover its radioactivity on standing after it has been thoroughly boiled, but never to its original value. In general, the recovery has been to about one-tenth of its original value.

Induced Radioactivity.

Just as radium emanation possesses the power of inducing radioactivity on bodies immersed in it, so the radioactive gas from water possesses the same property. To this is due the fact that when the gas is introduced into the electroscope the rate of leak increases for a time before it begins to decrease. The rate of decay of this induced radioactivity is the same when produced by the gas from tap-water as when produced by radium emanation. That is, it decays according to an exponential law, such that it falls to half its value in about 35 minutes.

This induced radioactivity is greater on a negatively charged conductor than on a positively charged one. When a conductor is not charged, there appears to be no induced radioactivity. When charged positively or negatively, the amount of induced radioactivity is at first proportional to the absolute value of the potential, and approaches a limiting value with high potentials. The rate of decay is the same for the induced radioactivity, whether produced by positive or negative electrification.

Experiments on induced radioactivity were made with air that had been circulated through water several times by passing through a water-pump. In this way the conductivity of the air in a large tank could be increased to about 80 times its normal value. Metal rods were suspended in this tank, and these could be charged to any desired potential. They were taken out after a definite time and suspended in another tank filled with ordinary air, and the current through this second tank, when the rod was connected to a quadrant electrometer, and the tank to a battery, compared with a similar rod which had not been made radioactive. In every case the voltage applied to the testing-tank was sufficient to give the saturation current. After each test the brass rod

was thoroughly sand-papered so as to remove all the induced radioactivity. The following table gives the relative amounts of induced radioactivity of a brass rod when charged to different potentials for the same length of time :—

Charged for 10 minutes at	Induced Radioactivity.
−1000 volts	32
− 800	28
− 600	23
− 400	18
− 200	12
+1000	5
+ 800	4·5
+ 600	3·5
+ 400	3·0
+ 200	2·0
0	0

To compare this with the effect of radium emanation, air was drawn through a weak solution of a radium compound into a closed brass cylinder of about 8 litres capacity, in which a brass rod could be suspended and charged to any desired potential. The rod was then tested for radioactivity in another similar vessel filled with room air, the rod being connected to the electrometer and the cylinder to the battery. The following results give the magnitude of the effect :—

Charged 10 minutes at	Induced Radioactivity.
+ 200 volts	26
+400	50
−200	150
0	6

It is thus seen that the induced radioactivity due to radium emanation on a rod at 0 potential is only about one-eighth of that on a rod at +400 volts. But in the case of the gas from water, this would have been too small an amount to measure. This probably accounts for the fact that with the radioactive gas from water it has not been possible to observe any induced radioactivity on a conductor at 0 potential. It is also seen that for the induced radioactivity due to both radium emanation and the water radioactivity, the effects are about six times as great for a negatively charged conductor as for a positively charged one.

Conclusion.

The above results lead to the view that the water radioactivity is due to a substance very similar to, if not identical

with, the emanation of radium, which is dissolved in the water. So far there is no indication of the presence of a sufficient amount of an emanating substance present in the water to account for the effects observed. The greater part of the emanation must be supposed due to radioactive emanating substances distributed near the source of the water.

It is a pleasure to express my sincere thanks to Prof. Thomson for the continual interest he has shown and help he has given during the course of this work.

Cavendish Laboratory, July 1903.

LXXII. *Preliminary Note on the Effect of Pressure upon Arc Spectra.* By J. E. PETAVEL *and* R. S. HUTTON*.

[Plate XXV.]

Introduction.

IN the course of an investigation on the effect of high gaseous pressures upon the chemical reactions occurring at the temperature of the electric furnace, it was found advisable to make experiments with electrodes of different materials ; the spectra as observed visually seeming to show some points of interest, the brief investigation of the subject herein described was undertaken. The shift of the spectrum lines caused by pressure has already been carefully investigated by Humphreys, Mohler, Jewell, and others †. A higher dispersion than that available in our experiments would be necessary to add anything further to this side of the question, which, therefore, we do not propose to deal with. In general, such results as are given and the explanations proposed must be regarded as preliminary in character ; but it is hoped that before long the work will be continued in the Physical Laboratories of the Owens College.

The apparatus consisted of a strong steel enclosure, suitable for withstanding pressures up to 450 atmospheres, having an internal diameter of 11 inches and a capacity of about 20 litres. The electrodes, which were much larger than is usual for such work, varied in diameter from $\frac{1}{4}$ inch to 2 inches, they were supported by feeding rods passing air-tight through stuffing boxes, and capable of adjustment by a powerful screw gear. Apart from the usual valves for admission or circulation of gases under pressure, the apparatus is provided with

† Jewell, Astrophys. Journ. iii. p. 89 (1896) ; Humphreys and Mohler, Astrophys. Journ. iii. p. 114 (1896); Mohler, Astrophys. Journ. iv. p. 175 (1896) ; Humphreys, Astrophys. Journ. iv. p. 249 (1896), vi. p. 169 (1897).

a glass window through which the arc can be observed or
the spectrum photographed*. In the course of these ex-
periments both direct and alternating currents were used,
the maximum electromotive force being about 500 volts.
With these voltages it was not found possible at the highest
pressure to maintain an unbroken arc of sufficient length for
satisfactory results to be obtained, and most of the photo-
graphs record the spectra of numerous successive flashes.
Another apparatus considerably smaller but constructed to
work with very much higher voltages has just been com-
pleted, and doubtless will prove more suitable for the
requirements of this work.

The spectroscope employed was provided with lenses and a
compound Rutherford prism of specially light flint glass, the
aperture of the lenses being about 5 cms., and the focal
length of the collimator 40 cms. The spectra were photo-
graphed on isochromatic plates 15 cms. long, the portion of
the spectrum studied being between wave-lengths 5500–3500.
The light from the arc was reflected by a mirror, an image
being formed on the slit of the spectroscope by means of a
lens. Upon removing the mirror the image of a spark
placed in line with the axis of the collimator, and at the same
distance from the slit as the arc, was projected on to the slit
by the same lens.

In all cases a reference spectrum was taken, this being a
spark of the substance investigated with self-induction and
capacity in the circuit, as fully described by Hemsalech†.

Carbon.

A number of photographs of the spectrum of the carbon
arc in air at 7, 40, 75, and 100 atmospheres, as also in a
mixture of air and hydrogen at 16 atmospheres pressure,
were taken. Owing to the rapid burning away of the carbons
in air under these pressures the gaseous atmosphere always
contained a considerable percentage of carbon monoxide, and
in addition, generally, carbonic acid gas. The red vapours
of oxides of nitrogen, referred to by Wilson and FitzGerald‡,
were conspicuous by their absence, and chemical analysis
failed to show any noticeable formation of nitric acid. As
far as could be judged the atmosphere remained fairly clear,

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* A view of the apparatus will be found in Journ. Instit. Electr. Eng.
vol. xxxii. p. 236 (1903).

† G. A. Hemsalech, *Comptes Rend.* vol. cxxix. p. 285 (1899); *Journ.
de Physique,* (3) vol. viii. p. 642 (1899).

‡ Proc. Roy. Soc. vol. lx. p. 378 (1896).

but after a long run the windows were clouded by a coating of finely divided carbon. During the time of exposure the arc was carefully observed by means of a direct-vision spectroscope, and was found to show very distinct alteration in appearance according to the conditions under which it was burning. When the arc was short and steady the difference from the spectrum of the normal carbon arc was not very marked, whereas when the carbons were drawn apart and the arc was on the point of breaking, a spectrum showing more sharply the fluted carbon bands flashed out vividly for a short time. Several photographs were taken with the object of studying this effect, the observer screening off the light by means of a photographic shutter except when this phenomenon was observed. The photographs taken at about 40 atmospheres pressure show a marked reversal of the five heads of the cyanogen band beginning at 3883. It may be noted that the analysis of the gas indicated that at the time these photographs were taken only a very small percentage of the oxygen remained uncombined. With a view to increasing the percentage of cyanogen compounds in the atmosphere, hydrogen was admitted step by step to the furnace charged with compressed air, care being naturally taken to allow time for combination to take place, so that the mixture should always be below the explosive limit.

With such a mixture the reversal was also photographed easily at 16 atmospheres pressure. The presence of hydrocyanic acid could be readily detected in the residual gases.

The accompanying table (p. 572) gives details of the conditions under which the experiments were carried out.

The reversal of the cyanogen bands has been previously observed by Liveing and Dewar*, but only under special conditions, and when cyanide of titanium or borate of ammonium were introduced into the furnace. As above stated, the photographs taken while the arc was burning normally, in most cases, do not show the reversal of the cyanogen band. A point, however, worthy of notice is that the calcium lines occur in nearly all the photographs, and are generally sharply reversed ; the blue line 4226 appears much more brilliantly than the H and K lines. The appearance of the calcium lines in those spectra which show the reversed cyanogen band is still more curious since here the calcium blue line 4226 is very strongly reversed, whereas there is no indication of the presence of the H and K lines.

* Liveing and Dewar, Proc. R. S. vol. xxxiii. p. 3 (1881).

Carbon Arc Spectra.—Condition of Experiments.

	Electrode.	Gas.	Initial Pressure.	E.M.F. (total volts on circuit)	Current.	Observations.
1.	+ve { 27 mm. diam. / 31 cms. long. −ve {	Originally air at time of exp. Oxygen reduced to 0·2 per cent. CO 43·6 per cent.	7 atm.	140 volts continuous.	250 amps.	Similar to normal carbon arc. Steady arc.
2.	Same as in No. 1.	Same as last.	8 atm.	400 alternating, Choking-coil and 12 ohms resistance in series.	20 amps.	Similar to above.
3.	+ve { 40 mm. diam. / 31 cms. long. −ve { 27 mm. diam. / 31 cms. long.	Gas originally air at time of exp. 7 p. c. CO₂ 19 p. c. CO. 0·5 per cent. Oxygen.	40 atm.	110 volts continuous.	20-30 amps.	A number of flashes as the arc broke. Shows reversal of cyanogen band 3883, &c. H & K invisible & 4226 Ca reversed.
4.	Same as No. 3.	Same as No. 3.	40 atm.	100 volts continuous.	40 amps.	Same. H & K invisible, 4226 Ca reversed.
5.	Same as 1 & 2.	Originally air. (Not all burnt out.)	76 atm.	400 alternating, Choking-coil and 12 ohms resistance in series.	20 amps.	Similar to normal carbon arc. H & K lines & 4226 Ca well reversed.
6.	Same as 1, 2, & 5.	Originally air. (Not all burnt out.)	83 atm.	300 alternating, Choking-coil and 2 ohms resistance in series.	50 amps.	Similar to normal carbon arc. H & K & 4226 Ca well reversed.
7.	Same as 1, 2, 5, & 6.	Air, freshly filled in.	100 atm.	300 alternating, Choking-coil and 2 ohms resistance in series.	50 amps.	Similar to normal carbon arc. H & K & 4226 Ca reversed.
8.	+ve { 18 mm. diam. / 30 cms. long. −ve {	Air and Hydrogen. CO₂ 0·2 p. c. OH₄ 3 p. c. O² 0·2 p. c. H 16·5 p. c. CO 18·8 p. c.	16 atm.	160 volts continuous.	50 amps.	Similar result to No. 3. Cyanogen band 3883 well reversed. No calcium lines visible.

Copper.

No very marked differences from the normal arc spectrum are noticeable in the intensity and character of the copper arc as photographed at 40 atmospheres pressure in air ; the spectrum corresponding line for line with that given by Kayser and Runge ; it is, therefore, not necessary to reproduce the spectrum.

An alternating current of 40 amperes and 320 volts was used.

Iron Arc-Spectrum.

Photographs of the iron arc were taken at 0·08 and 44 atmospheres, in each case with a comparison spectrum of the iron spark with self-induction and capacity. The relative intensity and general character are closely similar to the arc at atmospheric pressure as shown in the map of Kayser and Runge.

A point noticeable about the 44 atmosphere arc is that a number of the lines in the ultra-violet above 3800 are very strongly reversed.

Since these experiments were carried out an excellent paper on this subject has been published in the Astrophysical Journal *. Prof. Hale has used for this work a much higher dispersion than was employed in our experiments, and promises a further communication on the subject. It is therefore not necessary for us to deal further with this effect.

Iron Glow-Spectrum.

The spectra which we are about to deal with were taken in the large steel enclosure described above, the electrodes consisting, the one of a $1\frac{1}{4}$-inch iron bar, the other of a $\frac{3}{8}$-inch iron rod. The whole apparatus was exhausted. It was then noticed that when the length of the arc was increased to anything above 2 or 3 mm., the nature of the discharge completely changed. With the 2 mm. arc a thin light blue line was seen to pass in a practically straight line between the points of the two electrodes; the spectrum given under the conditions is substantially similar to the iron arc at all other pressures, and will not therefore be further discussed.

So soon as the distance between the electrodes was increased the whole gaseous mass contained in the enclosure commenced to glow brightly, the discharge no longer leaving the electrodes at their nearest points but being uniformly distributed over the last 3 or 4 inches of their length.

* Hale and Kent, Astrophys. Journ. xvii. p. 154 (1903).

The two iron rods soon became heated throughout the greater part of their length and a fused bead of metal occasionally dropped from the end of the smaller one. This glow, which was of a bright yellow colour, was very luminous and easy to photograph. The electromotive force required seemed to be practically independent of the distance apart of the electrodes. During most of the experiments this distance was between 1 and 2 inches, at which length the current remained fairly constant. The E.M.F. was between 550 and 650 volts on the terminals of the alternator, a non-inductive resistance of 45 ohms being placed in series with the arc. The current varied from 6 to 9 amperes.

The first spectrum was taken in air at a pressure of 17 mm., the pressure being increased for each successive photograph without interrupting the current. The exact values for each experiment are given in the subjoined table referring to Plate XXV. It is worthy of note that after the arc was stopped the pressure fell to half its former value.

So far as the spectra are concerned, the most noticeable difference from that usually observed lies in the fact that so few of the lines come out with any marked intensity. With the relatively large power used the entire discharge seems to be taken by the iron vapour, the gas-spectrum being altogether absent. Direct measurement and comparison with the ordinary iron arc shows that the simplification already noticeable in the self-induction spark is carried still further in the case of the glow-discharge, for whereas a small number of the brighter iron lines retain their intensity unimpared, the greater number are considerably diminished or are altogether absent. This is particularly remarkable in view of the recent work of de Watteville[*], who has shown that the spectrum of iron as obtained in the cone of the Bunsen flame presents an appearance, so far as the relative intensity and general character of the iron lines is concerned, essentially similar to that seen in the iron spark with self-induction, and thus is still much more complicated than the one with which we are now dealing. Although most of the iron lines found by Hartley [†] in the oxy-hydrogen flame-spectrum occur also in the glow-spectrum, the intensities and appearance are widely different and no systematic connexion between the two could be found.

Broadly speaking the effect of increase of pressure within these limits (17 to 105 mm.) is to still further simplify the spectrum. At the higher pressures, however, the first traces

[*] C. de Watteville, *Comptes Rendus*, vol. cxxxv. p. 1329 (1902).
[†] Hartley, Phil. Trans. 1894, A. p. 161.

of a continuous spectrum appear. A few lines form an exception to this general rule, and are more intense at the higher than at the lower pressure.

The accompanying table gives a list of all lines clearly visible on the photographs of the spectra taken at 17 and 105 mm. pressure (see Plate XXV. *a* and *e*). The relative intensities are given in columns 2 and 3 for these two pressures, and to more clearly emphasize the selective effect, the intensities of these same lines in the self-induction spark and ordinary arc are included in columns 4 and 5. The wavelengths and intensities of the arc are taken from Kayser and Runge ; the intensities of the self-induction spark from Hemsalech.

Table of Iron Glow Spectra.

	Pressure in mm. Hg.	Volts at Alternator Terminals.	Amperes.
a ...	17	550	6
b ...	33	450	7
c ...	46·	570	8
d ...	78	600	8
e ...	105	640	8
f ...	Self-induction spark at atmospheric pressure.		

Iron Glow Spectrum.

Wave-lengths.	Intensities *.				Wave-lengths.	Intensities *.			
	e.	*a.*	H.	K & R.		*e.*	*a.*	H.	K & R.
5455·80	4	1 n	8 s	10	5171·71 ⎱	3 n	1 n	2 s	8
5447·05	4	1 n	8 s	10	5167·50 ⎰				10
5434·66	5	1 n	4 s	8	5105·66	1	1	...	8
5405·91	4	1 n	5 s	10	4957·80 ⎱	1	4	12	8
5397·27	4	1 n	5 s	10	4957·43 ⎰				6
5371·62	4	1 n	4 s	10	4920·63	1	4	6 s	10
5328·15 ⎱	6 n	3 n	8 s	10	4919·11 ⎰			5 s	8
5324·31 ⎰			4 s	10	4891·62 ⎱	1	3	7 s	10
5269·65	7	4	10	10 n	4890·89 ⎰			6 s	8
5233·05	1	0 n	5 s	10	4872·25 ⎱	1	2	3 s	8
					4871·43 ⎰			5 s	8

* In the accompanying table n=nebulous, s=sharp.

Wave lengths.	Intensities.				Wave-lengths.	Intensities.			
	e.	*a.*	H.	K & R.		*e.*	*a.*	H.	K & R.
4859·86	00	5	3 s	8	3956·77	00	1	10 s	6
4528·78	2 n	7	10 s	10	3930·37	3	6	12	8
4494·67	1	4	8 s	8	3928·05	3	6	12	8
4482·35	4	2	5 s	8	3923·00	3	5	12	8
4476·20	00	2	5	10	3920·36	3	5	12	6
4469·53	00	2 n	3	8	3910·95 }	2	2	0	2
4461·75 }	6	2	5	6	3909·95 }			1	4
4459·24 }			6	8	3906·58	2	4	8 s	6
4447·85	0	2	5 s	8	3903·06	2	5	12	8
4443·30 }	0	3	3	8	3899·80	2	4	12	6
4442·46 }			4	8	3888·63 }	3 n	7 n	10	6
4430·74	7	1	2 s	8	3886·38 }			12	6
4415·27	7	6	20	10	3878·82 }	3	7	12	8
4404·88	9	9	20	10	3878·12 }			12	8
4383·70	10	10	20	10	3873·88 }	2	3	3 s	6
4376·04	5	2	5 s	8	3872·61 }			12	8
4337·14	1	00	6 s	10	3867·33	2	3	3 s	6
4325·92	10	9	20	10	3865·65	3	5	10	8
4315·21	0	3	7 s	10	3860·03 }	2	4	10	10
4307·96	10	9	20	10	3859·34 }			6	6
4299·42	2	4	6 s	10	3856·49	1	3	9	8
4294·26	4	4	6 s	10	3846·96	2	5	4 s	6
4282·58	1	4	5 s	10	3841·19 }	1	6	8	8
4271·93 }	9	10	20	10	3840·58 }			8	8
4271·30 }				10	3834·37	0	3	9	8
4260·64	3	9	15	10	3827·96	0	9	10	8
4250·93 }	6	6	10	10	3826·04	0	3	10	8
4250·28 }			10	10	3824·58	1	9	9	8
4233·76	1	5	6	10	3820·56	0	4	12	8
4227·60	10	5	5 s	10	3799·68 }	0	2	7 s	6
4216·28	4	00	2 s	6	3798·65 }			7 s	6
4202·15	6	5	12	10	3797·65 }			4 s	6
4199·19 }	1	4	10	10	3795·13	0	2	9	8
4198·42 }			10	10	3790·22	0	2	3 s	6
4187·92 }	1	4	7 s	10	3767·31	0	4	10	8
4187·17 }			9	10	3765·66	0	5	7 s	8
4172·20	2	3	1	8	3763·90	0	6	10	8
4154·95 }	00	1		6	3749·61	0	7	10	8
4154·57 }			6	6	3748·39	1	4	8	6
4154·04			6	6	3735·45 }	0	8	10 n	6
4143·96 }	6	7	12	10	3735·00 }				6
4143·50 }				10	3727·78	0	2	7	8
4134·77	4	4	6	10	3722·69	0	2	6	6
4071·79	6	9	12	10	3720·07	0	3	9	10
4063·63	6	9	15	10	3709·37	00	2	6	6
4045·90	6	10	15	10	3705·70	00	2	6	4
4034·59 }	6	4	8 s	6	3687·77 }	00	1	7	6
4033·16 }			9 s	6	3687·58 }				6
4030·84	6	4	10 s	6	3680 03	00	1	5 s	4
4005·33	4	4	15	8	3647·99	00	2	7 s	8
3998·16 }	00	1	4	6	3631·62	00	2	6	6
3997·49 }			10	6	3618·92	00	2	6	8
3971·41 }	5	5	4 s	6	3608·99	00	0	4	8
3969·34 }			12	8	3581·32	00	1	4	10

A glance at the Plate will show that the absorption due to the presence of iron-vapour &c. has resulted in a diminution of intensity, particularly towards the violet end of the spectrum. The phenomenon to which we wish to draw attention is independent of this effect and refers rather to the results which are observed when the spectrum is studied in detail. It is found that the simplification is purely selective in its occurrence, for whereas certain of the lines appear in this vacuum glow-spectrum with their characteristic intensity, neighbouring lines of almost equal importance in the ordinary arc and self-induction spark have so greatly diminished as to be quite invisible. The effect, therefore, is in no way connected with the actual exposure of the photograph or intensity of the light.

The present experiments are not sufficiently complete to form any general theory on the effect of high pressures. They indicate, however, that work in this direction would yield results of some spectroscopical interest.

With regard to the vacuum discharge, it would be of importance to extend the work to other metals and ascertain if the simplification found in the case of iron can be considered as general.

Our sincere thanks are due to Professor Arthur Schuster for the valuable advice he has frequently given during the course of this work; we desire also to record our indebtedness to Dr. G. A. Hemsalech, who has kindly assisted in several of the experiments.

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Owens College, Manchester.

LXXIII. *Photographic Reversals in Spectrum Photographs. By* R. W. Wood, *Professor of Experimental Physics, Johns Hopkins University**.

[Plate XXVI.]

THE importance of distinguishing between photographic and true reversals of lines in spectrograms has made it seem worth while to investigate with some care the conditions under which reversals due entirely to photographic action can occur. Professor Trowbridge has advanced the theory of selective reversibility of the silver salts in the sensitive film for certain wave-lengths, and it was in part to determine whether the tendency of a line to reverse was a

function of the wave-length, that the present investigation was undertaken.

As I showed several years ago, the Clayden effect, or the type of reversal giving rise to the phenomenon of dark lightning, results from the action of a light-shock on the plate before its exposure to diffuse light. The effect of this light shock, which must be of very brief duration, is to decrease the sensibility of the plate, resulting in a less energetic action during the subsequent illumination. I made no attempt at the time to determine the maximum duration of the light shock which would still give the Clayden reversal, but expressed the opinion that it could not exceed 1/10,000 of a second. This opinion was based on a single experiment, and I have since found that by a suitable adjustment of the conditions the duration may be as great as 1/1000 of a second, though only a very slight trace of reversal occurs under these conditions.

It appears to me now that there are at least four different types of photographic reversal, or perhaps five if we allow the chemical treatment of the plate between two exposures. As any one of these four types is liable to occur in any photographic work, when the proper conditions are fulfilled, it may be well to enumerate them at the beginning.

First Type.—The ordinary over-exposure reversal, which occurs when the plate is given three or four hundred times its normal exposure and then developed in the usual way.

Second Type.—The reversals produced by developing the plate in full lamp-light, the plate having been more or less over-exposed to begin with. This type has been extensively studied by Nipher, the results of his experiments being given in the Proceedings of the St. Louis Academy of Science.

Third Type.—This type must occur frequently, though I never remember to have seen it described. It happens when a normally or under-exposed plate is developed and then exposed to light for a minute or two before the hypo bath. The fogging, which is usually of a reddish-brown colour, does not occur on the portions of the plate where there is a developed image, and even if this image is very feeble it remains clean and almost transparent. I first noticed this effect in some photographs of spectra which showed strong reversals along the edges where the illumination must have been very feeble, and was unable to explain it. Further experimenting showed that it had resulted from turning up the light before the plate had been thoroughly fixed. Doubtless this effect has been described time and again in the photographic journals, but it was new to me, and may be

to some others. Reversals of this type will be seen in fig. 6 illustrating my paper on screens transparent to ultra-violet light*. The spectra (negative) have bright borders.

Fourth Type.—The Clayden effect, which is the type chiefly to be dealt with in the present paper. This occurs when an exposure of about 1/1000 of a second or less is given, and the plate subsequently fogged by exposure to diffuse light before development. If images of electric sparks are thrown on a plate and the plate then exposed to the light of a candle for a few seconds, the spark images will develop reversed, which is not the case if the exposure to candle-light precedes the impression of the spark images.

This effect is shown in fig. A. A series of spark images have been gradually reversed by exposing the plate in strips

Fig. A.
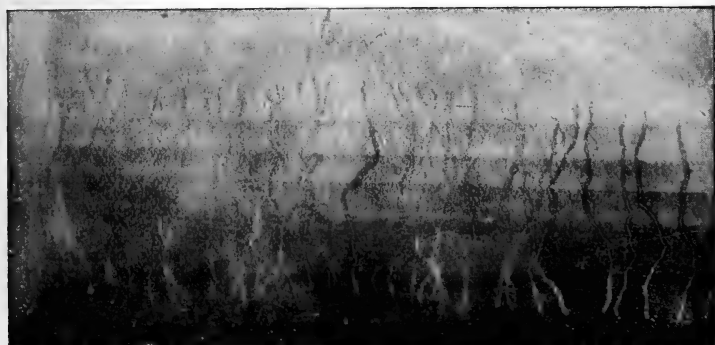
to the light of a candle. The upper strip was exposed 30 seconds, the following 20, 15, 10, 5, 2, 0 respectively.

Fifth Type.—I have found that the condition produced in the sensitive film by light shock can be imitated by treating the plate, after exposing portions of it to the action of a feeble light for a few seconds, to an oxidizing bath of bichromate of potash and nitric acid. If the plate is dried and then fogged by candle-light and developed, the previously exposed portions will come out reversed.

This effect is shown in Plate XXVI. fig. 4. A series of spark images was impressed on the plate, which was then covered with a piece of black paper in which a narrow slit had been cut. A number of images of this slit were then impressed on the plate by exposure to the light of a candle. If the plate in this condition were then fogged by candle-light and

* Phil. Mag. February 1903.

developed, the spark-images would come out reversed and the slit images not reversed. Before fogging it one end (the upper in the print) was immersed for a few minutes in a very dilute solution of bichromate of potash, slightly acidified with nitric acid. It was then dried, exposed to candle-light and developed. In the lower portion of the print we find the sparks black and the slit-images white; in the upper portion both sets of images come out dark, the reversal of the sparks being much stronger than on the untreated portion of the plate.

In the present paper I propose to discuss the Clayden effect not only in connexion with the selective reversibility hypothesis adopted by Professor Trowbridge to explain his spectrum photographs, but also in relation to the time factor, and radiations other than light, such as the Becquerel and Röntgen rays, which are quite different in their action from light.

I shall, in dealing with the subject, speak of the initial exposure of brief duration as the light-shock. The subsequent illumination which causes the reversal of the impression of the shock, I shall call the fogging exposure.

The first subject investigated was the relation of the phenomenon to the wave-length of the light. The light shock in this case was administered by exposing the plate to the spectrum of one or more sparks between cadmium electrodes by means of a small quartz spectrograph. Even with a small diaphragm the illumination by a single spark yielded a developable image of the spectrum down to the extreme ultra-violet. Six spectra were impressed on the same plate with different sized diaphragms; the plate was then exposed to the light of a candle for a few seconds and developed, the result being reproduced in Plate XXVI. fig. 1. It will be seen that the lines and the continuous background come out positive in the two upper spectra, showing that if the shock is too intense no reversal takes place, a circumstance in which the Clayden effect differs essentially from ordinary reversal due to over-exposure. Professor Trowbridge says that his reversals occur where bright lines fall on a continuous background, and considers the reversing action proportional to the product of the two effects. If his reversals are of this nature this cannot be the case, for by making one factor (the light shock) large, no trace of reversal appears. As I shall show presently this statement requires some modification, for, as we increase the intensity of the shock, we can by increasing the fogging exposure still get reversal. As I said in my previous note, it appears probable to me that in Professor

Trowbridge's photographs of spectra of single sparks, the shock was the almost instantaneous exposure to a bright-line spectrum of exceeding brief duration, followed by an exposure to a superimposed continuous spectrum of longer duration, which may have been due to incandescence of the inner wall of the capillary tube, or to phosphorescence of the gas. By employing a tube with a bore of about ·25 mm. I have obtained reversed lines in the blue with single discharges of a medium sized induction-coil and condenser.

The fact that the faint continuous spectrum of the spark is uniformly reversed shows that there is no selective reversibility so far as the initial light-shock is concerned.

I next endeavoured to determine whether the wave-length of the fogging-light had anything to do with the matter (in which case we should expect reversals in some parts of the spectrum and not in others), in the particular case where the fogging illumination was spread out into a spectrum as in Professor Trowbridge's photographs. Having already found that fogging the plate with X-rays never gave reversals of spark-images, it occurred to me that possibly ultra-violet light might act in a similar manner. Having impressed a number of spark-images on a plate, it was illuminated with light of wave-length in the neighbourhood of $\lambda = 23$, from a discharge between cadmium electrodes, a screen provided with a slit being placed in the focal plane of the quartz spectrograph, with the plate a short distance behind it. The spark-images were not reversed, and it appeared at first sight as if ultra-violet light of this wave-length acted like the X-rays. I was, however, not willing to accept this conclusion without further study, since the fogging illumination in this case consists in reality of a number of feeble light-shocks, that is, it is of much briefer duration than in the case of candle-light. In the first experiment the fogging illumination by ultra-violet light was produced by the passage of perhaps a score of sparks before the slit of the spectrograph. To get a feebler illumination of longer duration I moved the next plate to a distance of about two metres from the screen and let the coil run for about a minute. The room was absolutely dark, the spark-terminals and the front of the spectrograph being covered with heavy black cloth, so that the only light that reached the plate was of the wave-length above mentioned. On this plate the spark-images which had been previously impressed were strongly reversed, showing that the time factor comes in the fogging-light as well as in the light-shock, and that ultra-violet light is as efficient as any other, if it is not of too brief duration. This appears to

dispose of the idea of selective reversibility, at least so far as the Clayden effect is concerned.

Investigation of the Time Factor.—To determine the maximum duration of time which the light-shock may have and still reverse, the following method was used :—A disk of cardboard 50 cms. in diameter was mounted on the shaft of an electric motor, the speed of which could be determined by the tracing of a tuning-fork on a smoked metal plate mounted on the same shaft.   Near the rim of disk a number of narrow slits were cut, varying in width from 1 mm. to 5 mms.   An arc-light was focussed on the rim by means of a large condensing-lens, the image of the crater being about half a millimetre in diameter.   By driving the disk at a high rate of speed intermittent flashes of very brief duration were obtained as the slits passed across the arc's image.   A short distance behind the disk a rectangular metal tube was mounted provided with a slit 1·5 mm. wide, immediately opposite the point where the image of the arc fell on the disk.   Down this tube the plate was dropped, receiving in its passage before the slit light flashes of varying duration.   The plate was subsequently exposed to candle-light and developed.   The images of the slit in the case of the briefest flashes were perfectly sharp, in other cases they were broadened owing to the rapid motion of the plate.   This made the interpretation of some of the records difficult, and it was found better, when working with flashes longer than 1/2000 of a second in duration, to lower the plate down the tube with a thread.   A print from one of these plates is shown in fig. 2, Plate XXVI. In this case there were two 1 mm. slits on the rim of the disk not very far apart, then a slit somewhat wider further around, and after this a still wider one.   It will be seen that the slit images formed by the two very short flashes are completely reversed, while the others are only reversed on their edges.

If the plate moves during the exposure, as was the case in this photograph, it is obvious that the edges of the slit-image will receive less exposure than the centre, which accounts for the partial reversal.   As the result of exposing about two dozen plates, it was found that the duration of the shock could be as long as 1/1000 of a second, and still yield reversals.   It was only by carefully regulating the intensity of the fogging-light and the duration of the development that these reversals could be obtained.

When the duration is less than 1/2000 sec. reversals could be obtained without difficulty.   Flashes varying in duration from 1/15,000 sec. to 1/500 sec. were studied, and it was

found that as the duration of the shock was increased the reversals became weaker, the images finally failing to appear at all on the plate, notwithstanding the longer duration of the flash. On still further increasing the duration the images came out not reversed. There may be some connexion between the condition in which the light shocks fail to develop at all, and the zero condition of the plate described by Nipher.

The experiment in which the fogging of the plate was effected by exposure to ultra-violet light furnished by a quartz spectrograph shows that the time factor plays a role in the fogging-light as well as in the light-shock.

It was found that if the fogging light was rather intense but of short duration, the image of the light-shock did not reverse ; if the light was less intense, but of a little longer duration, no trace of the shock appeared, while if the light was still less intense, and of somewhat longer duration, the image came out reversed. With a suitable ratio of intensities and durations of time, it is possible to superpose two impressions on a photographic plate, only one of which appears on development.

This effect is shown in fig. 3, Plate XXVI. A series of spark-images of equal intensity was impressed on the plate, which was then fogged in sections, the lower strip being exposed to the light of one spark at a distance of a metre, the next to the light of four sparks at a distance of two metres, the next to nine sparks at three metres, and so on. The total amount of fogging-light was thus approximately the same in each case, though it was found that considerably greater action was produced by a large number of sparks at a considerable distance, than by a single spark close to the plate. It will be noticed that on the third strip from the bottom there is scarcely a trace of the spark-images. The ratio of the times of duration of the shock and the fogging-light was in this case about 1 : 9. On the two strips below this one, the sparks appear not reversed, while on all of the strips above reversal has taken place.

A more careful quantitative investigation of these effects is much to be desired, with apparatus of such design that the duration and intensity of both the light-shock and fogging-light can be accurately controlled.

I am of the opinion that the result of such an investigation would be the establishment of the fact that with very brief and intense light-shocks, comparatively intense fogging-light of short duration will yield reversals, while in the case of shocks of say 1/1000 sec. duration, the fogging-light must be

feeble and of long duration in order that reversals may be obtained. My plates appear to indicate this qualitatively, but quantitative data could doubtless be obtained with suitable apparatus. As I shall show later, it is possible to administer the shock in such a manner that it comes out reversed even when the fogging-light is the flash of a single spark.

I have tried to obtain some idea of the action of the light-shock by attempting to transform its effect on the plate into an effect similar to that produced by ordinary exposure, by means of the action of various chemical agents. These experiments were all failures, but the interesting fact was ascertained that an ordinary exposure appeared to be transformed into a shock exposure by the action of a dilute bath of bichromate of potash slightly acid with HNO_3. This effect is shown in fig. 4, Plate XXVI. A series of spark-images was impressed on the plate, and then a series of images obtained by illuminating the plate with the light of a candle shining through a slit in a piece of black paper. One half of the plate was then dipped into the bath, washed and dried, exposed to the light of a candle, and developed. A print from this plate is reproduced in fig. 4, Plate XXVI. On the upper portion, which was treated with bichromate, both the spark-images and slit-images appear reversed, on the lower the latter are not reversed. This experiment merely shows that a plate which has been exposed to light in certain places and then treated to the bichromate solution, is less sensitive to the action of subsequent illumination on the spots which have previously received light. The condition may appear at first sight to be similar to that produced by a light-shock, but there is in reality probably no connexion between the two, for while light-shocks not followed by fogging can be developed as not reversed images, the " bichromatized images " do not develop at all unless the plate is fogged before development.

The nearest approach which I have been able to make to the transformation of the effect of a light-shock into that due to ordinary exposure, is by the action of the X-rays. It was found that spark-images could not be reversed under any circumstances if the plate was fogged by these rays instead of candle-light. To prove that the case was not analogous to the one in which ultra-violet light failed to give reversals, owing to the comparatively brief duration of the illumination, long exposures were made with the X-ray tube at a considerable distance. Not only were reversals never obtained, but it was found that after a brief exposure to the rays fogging the plate by lamp-light failed to reverse

the spark-images. This seemed very remarkable, for it was subsequently ascertained that X-ray images could be reversed even when produced by long exposure to feeble radiation, by subsequent exposure of the plate to lamp-light. This effect is shown in fig. 5, Plate XXVI.

The spark-images were impressed first. The plate was then exposed in vertical strips to the action of X-rays for varying lengths of time, the left-hand strip receiving the longest exposure, while the right-hand strip was not exposed at all. Following this came an exposure in horizontal strips to lamp-light, the lower strip having the longest exposure and the upper none at all. It will be seen that there is no trace of reversal in the upper left-hand corner, where the fogging is due almost wholly to X-rays, while reversed edges appear on all of the sparks in the lower right-hand corner, where the fogging was due to light : moreover, in the lower left-hand corner, where the X-ray fog preceded the light fog, the images are not reversed. On investigating the matter further, I found that shocks administered by single powerful flashes of X-rays were reversed by subsequent exposure to lamp-light. In this case, however, the time element appears to be without much influence, for images formed by long exposure to very feeble X-radiation reverse in the same manner. This seems very remarkable, when we consider the fact that exposure to these rays changes the condition produced by light-shock in some manner, so that it is impossible to reverse it by further fogging.

The reversal by X-rays is illustrated in fig. 6. A plate

Fig. 6.

was wrapped up in black paper and exposed to the radiation for several minutes, a vertical iron rod shielding the centre strip of the plate. The plate was then exposed to lamp-light for different lengths of time in strips perpendicular to the shadow of the iron rod. On development it was found that on the end of the plate which had received the shorter exposures to light, the central strip came out lighter than the background, while on the opposite end of the plate it was darker. At a certain point near the centre of the plate, all trace of the shadow of the rod had disappeared, showing that exposure to X-rays for some time, and then to light for a certain time, produces an image no blacker than the light alone would have produced.

I next ascertained that if the plate be exposed simultaneously to light and X-rays, the latter inhibit the action of the former. A candle and an X-ray tube were set up at some little distance apart in front of a plate, the latter being much nearer the plate however, owing to its less energetic action. An iron rod mounted in front of the plate cast two shadows upon the sensitive film, one a light shadow, the other an X-ray shadow. After the double exposure the plate was developed, with the curious result that one shadow was darker than the background, the other lighter, showing that the light was more energetic in its action on the area screened from the X-radiation.

It may be worthy of mention that both this result and the preceding one were predicted before the actual experiments were tried. The prediction was the result of an attempt to apply Bose's strain theory of photographic action * to the phenomena in question. This theory seemed rather promising at first, especially as it enabled me to predict new phenomena, but it failed to account for so many things that I was finally forced to abandon it.

The action of other stimuli was next investigated. It has long been known that pressure-marks on the film can be developed. I found that if the plate was fogged by lamp-light before developing, the pressure-marks came out reversed. It then occurred to me try the effect of light shocks on pressure-marks, and I found to my surprise that the flash of a single spark was as effective in reversing the pressure-mark as the exposure to the lamp. The pressure-marks can also be reversed by exposure to X-rays.

As a result of numerous other experiments, I finally found that if we arrange the stimuli in the following order, pressure-marks, X-rays, light-shock, and lamp-light, an impression of any one of them can be reversed by subsequent exposure to any other following it in the list, but under no circumstances by any one preceding it. For example, pressure-marks can be reversed by any of the other three stimuli, while X-ray images are only reversed by light-shock and lamp-light.

Experiments with Becquerel rays have given rather uncertain results. Pressure-marks can be reversed by them, and they in turn can be reversed by lamp-light, but these were the only two cases in which reversals were obtained, which makes it difficult to fit the rays into the series and still have the rule hold.

These experiments show that the effects of the different kinds of stimuli on the sensitive film are quite different.

* J. C. Bose, Proc. Roy. Soc. June 19, 1902.

Much more experimental work will have to be done before any definite notion can be obtained as to the nature of the changes produced by the action of radiation of any sort, and it is hoped that the experiments described in this paper may prove suggestive to others. Doubtless an exhaustive study of the action of various chemical agents on the plate between the two exposures would throw much light on the cause of the reversals.

If I interpret the strain-theory correctly, the application to these phenomena would be to assume that the light-shock produces a negative strain, while lamp-light produces a positive strain, either of which yields an image on development. The reversal in cases where the lamp-light follows the impression of the light-shock could be explained by assuming that the negative strain has to be undone before the positive strain can begin, consequently these parts of the plate lag behind the parts which have not received the light-shock. We should then have to assume that the positive strain once started can be continued by a stimulus which, acting first, would have produced a negative strain, in order to account for the fact that exposures to lamp-light are not reversed by light-shocks. Moreover, it is difficu t to explain on the strain-theory that two different stimuli acting in succession may produce only the same effect as one of them acting alone. It appears to me that the strain-theory would lead us to suppose that the negative strain produced by the first stimulus might be exactly neutralized by a stimulus which produces a positive strain, the plate returning to its original condition, *i. e.* not darkening on development. This is never the case.

In cases where reversed lines appear in the spectrum, which are suspected of being pho ographic in origin, *i. e.*, not true absorption-lines, the following precautions should be taken. Repeat the exposure a number of times, using successively smaller diaphragms before the prism. If the reversal is ordinary solarization, due to over-exposure, it should disappear when the intensity of the light is sufficiently reduced. It seems to me that in such cases the reversed line should be bordered by bright edges, which does not seem to be the case in the photographs published by Professor Trowbridge. If the Clayden effect is suspected, the source of light should be examined with a revolving mirror, to determine whether a dual illumination is present. The speed of the mirror should not be too great, otherwise the phosphorescence, if it exists, may be spread out to such an extent that no trace of it appears. This may account for the failure to obtain evidence of a dual illumination in the case of heavy discharges in quartz tubes.

LXXIV. *An Experiment to Exhibit the Loss of Negative Electricity by Radium. By* Hon. R. J. STRUTT, *Fellow of Trinity College, Cambridge*[*].

SOMETIME ago M. Curie (*Comptes Rendus*, cxxx. p. 647) showed that the β or cathodic rays from radium carried away a negative charge from the substance, leaving a positive charge upon it. The experiment has not been easy to repeat with the comparatively weak radium preparations generally available, because a very delicate electrometer can scarcely be used under the conditions of a lecture experiment. In this note I shall describe a simple piece of apparatus which exhibits the effect in a permanent and striking manner.

A thin-walled glass tube a, hermetically sealed, contains the radium preparation. It is supported from above by the quartz rod b. From the bottom end hang a pair of gold or aluminium leaves $c\,c$. The glass tube a is smeared over with a conducting coating of phosphoric acid †, care being taken that the coating is in conducting communication with the leaves. The whole system hangs from the stopper d of the glass bulb f. Strips of tinfoil $e\,e$ at the sides of the bulb are connected to earth.

As long as air remains in the bulb no divergence of the leaves is observed, since the radium rays make this air a conductor, and consequently neutralize any difference of potential between the leaves and the tinfoil strips $e\,e$. But if a good vacuum be made in the vessel the leaves soon begin to diverge, owing to loss of negative electricity by the radium. This divergence increases until the leaves touch the tinfoil strips. When this happens they are discharged and collapse. The cycle then recommences.

In my apparatus the radium used was about $\frac{1}{2}$ gramme of a very weak preparation, one of those first on the market.

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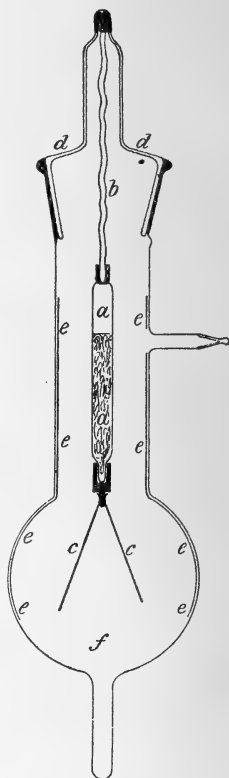
* Communicated by the Author.

† This is a very convenient method of making a glass surface conducting for electrostatic experiments. The phosphoric acid can be trusted not to dry up under the desiccating action of phosphoric anhydride.

Its activity is probably about 100 times that of uranium. The divergence of the leaves begins to be perceptible about three hours after complete discharge, and the full divergence is attained in about twenty hours. For lecture purposes it would be better to have a more active preparation, so that the complete cycle would be gone through in the course of an hour. If the apparatus is sealed off from the pump it will be available for exhibition at any time. But very complete exhaustion is necessary.

The α rays of radium, which are deflected by a magnet in the opposite sense to the β rays, should carry a positive charge. I have attempted to detect this by the same method. A brass strip, carrying an aluminium leaf, was supported on a quartz rod, and itself supported in turn a small dish, on which a radium preparation was spread. The whole was arranged in a vessel connected to earth, and exhausted. No deflexion was, however, obtained. In order to test the insulation a charge was given to the brass strip, causing the aluminium leaf to diverge. But even at the best vacuum which the Toepler-pump could produce, the charge was lost in an hour. The ionization produced in gases by the α rays is so strong that even at the lowest pressure of the mercurial pump it prevents the detection of the charge carried by these rays.

LXXV. *On Charging through Ion Absorption and its Bearing on the Earth's Permanent Negative Charge.* By GEORGE C. SIMPSON, *B.Sc.*, 1851 *Exhibition Scholar, Owens College*[*].

OF the numerous theories for the permanent negative charge on the earth's surface none has been so generally accepted by geophysicists as that due to Elster and Geitel, which ascribes the phenomenon to a charging of the surface through absorption of ions from the surrounding atmosphere. The explanation given by H. Geitel in his Hamburg address ("Ueber die Anwendung der Lehre von den Gasionen auf die Erscheinungen der atmosphärischen Electricität," published by Vieweg & Sohn, Braunschweig, 1901) was as follows:—Zeleny has shown (Phil. Mag. xlvi. p. 120, 1898) that when insulated conductors are in contact with Röntgenized air they become negatively charged in consequence of the greater mobility of the negative ions. This charging has a limit which is reached when the field set up is strong enough to make the number of positive and negative ions which reach the surface equal, by increasing the velocity towards the

conductor of the positive ions and decreasing the velocity of the negative ones.

As the air in contact with the conducting surface of the earth is always ionized one would expect the same process to take place in the atmosphere; *i. e.* the earth to gradually become charged until such a field is set up in the atmosphere around it that an equal number of positive and negative ions reach the surface in a given time. If this process takes place on the free surface it will do so to a much greater extent in all places where there is a protection against the earth's normal field, so that in all places having a covering of vegetation the absorption of negative ions will go on unhindered.

Others have found it necessary to enlarge this simple theory of Elster and Geitel. C. T. R. Wilson ('Nature,' vol. xlviii. p. 104, 1903) postulates a wind as necessary to remove the positive ions from the neighbourhood of the surface, and Riecke (*Nachrichten der k. Ges. der Wissensch. zu Göttingen, Math.- und Phys.-Klasse*, 1903, Heft 2) has considered that a greater proportion of negative than of positive ions which strike the surface may be absorbed, and so he introduces a special coefficient of absorption.

On the whole, it may be said that Elster and Geitel's theory has held its own; but, as recognized by Geitel in his address, it has the weak point of being based upon an experiment which has given a different result when repeated by Villari.

The great interest which belongs to this question of the origin and maintenance of the earth's negative charge led me to undertake a series of experiments to find the conditions under which a conductor can become charged, and to what extent, in consequence of ion absorption. The chief work which has so far been done on this subject is that of Zeleny, Townsend, and Villari, the result of which may be summed up as follows. Zeleny found that when air which had been ionized by means of Röntgen rays was passed along a tube, metals over which it streamed became negatively charged. From this and other experiments he drew the conclusion that when positive and negative ions *are in an electrical field* the negative ions move through the neutral molecules of air quicker than the positive ones, *i. e.* they have a greater mobility. Townsend (Proc. Roy. Soc. lxv. p. 192, 1899, & lxvii. p. 122, 1900) worked from another point of view. He assumed that positive and negative ions may be looked upon as the constituent parts of two separate gases. With this assumption the following is the process to be expected

when a mixture of ordinary air with positive and negative ions moves along a tube. Every ion which strikes the conducting wall is absorbed, this leaves an ion-free space within molecular distances of the wall, into this space positive and negative ions diffuse by the ordinary laws of gas diffusion. The greater the coefficient of diffusion the more ions will reach this space, and so a greater absorption of ions will take place during the passage of the mixture along the tube. By measuring the ratio of the quantity of positive ions in a given mass of gas before and after it had passed along tubes of different length, Townsend was able to calculate the rate of diffusion of positive ions, similarly for negative ions. The values of the coefficient of diffusion obtained by this method agreed very well with Zeleny's number for the "mobility."

Villari (Phil. Mag. [6] i. p. 535, 1901), contrary to Zeleny, found that metals exposed to a stream of ionized air became positively or negatively charged according as the air streamed over them with great or little friction.

Starting with the work carried so far, my first experiments were directed to finding an explanation for the apparent disagreement between Zeleny's and Villari's results. In order to do this it was necessary to repeat their experiments, but this was done by a slightly altered method. A straight tube 40 cms. long was connected to a box in which air could be ionized by means of Röntgen rays and then passed along the tube.
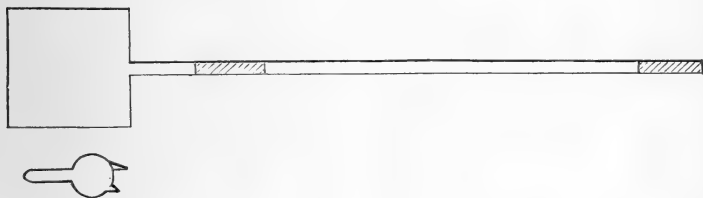
The tube was composed of four pieces: the first connected to the box being of glass and 10 cms. long, the second a metal tube 5 cms. long, the third a glass tube 20 cms. long, and finally another metal tube 5 cms. long. It was then found that when ionized air was passed through the tube the metal piece nearest the box became charged negatively, the one at the far end positively. This showed that for the first few moments after leaving the place of ionization the air gave up a negative charge, while later on it gave up a positive charge. This agrees with Zeleny's experiment, for the metal he used was situated so near the place where the air was ionized that it was passed over during the time the

air was giving up a negative charge. It also agrees with Villari's experiments; but in order to see this the arrangement of his apparatus must be considered.

In order to obtain a good insulation Villari placed the metal to be examined in the end of a glass tube 30 cms. long. Sometimes this metal consisted of a few filings or pieces of wire, at others the whole end of the tube was filled with tightly packed filings or strips of metal. He then found that when only a little metal was in the end it became negatively charged, while the tightly packed metal obtained a positive charge. A connexion between friction and charge appeared obvious. But in reality the friction itself had directly nothing to do with the phenomenon. The reason for the different charges being that as Villari worked *with a constant pressure driving the air through the tube,* the air took longer to reach the metal when there was much friction to overcome than it did when there was little. In the latter case the air reached the metal during the time it was giving up a negative charge; in the former this state had been passed, and a positive charging of the metal was the consequence. That this is the real explanation could be easily shown with my arrangement of the experiment, for on placing a plug of cotton-wool in the far end of the tube and keeping the same pressure in the box as before, the first metal tube which had always before become negatively charged became positively charged. That is, a positive or negative charge could be given to the metal tube by simply increasing or decreasing the time taken for the ionized air to reach it quite independently of the friction with which the air passed over the metal itself; in fact, the positive charge in my experiment was obtained when the air passed over the metal with the smaller friction, this being exactly contrary to Villari's result.

This brings Zeleny's, Townsend's, and Villari's results into line, for the process which leads to the charging may be considered as follows:—Imagine a small part of the stream of air separated from the rest, and follow it in its course along the tube. When it first enters, the negative ions, on account of their greater mobility, move quicker than the positive ions into the neighbourhood of the walls where they are absorbed. As this leaves a greater volume-density of positive than of negative ions behind, a state will soon be reached at which the greater number of positive ions balances the greater velocity of diffusion of the negative. Up to the point in the tube where this takes place the air has been giving up a negative charge to all bodies over which it streams; at the

point itself no charge is given up. The state of balance is, however, not maintained, for the ratio of positive to negative ions rapidly increases owing to the recombination of the positive and negative ions to build neutral molecules. From this point onwards more positive than negative ions reach the wall—a positive charging of the remainder of the tube being the result. Thus the negative charging found by Zeleny and the positive one found by Villari are both due to the same cause, viz. the greater rate of diffusion of the negative ions.

Having explained these experiments by assuming different coefficients of diffusion for the two kinds of ions, the question arises—are we justified in assuming, as Townsend has done, that positive and negative ions act as molecules of two separate gases mixed with the ordinary unionized gas from which they are formed, as such obeying Avogadro's law, and having different coefficients of diffusion quite independently of their electrical properties? Although Townsend has used this assumption in interpreting the results of his experiments, the ions in those experiments were not free from their electrical properties, for when an ion comes into the neighbourhood of the solid wall of a tube, it induces there an opposite charge, and under the influence of the field so formed moves towards the wall. But Zeleny showed that in a given field negative ions have a greater velocity than positive ones; hence the negative ions will move quicker than the positive ones toward the wall, so producing the apparently different coefficients of diffusion.

Thus we are not able to say, from Townsend's experiments, that negative ions really have a greater coefficient of diffusion than positive ones in the same sense that we say two gases have different coefficients of diffusion. Realizing that the only way to settle the point is to isolate a mass of ionized air and then to investigate the rates of diffusion of the two kinds of ions into the surrounding unionized air, an attempt was made to find a method of doing this. Vortex rings gave a simple means. A box from which vortex rings could be sent out was placed above a Röntgen tube so that the air within could be strongly ionized—a lead covering prevented the rays from ionizing the air in the room. On tapping the elastic side of the box, when the Röntgen tube was in action, a mass of ionized air was sent out into the unionized air of the room. In order to investigate the effect of diffusion the rings were directed into a gauze cage connected to an electrometer and properly protected against outside electrical fields. When the cage was 20 cms. from the orifice from

which the rings came, no charge could be observed on the
rings entering the cage ; but on reducing the distance
between the orifice and cage a distinct positive charge was
obtained from the rings. Thus during the time (about a
tenth of a second) the ionized rings had been passing through
the unionized air on their way to the cage, a diffusion of
the ions had taken place ; but as the rings entered the cage
with a positive charge we may safely say that the negative
ions had diffused away the more rapidly. The fact that
there was no charging of the cage when the rings had to
travel further than 20 cms., showed that within one-fifth of a
second both positive and negative ions had entirely diffused
away. In this way it was possible to show directly that
negative ions have a greater coefficient of diffusion than
positive ones.

Two interesting observations showing charging as a result
of ion absorption must be here described. The first was
observed when working with the ionized vortex rings. It
was then found that when the box from which the rings were
sent out had been filled with tobacco-smoke in order to make
the rings visible they entered the cage with a negative
instead of a positive charge.

The second observation was as follows :—Fine white sand
was allowed to run out of a metal funnel into a saucer some
distance below. As a result of the friction between the metal
and the sand, the latter was positively charged. A
ring wrapped round with cotton-wool soaked in spirits was
now so placed that the stream of sand passed through it on
its way from the funnel to the saucer. On lighting the spirits
the sand passed through a ring of flame. It was then found
that the sand carried a negative charge to the saucer. At
first it was thought that this change in sign was due to a
change in the frictional electricity, owing to the funnel
becoming hot, but that this was not so could be seen by
removing the flame, when at once the positive charge re-
turned although the funnel had had no time to cool.

The explanation of these two observations is the same.
The smoke ring on leaving the box consists of a mass of carbon
particles mixed with negative and positive ions. The sand
on falling through the flame carries with it a quantity of
the highly ionized gas of the flame. Thus in both cases
we have to do with a mixture of material particles with
positive and negative ions. As soon as this mixture enters
ordinary air diffusion takes place, but the ions in the middle
of the mass have to diffuse through the material particles.
We may consider the process as being similar to a diffusion

through a number of narrow tubes, so that if any ions pass out, as Townsend's experiments have shown, there is bound to be an excess of positive ones. The material particles are thus left behind with a negative charge. That this is the process by which the charge is obtained, and not that the sand and smoke leave the place of ionization negatively charged, was proved by the fact that it was possible to collect the positive ions which, streaming away from the sand, left it negatively charged. To do this a cylinder of metal was so placed that the sand passed through it immediately on leaving the flame. The positive ions diffusing out of the stream of sand were caught by the cylinder, which at once showed a positive charge. No charging of the cylinder could be detected when the sand passed through it without first passing through the flame.

In all the experiments so far described, we have obtained a charge as the result of ion absorption; but it has in every case been accompanied by the removal of air from the action of a strong ionizer to a place where diffusion could come into play. In the atmosphere such a removal does not take place, for we have strong reasons to believe that the natural ionization in the atmosphere is mainly due to a radioactive gas which the air carries with it; therefore none of the cases described above can be directly applied to the problems of atmospheric electricity. Even the charging of the metals in Zeleny's experiments, which has been used as the foundation of Elster and Geitel's theory, has no parallel in the atmosphere, for in Zeleny's experiments the air was first removed from the ionizer, then, as a result of diffusion, the charge appeared. Zeleny's experiments do not prove that an insulated conductor in air which is *constantly and uniformly* ionized will become negatively charged. Experiments which will now be described show that such an insulated conductor does not become negatively charged.

When working with a quadrant electrometer it was necessary to lead a long copper wire to it from apparatus a considerable distance away. In order to protect this wire from electrostatic induction it was led through a long wooden box covered on the outside with earth-connected tinfoil. It was then found that the wire although insulated became positively charged when left for any considerable time to itself. In order to see whether this charge had anything to do with the material of the wire, the copper one was replaced by one of zinced iron: this latter became negatively charged. To investigate the effect more thoroughly, a large cage, one and a half metres long and one metre high and wide, was

made of zinced iron wire netting. Inside this cage different
metals were hung on an insulating support and then con-
nected by a wire of the same material to an electrometer
outside. The metal to be investigated was first connected to
the earthed cage and a zero reading of the electrometer taken;
then, on insulating, it was found that the metal began to
become charged, until in a little over an hour a steady
deflexion of the electrometer-needle showed that the charging
had ended. The final voltages obtained varied in sign and
amount from metal to metal, being as follows :—

Copper ...	$+ \cdot 70$ volt.	Lead	$+ \cdot 23$ volt.
Iron	$+ \cdot 46$ „	Magnalium...	$- \cdot 28$ „
		(magnesium aluminium)	
Tin	$+ \cdot 25$ „	Sodium	$- \cdot 70$ „

It will be seen at once that these numbers are practically
those of the volta difference of potential between the different
metals and zinc. The zinced iron cage and the metal
within form the poles of a battery with the slightly ionized
air between acting as the electrolyte. The only effect of
greatly increasing the ionization of the air by means of
Röntgen rays was to increase the rate at which the final charge
was obtained, the charge itself remaining unaltered. Thus we
see that Zeleny's results do not apply to conductors in the
ionized air of the atmosphere—at least with the air at rest,
air in motion will be considered later. A conductor in
ionized air must be looked upon as a conductor in an electrolyte,
the charge it obtains being determined by its position in the
volta series and not by the mobility of the ions. It is easy to
see that a conductor in ionized air at rest will not become
" charged negatively until it produces in the surrounding air
an electrical field strong enough to neutralize the difference
in the mobility of the ions." Consider a space filled with
ionized air, there being an equal number of positive and
negative ions in a cubic centimetre. Now imagine a con-
ducting sphere suddenly placed in the middle of the space.
For the first instant more negative than positive ions will
reach it and be absorbed, but the corresponding positive ions
are all within molecular distances of the sphere so that no
field can be set up outside the immediate neighbourhood of
the surface. There may be something of the nature of a
double layer at the surface, but its presence cannot be
experimentally observed, for all conductors have the same
layer. We must accept then, as the result of both theory and
experiment, that in air at rest there is no tendency for an
insulated conductor to become charged negatively as a result

of the difference in the mobility of the ions. We can also no longer accept the theory that in all places protected against the formation of an electrical field, such as within conductors, there is a continual stream of negative ions out of the air, for the positive ions which this would leave behind are all within molecular distances of the surface, so hold an equal quantity of negative electricity as a "bound charge," no electricity being able therefore to appear as a free charge.

So far we have considered air at rest, we must now ask what will be the effect of setting the air in motion. At first sight it would appear that as fresh air is always moving past the conductor more negative than positive ions would reach the surface, and as a surface layer of positive ions would be unable to form, a negative charging could be expected. But when we consider conductors in air as being electrodes in an electrolyte it is difficult to see how motion could affect the potential which they had taken up when the air was at rest. In order to put this question to the test of experiment, air was driven past insulated metals within the cage described above, by means of a fan attached to an electric motor. Although the wind so produced had a velocity greater than six metres a second no change could be observed in the charge which the metal had attained before the fan was set in motion. Thus when a piece of the same netting as that of which the cage was made was hung within the cage, the needle of the electrometer connected to it remained at zero with or without the fan in motion. When copper was used as the insulated metal it took a positive charge as described above, and this charge was the same whether the fan was in motion or not. The electrical state of the fan itself was then investigated by carefully insulating it from the axis of the motor, but no charging, except a very uncertain positive one, could be observed, although the fan, which was 50 cms. in diameter, produced a wind with the velocity stated above. It must be stated that an uncertain positive charge appeared also sometimes on the metals which were subjected to the wind, but as it never produced more than a tenth of a volt potential, and could not be obtained at will, no conclusion can be drawn from it. The result of this experiment favours the second method described above of considering what will happen when air is in motion, but does not disprove the former, all that can be said for certain is that a conductor has not yet been charged as a result of its motion relative to the *naturally* ionized air of the laboratory.

Although it has been impossible to charge a conductor by this method, yet a very great absorption of ions takes place

from air which comes into contact with a conductor. To show this the fan was placed within a large closed box ($50 \times 80 \times 140$ cms.). By means of an Ebert aspiration apparatus the ionization of the air within the box was determined before and after the fan had been set in motion. The results showed a great decrease in the ionization as a result of the air being continually driven into contact with the walls of the box. Before the fan started 0·4 electrostatic unit was found in a cubic metre of air; after it had been in motion ten minutes, and while it was still in motion, a determination showed that this quantity had been reduced to 0·1. As far as the Ebert apparatus was able to show, its accuracy not being greater than 5 per cent., there was the same number of positive as negative ions in both experiments.

Applying now these results to atmospheric electricity and the earth's negative charge, it would be rash to say that they disprove the theory of the permanent charge being due to ion absorption, but they show that the process which Elster and Geitel consider is taking place in the atmosphere is not substantiated by experiment. We are also brought face to face with the fact that as yet no conductor has been charged through absorption of ions from the naturally ionized air of the atmosphere (charging due to the volta effect being excepted), and until this is done we cannot consider the problem of the earth's negative charge as being solved by the "absorption of ions from the atmosphere."

In conclusion I would like to express my thanks to Prof. Wiechert for the helpful interest he has taken in this work, and for placing every facility at my disposal for making these experiments in the Geophysical Institute in Göttingen.

LXXVI. *The Genesis of Ions by the Motion of Positive Ions in a Gas, and a Theory of the Sparking Potential.* By John S. Townsend, *Wykeham Professor of Physics, Oxford* [*].

IN a paper published in the 'Electrician,' April 3, 1903, I gave an outline of a method by which some properties of positive ions may be investigated. The present paper contains a fuller account of the theory of the genesis of ions by positive ions and its application to experiments which have been made with air and hydrogen.

The theory is founded on the determinations of the conductivity which takes place between parallel plate-electrodes when ultra-violet light falls on the negative plate.

By the action of the light a certain number, n_0, of negative ions are set free, which generate others in their passage through the gas. If the pressure p and electric force X are constant, then for the smaller values of $\dfrac{\text{X}}{p}$ and distances between the plates a, the number of negative ions reaching the positive electrode is $n_0 \epsilon^{\alpha a}$. The quantity α depends on the pressure and the electric force and denotes the number of ions that a single negative ion generates in moving through a centimetre of the gas.

For large values of $\dfrac{\text{X}}{p}$ and a the number of negative ions reaching the positive electrode is greater than $n_0 \epsilon^{\alpha a}$, showing that some other form of ionization has come into play. This stage is attained even when the potential between the plates is much lower than that required to produce a continuous discharge.

It will be seen from the following investigations that all the features of the new process of ionization which is introduced can be explained on the supposition that it arises from the action of the positive ions.

It is obvious that if both positive and negative ions give rise to others by collision in sufficient numbers we would get a continuous discharge. The investigations show how the potential required to produce a continuous discharge may be found on this theory, assuming that all the ionization is produced by collisions of positive and negative ions with neutral molecules in a uniform field of force. There is a very accurate agreement between the potentials thus calculated and the sparking potentials determined experimentally.

2. An investigation of the currents which would be produced between parallel plates, when both positive and negative ions generate others by collision, will show what experiments would be most suitable in order to test the theory.

If a number n_0 of negative ions start from the negative plate and move through a distance a to the positive plate they will generate others in the gas, and a number $n_0 \epsilon^{\alpha a}$ will reach the positive plate. The positive ions ($n_0(\epsilon^{\alpha a}-1)$ in number) produced in the gas will move in the opposite direction, and let it be supposed that these also have the property of ionizing the gas.

The results of the experiments may here be anticipated and it may be assumed that the positive and negative ions produced by the impact of a positive ion with a neutral molecule are identical with the positive and negative ions

produced by the impact of a negative ion.   In applying the theory, therefore, it is only necessary to consider one kind of positive ion and one kind of negative ion *.

Let n_0 be the number of ions starting from the negative plate, and let n be the total number arriving at the positive plate.   Of the number $n - n_0$ of each kind generated in the gas let p be produced in the layer of gas between the negative plate and a parallel plane at a distance x, and let q be produced in the layer between the plane and the positive electrode.

Then $n = n_0 + p + q.$

Let α be the number of ions of each kind produced by a negative ion in moving through a centimetre of the gas.

Let β be the corresponding number for a positive ion.

Considering the number of ions dp generated between the two planes at distances x and $x + dx$ from the negative electrode, we have
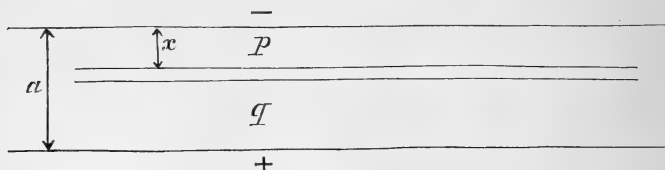
$\alpha(n_0 + p)dx$ new ions of each kind produced by the $n_0 + p$ negative ions moving through the distance dx, and $\beta q\, dx$ produced by the q positive ions moving in the opposite direction.

Hence $dp = \alpha(n_0 + p)dx + \beta q dx,$

or $\dfrac{dp}{dx} = (\alpha - \beta)(n_0 + p) + n\beta.$

Hence $p = \left[\dfrac{n\beta}{\alpha - \beta} + n_0\right](\epsilon^{(\alpha - \beta)x} - 1),$

and since $p = n - n_0$, when $x = a$ we have

$$n = \frac{n_0(\alpha - \beta)\epsilon^{(\alpha - \beta)x}}{\alpha - \beta\epsilon^{(\alpha - \beta)x}}.$$

The quantities α and β depend on the electric force and the pressure, so that if a number of experiments be made

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* [It is evident from the previous researches which Mr. Kirkby and I have made that the negative ion set free from a metal plate by ultra-violet light, or generated in a gas by Röntgen rays, is the same as the negative ion separated from a molecule of any gas by collision.   The positive ions only can be different in different gases.]

with different distances between the plates, the electric force and the pressure being unaltered, then the charges acquired by one of the plates will be given by formula 1, in which the quantities α and β are constant.

3. To illustrate the theory I may quote experiments made with air at a millimetre pressure, the electric force being 350 volts per centimetre in each experiment.

The distance a between the plates in centimetres is given in the first line of the following table ; the second line gives the charge q acquired by one of the plates determined experimentally * ; the third line gives the values of ϵ^{aa} where $\alpha = 5\cdot25$; and the fourth line gives the values of $\dfrac{(\alpha - \beta)\epsilon^{(\alpha - \beta)a}}{\alpha - \beta\epsilon^{(\alpha - \beta)a}}$ where $\alpha = 5\cdot25$ and $\beta = \cdot0141$.

a	0.	2.	4.	6.	8.	10.	11.
q	...	2·86	8·3	24·2	81	373	2250
ϵ^{aa}	1	2·86	8·2	23·4	66·5	190	322
$\dfrac{(\alpha - \beta)\epsilon^{(\alpha - \beta)a}}{\alpha - \beta\epsilon^{(\alpha - \beta)a}}$	1	2·87	8·3	24·6	80	380	2150

The numbers in the third line agree only with the experiments at the shorter distances. The numbers in the fourth line give the conductivities which would be produced if both positive and negative ions generate others by collisions, and it is evident that these numbers are in agreement with the experiments at different distances up to eleven millimetres.

A comparison between the numbers given by the two formulæ shows that for the larger distances positive ions give rise to a large increase in the conductivity, although the actual number of ions which they generate is in this case very small.

The formula can thus be easily tested by experiments on the conductivity produced by ultra-violet light between plates at different distances apart, and a number of values of α and β can be obtained for different forces and pressures.

4. The form of apparatus which was used for finding the conductivities is shown in fig. 1. The conductivity takes place in the gas between the two plates A and B. The

* [The values of q are multiplied by a factor so that their relative values may be compared with the numbers given by the formulæ.]

upper plate A was of quartz silvered over so as to obtain a
flat conducting surface. A series of fine parallel lines, half
a millimetre apart, was ruled on the silver near the centre of
the plate with the point of a needle so as to allow the light
to pass through and fall on the movable plate B which was
of zinc. The distance between the plates was adjustable by
means of the micrometer-screw to which the lower plate was
attached. The plates were four centimetres in diameter and
the light fell on a circular area of one centimetre diameter

Fig. 1.
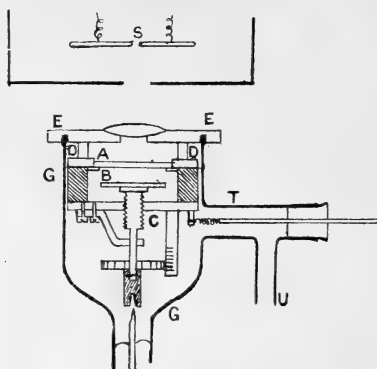
at the centre of the zinc plate. The conductivity thus takes
place in a field of uniform electric force, since the largest
distances between the plates in the experiments was not much
more than a centimetre. The brass plate C, in which the
micrometer-screw worked, was fixed firmly by four ebonite
pillars to the disk D which held the quartz plate. The disk
was fixed by four brass pillars to the large brass plate E, the
hole at the centre of E being covered with a quartz lens and
the joint made airtight with a plastic cement.

The apparatus was constructed very carefully by Mr. Bush,
instrument-maker in the laboratory, so that the plates were
accurately parallel to each other at all distances apart.

The glass cover G fitted into a groove in the plate E and
the joint was made airtight. A wide barometer-tube was
connected to the lower end of the glass cover, and the micro-
meter-screw was turned by a long rod coming up through
the mercury column.

Connexion was made with the plate B by a rod passing
through an ebonite plug in the side-tube T, and the tube U

served to connect the apparatus with a Toepler-pump and a MacLeod gauge. For the purpose of adjusting the pressure (which can be done only roughly by means of the pump) the apparatus was connected to the top of a glass bulb provided with a barometric tube leading to a mercury reservoir by flexible tubing in the ordinary manner. It was then possible to adjust the pressure of the gas in the apparatus to any required value by altering the height of the reservoir.

In order to find the conductivities between the plates it was found very convenient to use an induction-balance. The currents cannot be measured conveniently by the ordinary method of electrometer deflexions since the quantities vary over such a large range, and when this method is used it is necessary to make a correction for the capacity of the plate B, which varies with the distance between the plates.

The arrangement of the apparatus which was used for measuring the quantities of electricity acquired by the plate B is shown in fig. 2. A parallel plate condenser

Fig. 2.
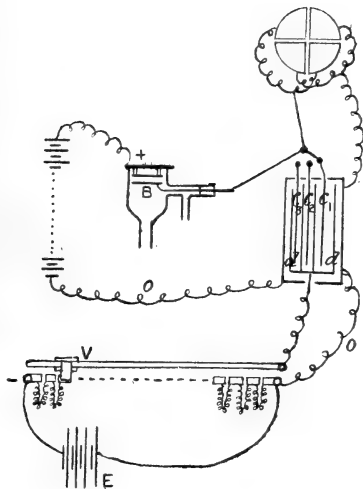
having seven plates was set up in a brass case which was maintained at zero potential. The plates c_1, c_2, c_3 were carefully insulated from the other plates, and a short connector from each plate terminated in an insulated mercury cup outside the case. The four other plates d were connected together and insulated from the case. When the plates d

were to earth the capacity of c_1 was approximately 60 electro-static units, and c_2 and c_3 120 each.   The ratios of the capa-cities of the plates were determined accurately by experiment. The rod connected to the movable electrode B was con-nected to the insulated quadrants of a sensitive electrometer which could be joined to any of the plates c_1, c_2, c_3.   At the beginning of an experiment the electrometer is at zero potential and insulated.   The potential of the plates d, which is varied during an experiment, is zero at the beginning. While the electrode B is acquiring a positive charge (the electrode A being at a high positive potential) the plates d are brought to negative potentials so as to keep the electro-meter reading approximately at zero during the experiment. At the end of the experiment the potential V of the plates d required to bring the electrometer reading exactly to zero is found.

The quantity of electricity Q acquired by the current through the gas is thus

$$Vc_1, \quad V(c_1 + c_2), \quad \text{or} \quad V(c_1 + c_2 + c_3)$$

according to the number of plates c joined to the insulated quadrants.

This method has many advantages.   The capacity of the electrometer need not be determined, and the variation of the capacity of the electrode B does not enter into the measurements.   The electrometer being used only as a de-tector no errors are introduced by the unequal values of the readings at the different parts of the scale which are noticeable in most electrometers.   In addition the method is useful in cases where it is desirable to maintain a constant force between the electrodes during an experiment, since the potential of the plates d is easily varied so as to keep the electrometer near the zero while the current is passing.

The potential of the plates d can be easily adjusted by a potentiometer method.   A set of 50 equal resistances, 20 ohms each, were arranged in series, and by means of a sliding con-tact it was possible to connect the plates with any of the junctions of the 20-ohm coils.   The two terminals of the 1000 ohm resistance were joined to a battery of known elec-tromotive force E, the positive terminal being connected to earth.

The value of V required to bring the electrometer to zero can be estimated accurately by observing the small deflexions on opposite sides of the zero when the plates are in connexion with two consecutive junctions of the resistances.

Quantities of electricity can be thus measured over a large range since the capacities c and the electromotive forces E can be easily altered in large proportions. In the following experiments 4 volts (E) and a capacity of 60 electrostatic units (c_1) were used to determine the smaller currents.

A battery of small Leclanché cells was used to establish the electric force X between the plates A and B. The electromotive force of these cells remained very constant as the largest currents which were used were small compared with the currents which would polarize the cells.

The conductivity between the plates was started by the action of ultra-violet light on the zinc plate. A spark-gap S (fig. 1) in a circuit through which a leyden-jar discharge takes place was used as the source of ultra-violet light, the spark passing between aluminium electrodes. The leyden-jar was charged from the secondary of a Ruhmkorff coil.

The apparatus used for generating the light was separated by a metallic screen from the rest of the apparatus in order to prevent induction effects on the insulated conductors.

In making the experiments I was assisted by Mr. Bennett as it requires two observers to make all the necessary observations in the course of an experiment.

5. The results of the experiments are given in the following tables.

The pressure of the gas at which experiments were made is given at the head of each table. The distance a between the plates in millimetres is given in the first line of the tables.

The electric force X is given in volts per centimetre. The quantity of electricity q which passed between the plates under the action of the force X while the light acted for ten seconds is given in arbitrary units. (The absolute value of q in electrostatic units may be found approximately by multiplying the numbers in the tables by ·03.)

The ratios of the quantities q are independent of the intensities of the light, the same relative values being obtained when the light falling on B was reduced so as to give one-twentieth of the usual conductivity.

The values of n calculated by the formula

$$n = n_0 \frac{(\alpha - \beta)\,\epsilon^{(\alpha - \beta)a}}{\alpha - \beta\,\epsilon^{(\alpha - \beta)a}}$$

in which $n_0 = 1$ are given in the line below the numbers q, the quantities α and β being chosen so that the values of n should as nearly as possible be equal to the values of q.

TABLE I.—Air, pressure 2 mms.

a	1.	2.	3.	4.	5.
X=700 q...	2·9	8·3	23·8	80	374
$a=10·5$ $\beta=·0282$ n...	2·87	8·3	24·6	80	380

TABLE II.—Air, pressure 1 mm.

a	2.	4.	5.	6.	7.	8.	9.	10.	11.
X=350 q...	2·87	8·4	...	25	...	80	...	372	2250
$a=5·25$ $\beta=·0141$ n...	2·87	8·3	...	24·6	...	80	...	380	2150
X=395 q...	3·2	10·4	...	34	...	165	588		
$a=5·8$ $\beta=·022$ n...	3·21	10·5	...	36	...	163	583		
X=437 q...	3·54	12·8	...	54·5	154	770			
$a=6·3$ $\beta=·0345$ n...	3·54	13·0	...	55·8	141	780			
X=480 q...	3·93	16·6	...	100	550				
$a=6·8$ $\beta=·048$ n...	3·93	16·5	...	97	557				
X=525 q...	4·34	20·6	51	248					
$a=7·27$ $\beta=·066$ n...	4·34	21	54	236					

TABLE III.—Air, pressure ·66 mm.

a	2.	4.	5.	6.	7.	8.	9.
X=350 q............	2·58	7·3	...	20·4	...	84	290
$a=4·8$ $\beta=·049$ n............	2·55	7·3	...	20·4	...	82	270
X=437 q............	3·1	10·4	...	43	157		
$a=5·56$ $\beta=·084$ n............	3·1	10·2	...	44	152		
X=525 q............	3·34	12·6	27·0	87			
$a=5·9$ $\beta=·12$ n............	3·31	12·4	27·8	89			

TABLE IV.—Hydrogen, pressure 8 mms.

a	1.	2.	3.	4.	5.	7.	8.
$X=525$ q...............	...	...	5·1	...	15·1	65	160
$a=5·28$ $\beta=·0475$ n...............	...	...	5·0	...	15·4	60	162
$X=700$ q...............	...	6·05	15·2	44·4	174		
$a=8·86$ $\beta=·059$ n...............	...	6·05	15·3	44·2	176		
$X=1050$ q...............	4·6	23·0	1000?				
$a=14·8$ $\beta=·0164$ n...............	4·55	22·7	800				

TABLE V.—Hydrogen, pressure 4 mms.

a	1.	2.	3.	4.	5.	6.	8.	10.
$X=350$ q......	...	2·5	...	6·05	...	14·9	43·7	177
$a=4·43$ $\beta=·0295$ n......	...	2·44	...	6·05	...	15·3	44·2	176
$X=525$ q......	...	4·6	9·9	22·3	66	1000?		
$a=7·4$ $\beta=0·82$ n......	...	4·55	9·9	22·7	67	800		
$X=700$ q......	2·65	7·25	25·3	960				
$a=9·6$ $\beta=·214$ n......	2·65	7·5	26	870				

TABLE VI.—Hydrogen, pressure 2 mms.

a	2.	4.	6.	7.	8.	10.
$X=262$ q......	...	4·6	10·0	...	22·7	65
$a=3·7$ $\beta=·041$ n......	...	4·55	9·9	...	22·7	67
$X=350$ q......	2·65	7·32	24·6	68	822	
$a=4·8$ $\beta=·107$ n......	2·65	7·5	26	65	870	

TABLE VII.—Hydrogen, pressure 1 mm.

a	2.	4.	6.	7.	8.	10.
X=262 q......	1·79	3·2	6·4	...	16·3	186?
a=2·75 β=·202 n......	1·79	3·2	6·4	...	16·2	196
X=350 q......	1·92	4·0	10·4	24·5		
a=3·1 β=·33 n......	1·92	4·0	10·6	23·8		

The hydrogen which was used in these experiments was very carefully prepared. The apparatus was tested before the hydrogen was admitted and it was found that when exhausted to about a twentieth of a millimetre, the increase of pressure during a day was only one three-hundredth of a millimetre, so that any impurity arising from leakage or air coming off the sides of the apparatus must have been very small. The hydrogen was generated by electrolysis from a solution of caustic potash and a special form of apparatus was used so that no small bubbles of oxygen should get mixed with the hydrogen. While the gases were being evolved the solution was kept hot in order that no gas should be absorbed by the solution and be thus conveyed from the neighbourhood of one electrode to the other. The hydrogen was kept at reduced pressure for 24 hours in an airtight receiver containing phosphorus pentoxide before it was admitted to the vessel containing the parallel plates. The apparatus was filled to a pressure of 10 centimetres with dry hydrogen and exhausted to a pressure of one millimetre several times so as to eliminate traces of air. In order to reduce as far as possible any error arising from the small leakage or air coming off the sides of the glass a fresh supply of hydrogen was introduced a short time before a set of experiments were made.

It will be seen that the values of n derived from the formula

$$n = \frac{(a-\beta)\epsilon^{(a-\beta)a}}{a - \beta\epsilon^{(a-\beta)a}}$$ agree with the numbers found for q experi-

mentally. Since q is expressed in arbitrary units it is possible to find values for a and β which give values of n equal to values of q at three different distances. It is evident that the formula gives the correct conductivities for a large range of distances.

6. Further considerations in favour of the theory can be

obtained by comparing the values of α and β at the different pressures, and the investigation leads to a very simple method of representing the values of these quantities corresponding to the different forces and pressures. It has already been shown that x is given in terms of X and p by a formula of the form $\dfrac{a}{p} = f\left(\dfrac{X}{p}\right)$. The theory also requires that β should be connected with the variables by a similar formula. These results may be obtained from the following simple considerations. If the pressure is increased from p to xp all the free paths are reduced to $\dfrac{1}{x}$ of their original value. If the force remained unaltered the velocities on collision would be reduced, but if the force X is increased to $x \times$ X the velocities will be restored to their original values and the number of ions arising from a given number of collisions will be the same as before. Since the total number of collisions per centimetre is increased by increasing the pressure, we see that when X and p are altered in the same proportion then α and β must be altered in similar proportions.

The results of the experiments show that β, as well as α, is connected with the quantities X and p by a relation of the form $\dfrac{\beta}{p} = \phi\left(\dfrac{X}{p}\right)$. This may be seen from the accompanying diagrams (fig. 3) in which the points whose coordinates are $\dfrac{\beta}{p}$ and $\dfrac{X}{p}$ are marked, p being in millimetres of mercury and X in volts per centimetre. The pressure of the gas in the experiments from which the value of $\dfrac{\beta}{p}$ was derived is given by the number beside each point. Some of the values of $\dfrac{\beta}{p}$ were obtained from experiments at two or three different pressures. It is evident that a single curve passes through the points derived from all the experiments.

The properties of the negative ions as shown by the curves $x = f(y)$ for different gases have already been investigated in the previous researches. The values of $\dfrac{\alpha}{p}$ corresponding to the smaller values of $\dfrac{X}{p}$ as previously obtained agree very accurately with the values derived from the present experiments. The method of finding α and β simultaneously, as

has just been described, is more accurate for the determination of the larger values of $\frac{\alpha}{p}$ than the method which was applied to the previous experiments. The values of $\frac{\alpha}{p}$ which were found in the earlier determinations for the larger values of $\frac{X}{p}$ are somewhat too large since the positive ions contributed slightly to the increases of conductivity which were observed. Thus with air the value of $\frac{\alpha}{p}$ corresponding to $\frac{X}{p} = 787$ was given as 9·3, whereas the value obtained by the more correct method is 8·9. In the case of hydrogen the errors which were introduced are larger.

Fig. 3.
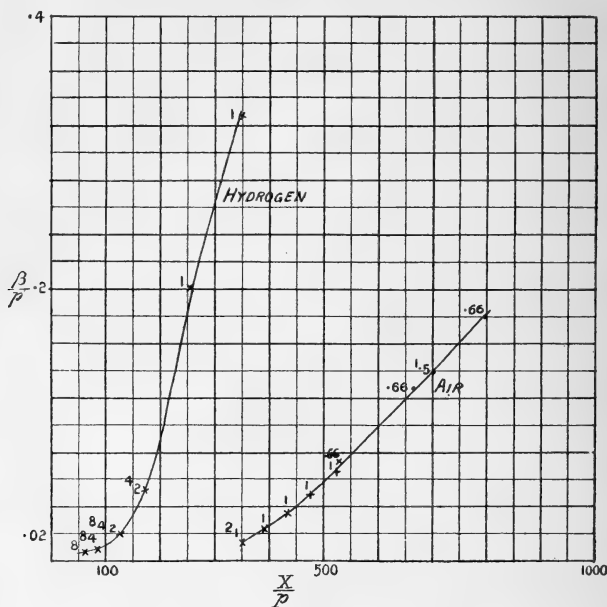
It may be possible to determine accurately the values of α and β corresponding to forces larger than those which were used in the experiments which I have described in this paper, but since the researches have not yet been carried far in this direction a more complete investigation dealing with large forces and small pressures may more conveniently be reserved for future investigation.

7. There are a number of general properties of the conductivity which are illustrated by the experiments which have been already given. The experiments show that when a given number of negative ions start from the negative plate the number reaching the positive plate depends only on the potential-difference $X \times a$ and the quantity of gas $p \times a$ in the space between the plates. Thus, in Tables I. and II. the currents obtained in air at a pressure of 2 millimetres with an electric force of 700 volts per centimetre, and with distances of 1, 2, 3, 4, and 5 millimetres between the plates, are the same as the currents in air at one millimetre pressure with an electric force of 350 volts per centimetre, with distances 2, 4, 6, 8, and 10 millimetres between the plates. Similar instances of the coincidences of currents in hydrogen at 8, 4, and 2 millimetres pressure are to be found in Tables IV., V., and VI. Thus the currents obtained at 2 millimetres pressure with a force of 262 volts per centimetre with the distances 4, 6, 8, and 10 millimetres between the plates are equal to those at 4 millimetres pressure with a force of 525 volts per centimetre with the distances 2, 3, 4, and 5 millimetres between the plates, or at 8 millimetres pressure with a force of 1050 volts per centimetre with the distances 1, 1·5, 2, 2·5 millimetres between the plates.

This result can also be established by the theory. The quantity $\dfrac{X}{p}$ is equal to $\dfrac{v}{m}$, where v is the potential-difference between the plates and m the mass of the gas in the space between the plates. Hence

$$a \times \alpha = m f\left(\frac{v}{m}\right) \text{ and } a \times \beta = m \, \phi\left(\frac{v}{m}\right),$$

so that the ratio

$$\frac{n}{n_0} = \frac{a(\alpha - \beta)\epsilon^{(\alpha - \beta)a}}{a\alpha - a\beta\epsilon^{(\alpha - \beta)a}},$$

which is a function of $a \times \alpha$ and $a \times \beta$, can be expressed as a function of v and m. The same conclusion follows from elementary considerations. When an ion passes from one plate to another the number of collisions that occur depends on the quantity of gas m between the plates; and if this quantity and the potential-difference v are constant, the fall of potential along the path between any two collisions will be fixed. Under these conditions the velocities of the ions at collision and the total number of collisions will be unchanged when the distance between the plates is altered. Hence for each ion starting from the negative plate the number of new ions produced in the gas will be fixed when v and m are given.

8. A comparison between the curves shows that it requires much larger forces to develop new ions by the motion of positive ions in air than in hydrogen. If the positive ion be regarded as being approximately the same size as the molecule of the gas in which it is generated, the free paths of a positive ion in hydrogen would be 1·85 times as long as the free paths of a positive ion in air at the same pressure, this number being the ratio of the mean free paths of molecules in hydrogen and air (Meyer, ' Kinetic Theory of Gases '). It is interesting to compare the number of ions that would be produced per centimetre by a positive ion in hydrogen at 1·85 millimetre pressure and the number that would be produced per centimetre in air at 1 millimetre pressure when the same electric force is acting. The lengths of the free paths and the number of collisions per centimetre are the same in the two gases; also the falls of potential along the free paths would be the same if the same electric force is applied. As an example the force of 370 volts per centimetre may be taken. For air at a millimetre pressure $\beta = ·018$ when $X = 370$ and $p = 1$. For hydrogen $\dfrac{\beta}{p} = ·09$ when $\dfrac{X}{p} = 200$; so that when $X = 370$ and $p = 1·85$, β is ·166. The ratio of the values of β gives the ratio of the numbers of ions produced per centimetre in the two gases; so that for this particular force there are about nine times the number of ions produced in hydrogen as in air for the same number of collisions and the same falls of potential between the collisions. This result justifies the assumption that the positive ion in hydrogen is of smaller mass than the positive ion in air, so that the former acquires a greater velocity in travelling a given distance under a given potential-gradient. The result could also be explained on the hypothesis that the positive ions are of the same mass in air and in hydrogen, and that the velocity required to generate ions from molecules of air is greater than that required to produce a similar effect on molecules of hydrogen. This explanation is not in accordance with what we should expect from considerations of the action of the negative ions in the two gases.

It can be seen from the previous researches [*] that the molecules of air are ionized by negative ions with somewhat smaller velocities of impact than the velocities required to ionize molecules of hydrogen to the same extent.

9. The fall of potential necessary to give sufficient velocity to a positive ion in order to produce new ions by collision is

* J. S. Townsend, Phil. Mag. April 1903, p. 397.

smaller than the potential between the electrodes which is required to give an effect arising from this cause which can be easily detected experimentally. The order of that potential may be found if we suppose that the mean free path of a positive ion is a quarter of that of a negative ion; so that in going through a centimetre of air at a millimetre pressure a positive ion would make 60 collisions. Taking, for example, the case in which a force of 700 volts per centimetre is applied, the value of β is ·14 when the pressure is 1 millimetre; so that on the average only 14 new ions are generated as the result of some 6000 collisions. Let it be supposed that the new ions are generated at the collisions that terminate the 14 longest paths. In general, out of N paths of mean length l the number y which exceed the distance x is given by the formula

$$y = N\epsilon^{-\frac{x}{l}}.$$

Hence the fourteen longest paths when $N = 6000$ and $l = \frac{1}{60}$ exceed ·102 centimetre, so that new ions are generated when the velocity corresponds to a fall of potential of about 70 volts.

A similar investigation shows that new ions may be produced in hydrogen when the fall of potential is about 20 or 30 volts. The voltages obtained by this method would be somewhat smaller if the number of collisions per centimetre is taken from the tables given by Mayer (*loc. cit.*), since the length of the free path of a molecule in air at 1 millimetre pressure deduced from experiments on viscosity is less than $\frac{1}{60}$ centimetre.

10. The tables show that the currents in the gas increase very rapidly for small increases in the distance a when the plates are far apart. When the plates were separated by one millimetre more than the largest distances given in the tables, the force X being maintained at the same value, a current was usually obtained which continued after the light from the external source ceased to act and a glow was seen in the space between the electrodes. From the values of α and β which have been found it is easy to calculate the distance a' for the given forces and pressures at which a continuous discharge might be expected, since the value of n then becomes infinite. This condition is satisfied when the quantity $\alpha - \beta\epsilon^{(\alpha-\beta)a}$, the denominator in the expression for n, vanishes, so that the required distance a' is equal to $\dfrac{\log \alpha/\beta}{\alpha - \beta}$.

It may be seen from this result that the sparking potential $v'[=a'X]$ depends only on the quantity of gas $a' \times p$ between the electrodes.

Substituting in the above expression the general equations found for α and β in terms of the force and pressure, the following value of a' is obtained :

$$a' = \frac{\log f\left(\dfrac{X}{p}\right) - \log \phi\left(\dfrac{X}{p}\right)}{p\left(f\left(\dfrac{X}{p}\right) - \phi\left(\dfrac{X}{p}\right)\right)}$$

or

$$a'p\left[f\left(\frac{v'}{a'p}\right) - \phi\left(\frac{v'}{a'p}\right)\right] = \log f\left(\frac{v'}{a'p}\right) - \log \phi\left(\frac{v'}{a'p}\right),$$

which shows that v' is a function of the product $a'p$.

This property of the sparking potential for a gas between parallel plates has been well established experimentally. The researches made recently by W. R. Carr (Phil Trans. vol. cci. 1903) over a large range of pressures and potentials confirm this result, which was obtained previously by De La Rue and Muller (Phil. Trans. vol. clxxi. 1879) and others over smaller ranges of pressures and potentials.

Since the functions f and ϕ have not yet been determined in simple algebraical forms, the relation between v' and $a'p$ as found from theoretical considerations may be shown by means of a curve.

Thus in Table II. for air at one millimetre pressure when $X = 350$, $\alpha = 5\cdot25$, and $\beta = \cdot0141$, the sparking distance in centimetres is given by the formula

$$a' = \frac{\log 5\cdot25 - \log \cdot0141}{5\cdot23} = 1\cdot13,$$

so that $v' = 350 \times 1\cdot13 = 395$, and $pa' = 1\cdot13$.

The values of v' and pa' deduced in a similar manner from the other experiments with air are marked on the accompanying diagram (fig. 4) by the small crosses. The values of v' and $p \times a'$ calculated from the values of α and β for hydrogen are represented in the diagram fig. 5 (p. 617). The curves passing through these points correspond very accurately with the curves found experimentally. Some values of sparking potentials were determined experimentally with the same apparatus which was used to determine the quantities q. When making the experiments it was found necessary to have a large resistance (50,000 ohms) in the battery circuit, in order that the current should not rise too high

during the discharge. Considerable inconvenience was caused once when this precaution was not taken, as the thin layer of silver on the plate A got burnt away.

TABLE VIII.—Air.

p.	X.	a'.	b_1.	b_2.
1	350	1·13	1·15	1·16
1	437	·832	·82	·84
·66	350	·965	·95	·975
·66	437	·766	·76	·78

Fig. 4.
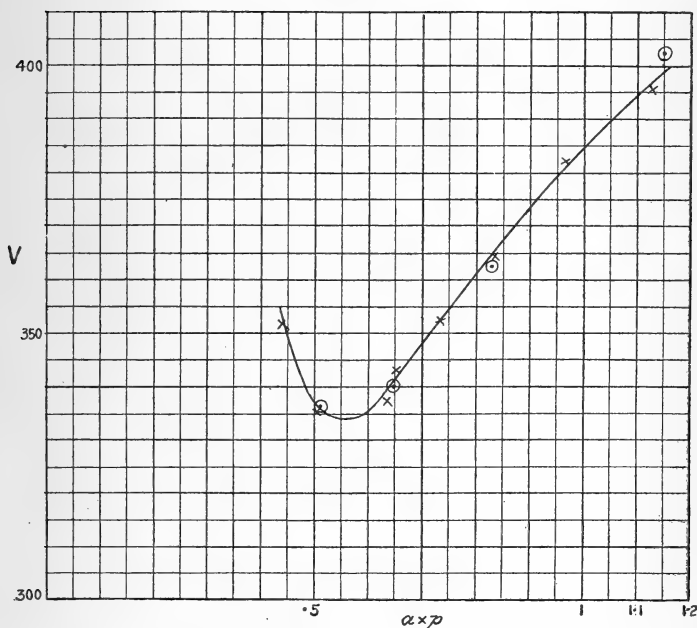
The experiments with air were made by increasing the distance between the plates by small steps until a glow appeared in the gas, corresponding increases being made in the potential so as to keep the electric force the same as that for which the distance a' was determined.

The results given in Table VIII. were obtained. At

the distance b_2 a spark appeared, and at the shorter distance b_1 no spark appeared when the ultra-violet light was acting. a' is the sparking distance for the force X and pressure p, as calculated from the values of α and β. The quantities a', b_1, and b_2 are in centimetres.

The sparking potentials $X \times \dfrac{b_1 + b_2}{2}$ determined experimentally are plotted on the curve, fig. 4, and marked by small circles to distinguish them from the points determined by calculation.

The experiments with hydrogen were made a little differently. The values of a' were first calculated for the different forces and pressures from the numbers given for α and β. The plates were then set at this distance apart and the potential at which a spark appeared was found experimentally. A comparison between the potentials thus found and the potentials $v' = X a'$ gives a method of testing the theory. The potentials V found experimentally are given in Table IX., and are also marked on the curve, fig. 5.

TABLE IX.—Hydrogen.

p.	X.	a'.	v'.	V.
8	525	·90	472	465
8	700	·57	398	385
4	350	1·14	398	385
8	1050	·306	322	322
4	525	·613	322	323
4	700	·405	283	282
2	350	·810	283	287
2	525	·501	269	273
1	350	·806	282	289

It will be seen from the diagrams that the sparking potentials as given by the theory are in accurate agreement with the potentials determined experimentally. The latter determinations were made while the light from the external source was acting on the negative electrode, so that the potentials are a little lower than those which would have been obtained if no radiation had been falling on the electrode.

This effect of ultra-violet light on the sparking potential was first observed by Hertz in air at atmospheric pressure. At pressures of a few millimetres the effect of ultra-violet light on the sparking is not so great as the effect at high pressures.

In determining the potentials experimentally, the light from the external source should be of small intensity since large currents would give rise to effects which are not taken into consideration in the theory. A large current in a gas tends to reduce the resistance and disturbs the uniformity of the electric field between the electrodes. Thus the potential-difference required to maintain a discharge in a gas is much less than that which is necessary to start a discharge in a gas

Fig. 5.
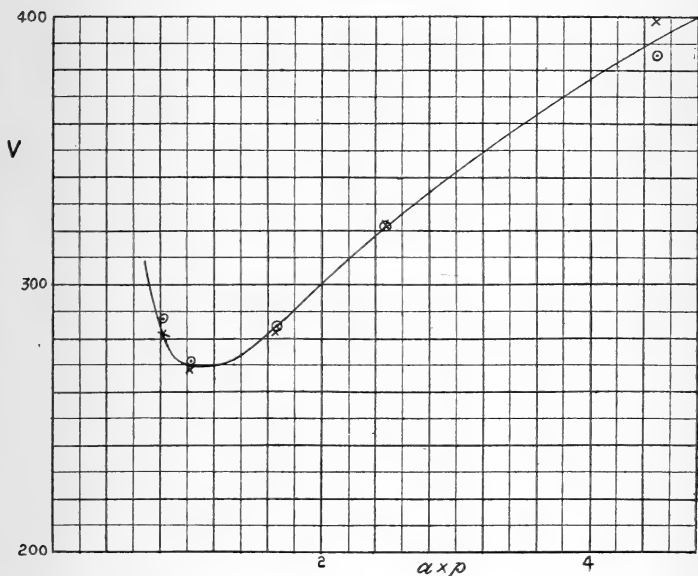
which had been acting as an insulator. If the intensity of the light is very strong, the current might be sufficient to break down the insulation when the quantity $\alpha - \beta \, \epsilon^{(\alpha-\beta)a}$ is small, but does not vanish. The currents used in the experiments were not large enough to produce these effects, since the ratio of the currents for the different distances between the plates did not alter with the intensity of the light.

The spark which takes place when no radiation is falling on the electrodes may be attributed to the small number of ions which, as Elster and Geitel[*] and C. T. R. Wilson[†] have shown, are always present in gases. These would eventually

[*] *Physikalische Zeitschrift,* ii. Jahrgang, No. 8, Nov. 24, 1901.
[†] Proc. Roy. Soc. vol. lxviii. (1901).

produce large currents when the potential is sufficiently high
to cause new ions to be generated in sufficient numbers by
collisions. In this case the sparking could not take place
unless the quantity $\alpha - \beta \, \epsilon^{(\alpha-\beta)a}$ vanished, and possibly it
might require a slightly larger potential than that which
corresponds to this condition in order to ensure the develop-
ment of continuous currents from such a small number of
ions. There are a number of points in connexion with these
subjects which require further investigation, and I hope to
make some other experiments in this direction.

The experiments which I have given on sparking potentials
were made with a source of ultra-violet light of small in-
tensity, which only reduced the sparking potential very
slightly below that which was obtained when no external
radiation acted on the negative electrode, and the agreement
between these potentials and the potentials found by calcu-
lation shows that a satisfactory explanation of the sparking
potentials in gases may be obtained from the theory of the
genesis of ions by collision.

LXXVII. *Some Experiments on the Electrical Discharge from
a Point to a Plane.* By NORMAN R. CAMPBELL, *B.A.,
Scholar of Trinity College, Cambridge* *.

A MONG the many investigations on the discharge from a
point are those of Warburg † and Tamm ‡, who have
measured the relation between the current and the potential-
difference : the production of ozone and other chemical
reactions produced by the discharge have been studied by
Warburg §, Cook ‖, and several others; but no researches appear
to have been made with the object of elucidating the mechanism
whereby the current is transferred between the electrodes.
It was with the hope of throwing some light upon this
problem that the following experiments were undertaken.

The principle of the method adopted depends on the
results obtained by Mr. C. T. R. Wilson in his work on the
Condensation Nuclei produced in gases by various agents ¶ :
these experiments are, in fact, an extension of those described
in pp. 441–444 of that paper. By producing a known
adiabatic expansion in a vessel through which the discharge

---

is passing, it is possible to ascertain whether any particles larger than ions are being produced : it may be noted that the presence of positive ions cannot be detected by this method, if negative ions are also present.

Apparatus and General Results.

The form of the apparatus eventually found most suitable is drawn approximately to scale : the electrical connexious are merely represented diagrammatically.
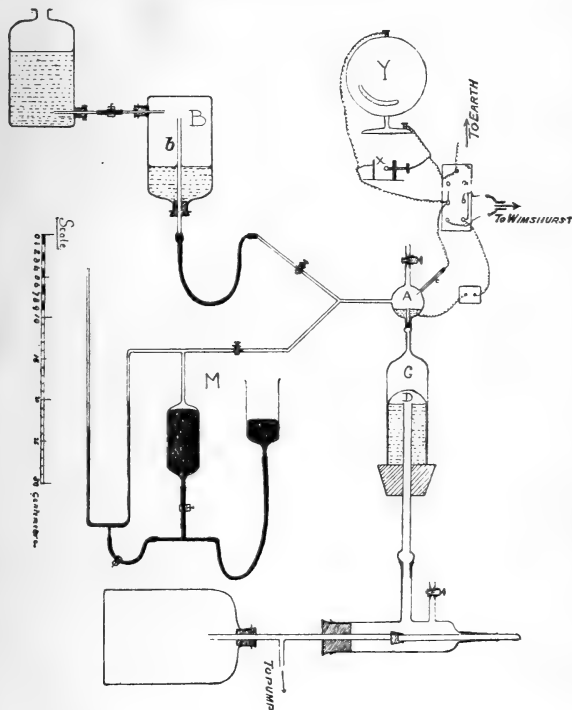
The discharge passed between a point and the water surface in the bulb A, which was connected by a sealing-wax joint to the expansion apparatus below, to the apparatus for regulating the pressure, and to the aspirating bottles for filling with the required gas. The measuring tubes being full of mercury, the tube *b* was pulled beneath the surface of the water in the bottle B containing the gas, and water allowed to flow into the bulb and expansion-tube below. The wire bearing the point was then inserted into the side tube C, by means of sealing-wax, the tube *b* pushed up into the gas, and the piston

D raised from beneath: the water flowed out of the bulb, leaving it full of the gas except for a layer of water which served as one of the electrodes for the discharge. By warming the side tube C excellent insulation could be established between the point and the water.

The two electrodes were connected to a Wimshurst machine through a key arranged so that either of them could be made positive or negative, and either or both of them earthed at will. An adjustable spark-gap x, inserted in parallel, regulated the potential-difference, which was measured by the Braun electrostatic voltmeter Y. It was soon found that the effects observed were independent of the potential-difference within the range available (*i. e.*, from the minimum potential of 2000 volts up to 5000 volts) : in no case were any nuclei produced unless a visible discharge had passed. It was also quite immaterial whether the point or the water were connected to earth, so long as the direction of the electric force remained unaltered. The experiments were conducted with the water connected to earth and the point raised to about $+3500$ volts.

Under these conditions, in air, oxygen, hydrogen, and helium, the discharge from the positive point produced a small faintly luminous fan-shaped " brush :" when negative the point was tipped with a much lighter star-like glow. In nitrogen, the negative discharge had the same appearance, but the positive " brush " was extended in the form of a long violet " tail," reaching to the surface of the water. In the hydrocarbons employed, as well as in CO, the light from the discharge was greenish, and the angle of the " fan " on the positive point considerably greater than in the other gases.

Mr. Wilson has pointed out in the paper mentioned above, that the nuclei produced appear to grow after the discharge has stopped : it is therefore necessary to observe the results of expansion while the discharge is passing. The fog-dispersing power of the discharge renders the detection of very fine showers somewhat difficult, and hence great accuracy in determining the value of the expansion required to give a fog is impossible.

Each form of apparatus used was standardized by means of Röntgen rays or radium rays as described by Mr. Wilson[*]. The form shown in the figure required no correction : only corrected values are given below.

It will be necessary here to mention with some detail the chief difficulty encountered in the course of this research.

* C. T. R. Wilson, Phil. Trans. 193 A. p. 292 (1899).

In examining by means of an expansion apparatus the ionization produced in a gas by the simple ionizing agents, such as Röntgen and Becquerel rays, and also in cases of " spontaneous ionization," there is no difficulty in obtaining regular and consistent results, even in the presence of considerable impurities in the gas. But when more complicated ionizing processes are employed, such as ultra-violet light or the discharge here investigated, there appear irregularities and inconsistencies which render it at first sight hopeless to obtain a definite result. Slight impurities in the gas, or some other accidents whose nature it is difficult to ascertain, exercise an effect on the results altogether out of proportion to their own magnitude. But after some weeks of work it was found that the experiments could be divided into two classes :—

> (1)   Regular,
> (2)   Irregular.

(An " experiment " includes all the observations taken without refilling the apparatus with gas or changing the point from which the discharge took place.) In the " regular " experiments a perfectly definite value could be assigned to the minimum expansion requisite to give a fog with a given discharge : the value remained constant however often the observations were made, in whatever order they were made, or however long a time elapsed between successive observations : the minimum expansion was invariable. It was also found that these regular experiments agreed with each other. For the same gas, the same point, and the same sign of the point, *all* regular experiments gave the same result. *Experiments which were regular were also consistent.*

It was entirely different with the " irregular " experiments. Here no definite value could be assigned to the expansion required to give a fog : of several expansions of the same amount made in succession without stopping the discharge, some would give a fog, some would not. At other times it was thought that definite results had been attained, but on waiting a few minutes they could not be repeated. These irregular clouds were always obtained with expansions less, usually much less, than the minimum value found in the regular experiments. It was occasionally found that on leaving the apparatus for a few hours without refilling with gas, a series which had been irregular would become regular. A change from regular to irregular results was never observed except on the introduction of a new point or a new portion of gas.

In the conclusions given below it is assumed that the regular and consistent experiments give the normal result, and that the irregularity of the others is due to some unessential disturbance. The mutual agreement of the former series affords a logical inductive proof of the correctness of this assumption.

No single sufficient cause for the inconsistencies can be suggested, but some of them at least may be due to hydrocarbons present as impurities, for the introduction of a mere trace of coal-gas, ether, or benzoline was always sufficient to produce them. All gases before passing into the apparatus were bubbled through $KMnO_4$: this procedure minimised, though it did not entirely prevent, the annoyance. There seems, in these cases, to be some deposit formed on the glass of the apparatus, for, when the irregularities had once appeared in a series of experiments, it was necessary to clean all the glass with chromic acid before regular results could be again attained. Even this was not sufficient after the hydrocarbons had been introduced ; a new bulb had to be blown from fresh glass.

After these general observations, the results will be described in greater detail.

Air.—The air was drawn from outside the window of the laboratory and purified by passing through a solution of potassium permanganate and a plug of cotton-wool.

Points of the following metals were employed : Pt, Au, Ag, Cu, Ni, Al, Zn, Steel, German-Silver, Carbon, Mg, and Copper coated with Sodium Amalgam. All these metals except the last two gave the same result. Out of 34 experiments, 19 were irregular ; the others gave :—

Discharge from positive point—Fog with expansions for which $\frac{v_2}{v_1}$ lay between 1·254 and 1·251.

Discharge from negative point—Fog with expansions for which $\frac{v_2}{v_1}$ lay between 1·252 and 1·247.

Twelve experiments made with the Mg and Sodium Amalgam points were all quite irregular and inconsistent. This will be referred to later.

Hydrogen.—Prepared from commercial zinc and HCl and passed through a solution of $KMnO_4$. At first great care was taken to procure perfectly pure gas, but it was found that if these precautions were omitted and the discharge passed for five minutes before making observations, the same results were obtained as with the pure gas.

The metals employed were Pt, Ag, Cu, Ni, Al, Mg, Carbon, and Sodium Amalgam. All these without exception gave the same effects. Out of 32 experiments, 6 were irregular, the remainder gave :—

Positive discharge—Fogs for $\frac{v_2}{v_1}$ between 1·253 and 1·247.

Negative discharge—Fogs for $\frac{v_2}{v_1}$ between 1·244 and 1·239.

Nitrogen.—Prepared from air by passing over KOH and red-hot copper. Points of Pt, Cu, Al, Mg, were used—with the same result in every case. 15 experiments, 4 irregular.

Positive discharge—Fogs for $\frac{v_2}{v_1}$ between 1·252 and 1·251.

Negative discharge—Fogs for $\frac{v_2}{v_1}$ between 1·251 and 1·250.

A few experiments were performed with a Pt point in a mixture of about equal volumes of nitrogen and hydrogen, to try whether the chemical action resulting in the formation of ammonia would produce any effect. The results were the same as for hydrogen.

Oxygen.—Prepared in three ways :—

        (1)   from $KMnO_4$,
        (2)   from $KClO_3$,
        (3)   from electrolysed potash.

The metals employed were Pt, Ag, Ni, Cu, Al, Mg. 36 experiments were performed. With the positive discharge 17 of these were irregular ; 19 gave the same results as for air and hydrogen, except with a magnesium point. The experiments with a positive brush from the latter point, and all the experiments with a negative brush, were perfectly irregular and inconsistent.

Acetylene, Coal-Gas, and Carbon Monoxide.—Pt and carbon points used. These gases were employed to discover what effect a trace of hydrocarbons might be expected to have on other gases : they also exhibit strong absorption of ultra-violet light, a property which might affect their behaviour in the present experiments.

It was, however, found that the discharge produced decomposition in the gases, and that a cloud was formed with very slight expansions, sometimes with no expansion at all, the clouds being much denser and totally different in appearance to those in other gases. Similar results were obtained in air mixed with a little benzoline vapour.

General Conclusions.

It will be at once noticed that the minimum expansion required to give a cloud while the discharge is passing has the same value for both signs in air and in nitrogen, and for the positive discharge in hydrogen and oxygen, and that this value agrees very approximately with that found by C. T. R. Wilson for condensation on negative ions in these gases, namely, 1·25. We may therefore conclude that in these cases the largest nuclei produced by the brush are the negative ions.

The corresponding value for the negative brush in hydrogen is somewhat less. This agrees with Mr. Wilson's result that the expansion required with the positive discharge is slightly greater than that required with the negative discharge; which he attributes to the greater fog-dispersing power of the former, since this property renders very fine showers difficult to observe. It is remarkable, however, that in these experiments the value for the negative discharge is less than 1·25, while that for the positive is only slightly greater. Hence Mr. Wilson's explanation seems hardly admissible. Slight impurities in the gas, which are notably hard to remove in the case of hydrogen, might cause the peculiarity, but it is surprising that they should give so consistent and so definite a value. In this connexion it may be noted, that the experiments on the negative brush seemed always more liable to disturbance by slight impurities than those on the positive.

It is possible that the slightly smaller value of $\dfrac{v_2}{v_1}$ for the negative discharge may be due to "growth" of the ions while diffusing from the point to the sides of the bulb : this is hardly probable, however, for it would be expected that the ions would diffuse faster in hydrogen than in air, and hence would have less time to grow.

The irregularities of the results in oxygen, with which may be classed those with magnesium and sodium amalgam in air, call for some explanation. The clouds produced in these instances were not visible at the ordinary pressure, but never required an expansion greater than 1·08 to ensure their appearance ; between these limits the value varied in a perfectly irregular manner.

It is probable that this effect may be attributed to the phenomena discovered by Meissner, in connexion with the production of ozone[*], which are generally considered

[*] Meissner, *Untersuchung über den Sauerstoff* : Hamburg, 1863.

as due to the presence of H_2O_2. Meissner found that if ozone is bubbled through a solution of KI, or passed over some of the more oxidizable metals, and then brought into contact with water vapour, clouds are produced. Now the oxygen or air in the bulb had a distinct smell of ozone after the discharge had passed, and the addition of a little KI to the water rendered the clouds produced much thicker, so that they sometimes appeared without any expansion at all. Again, in air, it was with Mg and Na, the two most oxidizable metals used, that the irregularities occurred. Meissner's experiments prove that the mere presence of these metals in company with ozone and water vapour would be sufficient to produce a cloud.

The absence of these peculiarities from the positive discharge may be explained by the observations of Warburg*, confirmed by Cook†, that the negative discharge is more active in the production of ozone than the positive. In air, the smaller proportion of oxygen present will account for the absence of Meissner clouds in a similar manner.

If a few minutes were allowed to elapse between the cessation of the discharge and the production of the expansion, the clouds were much less dense than if produced during the discharge. The effect of passing the discharge in the bottle B, before admitting the gas to the apparatus, was also tried : it was found that no clouds were formed on expansions until the discharge was passed in the bulb, when the results were the same as for normal oxygen. It appears therefore that the abnormal clouds are due to some process occurring during the passage of the current, the effect of which vanishes quickly after its cessation. It may be that H_2O_2, to which the Meissner effects are attributed, is produced together with ozone by the discharge, and disappears rapidly afterwards by reaction with the ozone. The presence of a trace of H_2O_2 in the presence of ozone would be impossible to detect by chemical tests.

It was at first thought that these abnormal effects might be due to ultra-violet light, which is known to produce large nuclei in oxygen. If this were the case, it would be expected that the discharge from Al or Zn points, which is rich in radiation of short wave-length, would have given these effects, to at least as large an extent as magnesium, but this exception is contradicted by experiment. As an additional objection to this view, it may be mentioned that Cook's experiments seem to show that more ultra-violet light is

* E. Warburg, *Berl. Ber.* 1900, pp. 712–721.
† Cook, see above.

produced by the positive than by the negative discharge. It is more probable that the clouds produced by ultra-violet light and by these discharges are due to a common cause— the Meissner effect.

We thus arrive at the following conclusions :—

(1) A discharge from a point in a gas in which it induces no chemical reaction, produces no nuclei larger than negative ions. The current is carried entirely by ions and by no larger particles of the electrodes or gas.

(2) No discharge from a point can be obtained in which positive ions exist unaccompanied by negative ions.

The importance of this second conclusion should be noted : it is certainly worthy of remark that there should always be negative ions present in the field of the positive discharge.

Helium.—A few experiments were undertaken with this gas to ascertain whether the phenomena of the discharge were in any way different in character for a monatomic gas. In estimating the expansion required to give condensation on negative ions in such a gas, account must be taken of the dependence of the cooling effect due to expansion, which causes condensation, on the ratio of the specific heats of the gas.

If T_1, T_2 are the temperatures before and after expansion we have

$$\left(\frac{v_2}{v_1}\right)^{\gamma-1} = \frac{T_1}{T_2}.$$

Hence to get condensation on negative ions in a gas with a different value of γ, we must keep $\frac{T_1}{T_2}$ constant and have

$$\left(\frac{v_2'}{v_1'}\right)^{\gamma-1} = \left(\frac{v_2}{v_1}\right)^{\gamma-1}.$$

On standardizing the apparatus with radium rays the value 1·15 was obtained for $\frac{v_2}{v_1}$ to give condensation on negative ions in helium.

Then $(1\cdot15)^{\gamma'-1} = (1\cdot25)^{\gamma-1}$, when γ refers to air or hydrogen, and γ' to helium. If $\gamma = 1\cdot41$, this gives $\gamma' = 1\cdot66$, a value deduced for helium on other grounds. If an accuracy greater than 1 per cent. is not desired, there is probably no simpler or more efficient method than this for determining γ for any given gas.

On making experiments on the discharge of the same nature as those on the other gases tried, the results obtained

were the same as for air, &c. The largest nuclei produced by the discharge are negative ions.

Some rough measurements were made on the comparative values of the potential required to give a discharge in air, hydrogen, and helium. The following results were obtained with a given platinum point:—

Air.	Hydrogen.	Helium.
+1850	1750	1200 volts.
−1600	1500	800 „

The electrometer used was not suitable for accurate measurements, but it showed sufficiently plainly that the minimum potential in helium was a great deal less than that for the other gases used, and differed more from values for the diatomic gases than these differed among themselves.

My best thanks are due to Professor Thomson and Mr. C. T. R. Wilson for the generous advice and assistance which they have given me.

LXXVIII. *Notices respecting New Books.*

La Compressibilité des Gaz Réels. Par L. Décombe, *Docteur ès Sciences.* Paris: C. Naud. 1903. Pp. 99.

IN this monograph, which forms No. 21 of the "Scientia" series, the author gives a very full and up-to-date account of what is known regarding the compressibility of gases. We regret to see Chapter I., which deals with the experiments relating to moderate pressures, headed "La Loi de Mariotte." The author gives references to the publications of both Boyle and Mariotte, mentioning their respective dates, but throughout the chapter never condescends to give Boyle the credit of his discovery, and persists in speaking of "la loi de Mariotte." Chapter II. deals with the behaviour of gases under very high, and Chapter III. under very low, pressures. In Chapter IV. is given a very interesting account of the experimental researches relating to the variation of the curve connecting the product $p \cdot v$ with p at different temperatures. Chapter V. deals with the critical point, and Chapter VI. with the characteristic equation of a gas. In Chapter VII. is given an outline of the theory of corresponding states, and the concluding Chapter VIII. is devoted to gaseous mixtures. Numerous references to original papers greatly enhance the value of this monograph.

Principles of Inorganic Chemistry. By Harry C. Jones, *Associate Professor of Physical Chemistry in the Johns Hopkins University.* New York: The Macmillan Company. London: Macmillan & Co., Ltd., 1903. Pp. xx+520.

THE author of this work is already well known as one of the most

active and enthusiastic exponents of recent advances in physical chemistry, and as the author of one or two books which have done a good deal towards popularising modern electro-chemical theories amongst his fellow-countrymen. The publication of the new book under review will, we feel convinced, add largely to his growing reputation. It is a text-book brightly and clearly written, thoroughly up-to-date, and well within the mental reach of any reader with a good general education.

One of the most striking features in connexion with almost all recent advances in chemistry lies in the fact that this science is daily becoming more and more mathematical. The days when the teaching of chemistry consisted in the enumeration of a certain number of chemical facts, whose dreariness was only relieved by the possibility of devising striking lecture experiments in illustration of some of them, are now gone for ever. Such generalizations as Guldberg and Waage's Law of Mass Action, Willard Gibbs's Phase Rule, and the theory of Electrolytic Dissociation, have invested chemistry with an interest and a vitality which no mere collection of experimental facts can ever possess.

It was with the object of familiarising the ordinary student with the fundamental conceptions of modern chemical theories that the book under review was written, and one cannot help admiring the skill with which the author has interwoven the new theories with the standard methods of treatment followed by the older text-books. It is unnecessary to give any detailed account of the contents of the volume ; in its general features, it does not differ essentially from any other similar text-book on the subject. Its main feature, as we have already pointed out, lies in the *method* of treatment rather than in the arrangement of the subject-matter, and we do not know of any other *elementary* text-book which we could more confidently recommend to anyone wishing to gain some insight into recent advances in chemical theory. As is the case with all American books, an excellent index at the end of the volume greatly enhances its value for purposes of reference.

Il Moto dei Ioni nelle Scariche Elettriche. Con tre tavole e nove figure intercalate nel testo. Bologna : Ditta Nicola Zanichelli. 1903. Pp. 66.

THIS pamphlet contains the substance of a lecture delivered by Prof. Augusto Righi at the Bologna meeting of the Italian Electrotechnical Society. It is a most interesting general account of the electron theory and allied matters, and will be found useful by those who, while interested in the recent developments of electrical theory, cannot afford the time for a detailed study of this fascinating subject.

LXXIX. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 548.]

April 29th, 1903.—J. J. H. Teall, Esq., M.A., F.R.S.,
Vice-President, in the Chair.

THE following communications were read:—

1. ' The Age of the principal Lake-Basins between the Jura and
the Alps.' By Charles S. Du Riche Preller, M.A., Ph.D., A.M.I.C.E.,
M.I.E.E., F.R.S.E., F.G.S.

1. In a paper read before the Society last session, the author
showed, on the evidence of extensive high-level deposits of Decken-
schotter in Subalpine France and Switzerland, that the principal
Swiss lake-basins could not have existed at the time when those
deposits were formed, during and after the first or Pliocene glaciation
of the Alps. In the present paper he deals with the question reserved
in the preceding one, that is, to which subsequent period the forma-
tion of those lake-basins should be assigned. By the light of further
recent investigations in the different localities, he first considers
the conditions of the Zurich lake-valley, where the successive
glacial and fluviatile deposits are clearly defined, and then applies
his conclusions to the other principal lake-basins lying in the same
zone along the edge of the Alps.

2. The hitherto generally accepted view that the lake-basins are
pre-Glacial in the old sense, or were formed during the first inter-
Glacial period, rests, in the main, on two arguments : (1) that the
alluvia at the lower ends of the lakes are all Glacial, not only from
their appearance, but because the materials composing them could
only have been transported thence by glaciers, which either passed
over the lakes by bridging them, or through them by completely
filling them with ice ; and (2) that the zonal bending of the Molasse
along the edge of the Alps, to which the lake-basins owe their
existence, occurred before the second or maximum glaciation,
because at a point in the Lorze ravine (near the Lake of Zug) the
Deckenschotter conglomerate dips reversely, that is, up the valley,
while the overlying, younger, loose gravel dips in the opposite
direction.

3. The author adduces evidence to show that the deep-level
gravel-beds in the Limmat Valley near and below Zurich are essen-
tially fluviatile, composed of the characteristic Alpine material
of the Rhine and Linth drainage-areas, and in all other respects
similar to the gravel carried by the River Sihl at the present day.
These gravel-beds rest upon Glacial clay of the second glaciation,
which fills the Molasse-bed of the valley to a great depth, and are
overlain by the moraine-bars and secondary products of the third
glaciation, the latter being overlain by, and mixed with, the
post-Glacial alluvia of the Sihl.

4. He further argues that it is, on mechanical grounds, difficult
to conceive how glaciers could either bridge, or completely fill with

ice, such extensive basins as those of the principal Alpine lakes, from 2 to 8 miles in width and from 470 to 1020 feet in depth, the quantity of water to be displaced and expelled in the individual cases ranging from 3500 million to 90,000 million cubic metres or tons.

5. As regards the more recently-enunciated argument of the Deckenschotter and overlying gravel-exposure in the Lorze Valley, the author points out that, apart from the difficulty of differentiating the second and third glaciation-materials in that locality, it is obviously hazardous to deduce from a purely-local phenomenon of this kind, and more especially from any dip of loose gravel—in contrast with rock or compact conglomerate,—the date of the zonal bending affecting six valley-systems, and extending over more than 200 miles along the edge of the Alps.

6. The author's investigations point to the conclusion that the deep-level Limmat gravel-beds, overlain by the moraine-bars of the third glaciation, were deposited by a river during the second inter-Glacial period; that the lowering of the valley-floor was initiated in the course of the third glaciation, probably when the glacier had already reached its maximum extension, about 10 miles below Zurich; that the zonal subsidence continued throughout the retreat of the ice; and that the simultaneous formation of the lake-basin should therefore be assigned to the end of the Glacial Period, after which the original basin was, notably at its upper end, restricted to its present dimensions by post-Glacial alluvia.

7. In conclusion, the author shows that the same arguments apply, in the main, also to the origin and age of the other principal zonal lake-basins, which he illustrates by longitudinal sections. In his view, the position and depth of these basins, as well as the intervening ground, point to the probability that the bending took place not only along one line, but along several, more or less parallel, not always continuous lines within the zone between the Alps and the Jura; that the bending was by no means of uniform depth; and that, therefore, the Alps did not subside as a rigid mass, but that the zonal bending along their edge merely extended locally for some distance from the deepest points of the lake-basins along the floors of the principal Alpine river-valleys.

2. 'On a Shelly Boulder-Clay in the so-called Palagonite-Formation of Iceland.' By Helgi Pjetursson, Cand. Sci. Nat.

There is no equivalent in the Tertiary basalt-plateaux of Britain of the great palagonite-formation of Iceland, which Prof. Thoroddsen has shown to be younger than the basalt-formation of the latter island. The basement-layer of the breccia-formation, resting directly upon the basalts, contains glaciated blocks of all sizes, up to 6 feet and more in diameter. These ground-moraines are followed by tufaceous sandstones, conglomerate, columnar basalts, other ground-moraines and volcanic tuffs and breccias. At Búrlandshöfdi a shelly Boulder-Clay, 70 to 80 feet thick, rests upon the fundamental basalt, which here shows a glaciated surface. Unbroken shells are very rare. *Astarte borealis* is the most common shell, and *Saxicava*

arctica and *Mya truncata* are less common, indicating that some of the older moraines are of Pleistocene age. The author concludes that volcanic activity did not pause in Iceland during the Glacial Period, but that it was especially active at the beginning and the close of glaciation, building up bulky hills of slags and ashes, some of which have survived the Glacial Period as volcanoes, while others have become extinct. Volcanic activity had died out in Britain at this time, and hence the palagonite-formation is unrepresented in that country.

<div style="text-align:center">

May 13th.—Edwin Tulley Newton, Esq., F.R.S.,
Vice-President, in the Chair.

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The following communications were read:—

1. ' On some Disturbances in the Chalk near Royston (Hertfordshire).' By Horace Bolingbroke Woodward, Esq., F.R.S., F.G.S.

A ' line of flexure ' is marked on the Geological-Survey map from Therfield, south-west of Royston, in Hertfordshire, to near Heydon in Cambridgeshire, a curved line a little below the crest of the Upper Chalk-escarpment. The author in 1902 found evidence which satisfied him that the disturbances, previously supposed to be an anticline, were due to Glacial action, a view confirmed during the present year. Four sections are described: Great Chishall, Pinner's Cross, the Limekiln south-west of Newsell's Park and north of Barkway, and north of Reed. The disturbed Chalk near Royston, with its fractured and displaced flints, occurs in conjunction with Boulder-Clay, and the latter is found beneath a considerable thickness of disturbed Chalk. This is compared with similar phenomena near Trimingham, and at Litcham in Western Norfolk. While Boulder Clay occurs along the high ground bounding the disturbed area to the south, the vale and undulating downs immediately to the north are devoid of this Glacial Drift. The facts were to be explained, on the land-ice theory, if the ice were at first welded to the rubbly surface-strata in regions north of the escarpment, and, when movement set in, there were overthrusts of débris-laden ice, and upper layers of ice were rent asunder from and moved over lower ones; while to the thrust or long-continued pressure of ice along shear-planes at the higher levels may be attributed the belt of disturbed strata. Certain patches of esker-like gravel in Wardington Bottom might be explained by streams due to the melting of the ice banked up against the scarp; and we might go some way with Sedgwick in believing that the outlines of the combes ' do not appear to have been produced by a long-continued and slow process of erosion; but rather to have been cleanly swept out by rapidly-descending water-floods.'

2. ' On a Section at Cowley, near Cheltenham, and its Bearing upon the Interpretation of the Bajocian Denudation.' By Linsdall Richardson, Esq., F.G.S.

According to Mr. Buckman's map, published in 1901, the Upper *Trigonia*-Grit should have been seen at this spot to rest directly,

and non-sequentially, upon the Upper Freestone, whereas observation shows the intervention of at least the *Buckmani*-Grit, part of which thins out from 4 inches at the north-eastern end, to nothing at the other end of the quarry, which is in the direction of the anticlinal axis. The error is not one of fact but of inference, and the present evidence rectifies portions of those limits which were drawn theoretically. A section near the ' Air-Balloon ' Inn, on the road from Birdlip to Cheltenham, shows the Lower *Trigonia*-Grit covered by *Buckmani*-Grit and underlain by the Upper Freestone. There is only one section where the Upper *Trigonia*-Grit is seen to rest directly upon the Lower *Trigonia*-Grit, the latter being only 3 feet 2 inches thick. The causes producing the Bajocian denudation appear to have been forces so acting as to effect a repetition of flexure along old lines of weakness (Aalenian); and thus in the Birdlip area an anticline may be again located, but the elevation was this time much greater: indeed, the level of the Aalenian denudation was passed by the Bajocian. Other sections near Brimpsfield and in Cranham Wood are given in connection with the location of the anticlinal axis. The exact location of the anticlines and synclines of the Inferior-Oolite rocks in the Cotteswolds, where sections are numerous, may afford some important working hypothesis for unravelling the structure of the Vale of Gloucester, where excavations are few.

3. ' Description of a Species of *Heterastræa* from the Lower Rhætic of Gloucestershire.' By Robert F. Tomes, Esq., F.G.S.

<div align="center">May 27th.—Edwin Tulley Newton, Esq., F.R.S.,
Vice-President, in the Chair.</div>

The following communications were read :—

1. ' An Experiment in Mountain-Building.' By the Right Hon. the Lord Avebury, P.C., F.R.S., F.G.S.

Various observers have endeavoured to throw light on the origin of mountains by compressing pieces of cloth, etc. In these cases, however, the pressure was only in one direction. The author wished to obtain a method of obtaining compression in two directions at right angles to one another ; and, accordingly, he had an apparatus constructed, consisting of four beams of wood, which could be approximated by means of screws. In the space, 2 feet across and 9 inches in depth, were placed pieces of carpet-baize and layers of sand, each about $1\frac{1}{2}$ inches deep. The beams were then caused to approach one another until the sand rose in the centre into contact with the glass cover, against which it was flattened out. Casts were made of the surfaces of the different baize-layers, and it was found that in the lower layers the ridges were narrower, shorter, more precipitous, and more broken up, than in the higher layers. A second series of casts was exhibited, with the sand and baize having been arranged as before, but with the weight placed on one side. The ridges followed the edges, though not closely, leaving a central hollow. There was a difference

between the higher and lower layers, similar to that seen in the first experiment.

2. 'The Toarcian of Bredon Hill (Worcestershire), and a Comparison with Deposits elsewhere.' By S. S. Buckman, Esq., F.G.S.

The Upper Lias (G 3) of Bredon Hill is shown on the Geological-Survey map as more than 300 feet thick, whereas at Wotton-under-Edge it is said to be only 10 feet thick. But at the former locality the Inferior Oolite is represented as resting directly on Upper Lias, while at the latter the 'Midford Sands' intervene. It is shown that this 'Upper Lias' at Bredon contains strata of the following hemeræ :—*Lilli, Variabilis, Struckmanni, Dispansi, Dumortieriæ*, and *Moorei*, in addition to those of the hemeræ *Falciferi* and *Bifrontis*, which at Wotton have been called Upper Lias, where the rest have been mapped as the 'Midford Sand.' These sands are 210 feet thick, and hence the Toarcian strata of the two places are 220 and 380 feet thick, respectively. Thus the so-called 'Upper Lias' is really the equivalent of the Upper Lias, Cotteswold Sands, and Cephalopod-Bed of the Cotteswolds ; of part of the Junction-Bed, the Upper Lias, and Bridport Sands of the Dorset coast ; and of the Toarcian of Normandy. Measured thicknesses of the strata at four localities in the Cotteswolds are given, and sections to show that an anticline was formed, and penecontemporaneous erosion took place at Birdlip before the *Scissi* hemera. A table of comparative thicknesses of deposits laid down during similar times in the Cotteswolds and Dorset is also given, and a section of the Toarcian at May-sur-Orne, and of the Toarcian and Aalenian strata at Tilly-sur-Seulles (Calvados), where the Toarcian is reduced to a thickness of only some 23 feet. The chronometry of the Toarcian is discussed, and the approximate maxima of deposit formed during the hemeræ *Falciferi* to *Moorei* are given, amounting to a total of 719 feet. This time is divided into nine hemeræ, so that the time-value of a hemera is equal to the average time taken to deposit about 80 feet of strata. A concluding table gives the sequence and correlation of the following deposits :—The Cotteswold Sands, the Sands at Sodbury, the Midford Sands, those of Cole (Somerset), the Yeovil Sands, the Bridport Sands, and the Northampton Sands.

3. 'Two Toarcian Ammonites.' By S. S. Buckman, Esq., F.G.S.

June 10th.—J. J. Harris Teall, Esq., M.A., F.R.S.,
Vice-President, in the Chair.

The following communications were read :—

1. 'On Primary and Secondary Devitrification in Glassy Igneous Rocks.' By Prof. T. G. Bonney, D.Sc., LL.D., F.R.S., F.G.S., and John Parkinson, Esq., F.G.S.

Part I.—By John Parkinson, Esq.

In this part the types of primary devitrification as found at Obsidian Cliff are briefly reviewed, with the order in which they

appear in the crystallizing glass. 'Porous spherulites' are once more mentioned, in order to call attention to the 'feather-like' crystals which often distinguish them, and of which an explanation is given in Part II. Reference is made to the conditions which favoured primary devitrification at Obsidian Cliff; and the author, leaving general principles to be discussed in the second part, mentions one or two special types of primary devitrification. These are concerned with the probable formation of eutectic zones, or patches; either following the crystallization of an overplus of any given material, or as a residuum. After a brief reference to secondary devitrification, this part of the paper concludes with a summary in which the several relations of secondary to primary devitrification-structures are given.

Part II.—By Prof. T. G. Bonney.

Crystallization in a colloid mass involves an orientation and commonly a separation of the molecules; a process illustrated in an early stage by the formation of microliths in a glass, and the devitrification of the latter when it is heated without actual melting, or by a metal becoming crystalline under strains. Certain conditions, such as slow cooling, supersaturation, and the presence of inclusions—anything causing discontinuity—are favourable to crystallization, some special cases of which are discussed in the paper. The structures thus formed in rocks may be classified as (1) the linear, and (2) the granular; and the former may be subdivided into (a) the rectilinear, (b) the curvilinear. Spherulitic structure in its simpler form falls under (a), and is at first little more than a radial grouping of molecules, the process becoming, as described, gradually more complicated. Of this, 'graphic' or 'pegmatitic' structure is a final stage, where two minerals are crystallizing out of a solution, and one has slightly the advantage over the other, so that it virtually forms a skeleton-crystal. Into this the ordinary radial growth of a spherulite may be seen to pass; likewise also examples of (a) into those of (b): the latter being due to the 'leading' mineral meeting with a rather stronger resistance, as if a crystal were forming in a very tough jelly. An experiment of Messrs. J. l'Anson & E. A. Pankhurst (Min. Mag. vol. v, 1884, p. 34) on the formation of tubes of colloid silica from a fluid alkaline silicate, affords a good illustration of this curvilinear growth. Resistances, as the author has pointed out in an earlier paper, are favourable to actinolitic and branching growths, and the various types of structure mentioned above can be shown to be dependent on them.

The granular structure is next discussed, and explanations are offered of its varieties. This, on a microscopic scale, is often a result of devitrification, where (so far as is known) there has been no marked rise of temperature; and the author shows how this is affected by greater or less freedom of molecular motion, discussing also cases in which a crystalline mass, like a spherulite, has undergone a later re-arrangement.

In conclusion, the relation of some of these structures to an eutectic composition is discussed. It is not, however, easy, owing to the complexity of the conditions, to come to any very definite conclusions in the case of old rock-masses.

2. 'Geology of the Ashbourne & Buxton Branch of the London & North-Western Railway :—Crake Low to Parsley Hay.' By Henry Howe Arnold-Bemrose, Esq., M.A., F.G.S.

The present paper is a continuation of one published in 1899, and deals with the geology of the next 8 miles of this railway. After passing through Yoredale Shales in the second cutting (No. 10), the railway enters the thick beds of Mountain Limestone, in which it continues as far as Buxton. The latter rock is frequently folded, and owing to this no very great thickness of limestone is seen. It was not found possible to correlate the beds in the different cuttings. The following cuttings are described:—(9) Crakelow Farm, (10) Newton Grange, (11) Moat Low, (12) New Inns South, (13) New Inns, (14) Alsop-en-le-Dale, (15) Nettly Low, (16) Cold Eaton, (17) Cheapside, (18) Bank House, (19) Heathcote, (20) Hand Dales, (21) Caskin Low, (22) Lean Low, and (23) Parsley Hay; and measured sections are given of several of them, with an account of the folding and other features displayed. The Newton-Grange cutting shows 6 feet of tuff, probably a thin representative of the 140-feet seam in the Tissington cutting. The limestones are in places granular, oolitic, or dolomitized, and microscopical accounts are given of the several varieties, as well as of the encrinital limestones, pellets and pebbles in the limestones, and the calcareous tuff.

June 24th.—Sir Archibald Geikie, D.Sc., LL.D., F.R.S., Vice-President, in the Chair.

The following communications were read :—

1. 'On a Transported Mass of Ampthill Clay in the Boulder-Clay at Biggleswade (Bedfordshire).' By Henry Home, Esq.

The section described was exposed in the construction of a well 2 miles south-south-east of Biggleswade Railway-Station. Under $10\frac{1}{2}$ feet of soil and Boulder-Clay, the Ampthill Clay was penetrated for 67 feet, resting on Chalky Boulder-Clay, fine silty clay, disturbed Gault, and Lower Greensand. The clay is lithologically identical with the Ampthill Clay with its selenite-crystals, and contains *Ammonites excavatus*, often covered with *Serpulæ*, but no abundant examples of *Ostrea deltoidea*. The boulder was probably an outlier, situated in Oxford Clay at a level high enough to be ploughed into by the agent which formed the Glacial Drift. The distance from which it was moved may not have been greater than a mile or two, but on this point no definite opinion can be expressed. The septaria in it have a dip of 9°. The extent of the mass has not yet been ascertained.

2. 'The Rhætic and Lower Lias of Sedbury Cliff, near Chepstow.' By Lindsall Richardson, Esq., F.G.S.

The chief portion of the cliff-section described has a direction north-east and south-west; the dip of the beds does not exceed 3° to the south-south-east. The section is as follows:—

			Feet	inches.
Lower Lias...		Based on a conglomerate composed of fragments of Cotham Marble	37	1
Upper Rhætic.	2.	Greenish-grey, chiefly laminated shales. 2 ft. to	3	4
	3.	*Estheria*-bed	0	4 to 12
	4.	Greenish-grey shales................................	3	5
	5 a.	Black, laminated shales	2	4
	5 b.	Hard, blackish-blue limestone	0	5
	6.	Black, earthy shales, full of shell-débris.........	0	6
	7.	Hard, slightly pyritic limestone	0	6
Lower Rhætic.	8. 9. 10.	} Black, selenitic shales	5	4
	11.	Firm, black, thinly-laminated shales	1	2
	12.	Black, earthy shales	0	6
	13. 14.	} Alternating layers of sandstone and shale, the former with small quartz-pebbles .... }	0	8
	15.	Sandstone (Bone-Bed), coarse and calcareous, with masses of 'Tea-green Marl'	0	4
Upper Keuper.	I.	'Tea-green Marls,' with hard bed of marlstone.	10	3
	II.	Red Marls, angular fracture	56	0

3. 'Notes on the Lowest Beds of the Lower Lias at Sedbury Cliff.' By Arthur Vaughan, Esq., B.A., B.Sc., F.G.S.

The author examined this section in company with Mr. Richardson. The two chief points of interest are, the relation of the basal conglomerate to the Cotham Marble and White Lias of neighbouring districts, and the examination of the faunal sequence, with a view of testing the absolute value of ammonite-zones. The conglomerate resembles the so-called 'False Cotham,' and both may be explained by the breaking-up of Cotham Marble, in one case at intervals during a continuous phase of deposition, in the other after the phase of deposition which produced it had entirely ceased at that place. The break thus represented in the Sedbury area may be considered to correspond roughly with the time of deposition of the White Lias in the areas to the south and east. The succession of events appears to require a tilting, the axis of rotation being a nearly east-and-west line a little south of Sedbury, with gradual and uniformly-increasing depression towards the south, followed by a period of horizontal equilibrium. On the other hand, the succeeding *Psilonotus*, *Angulatus*, and *Arietes*-zones indicate a gradually increasing depression towards the north with a change of axis.

A 'range-graph' is given, showing the times of appearance and disappearance, the abundance or rarity, of several fossils within and below the zone of *Ammonites psilonotus*; and on account of the beginning of five forms at a given horizon and the disappearance of several forms immediately below it, this level is chosen as the base of the zone of *A. psilonotus*, rather than the point of appearance of *A. planorbis* 4 feet higher up. It is hoped that the construction of 'range-graphs' will be of use in testing the value of a series of ammonite-ages as divisions of relative time.

Fig. 3.
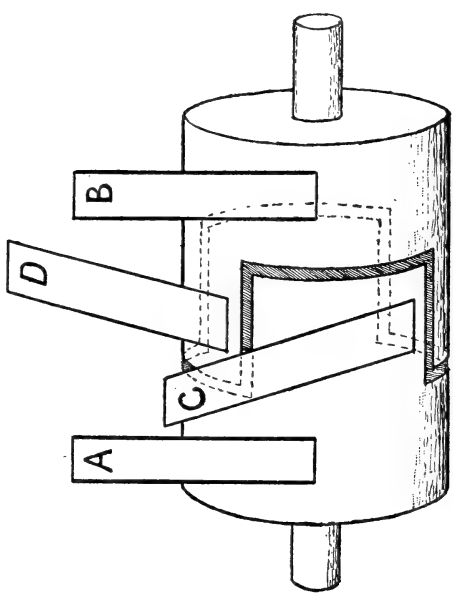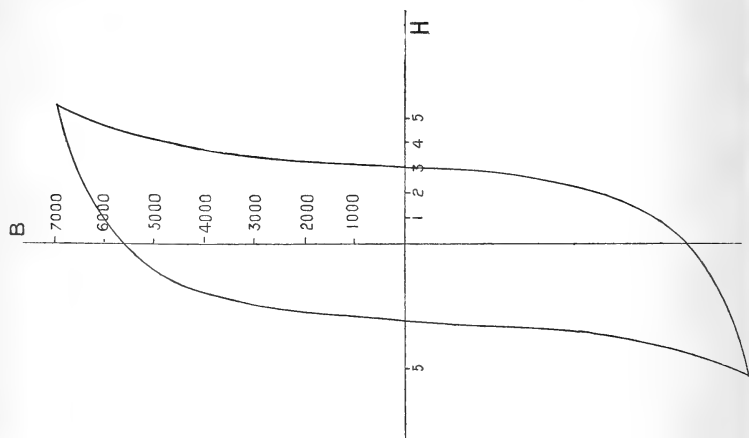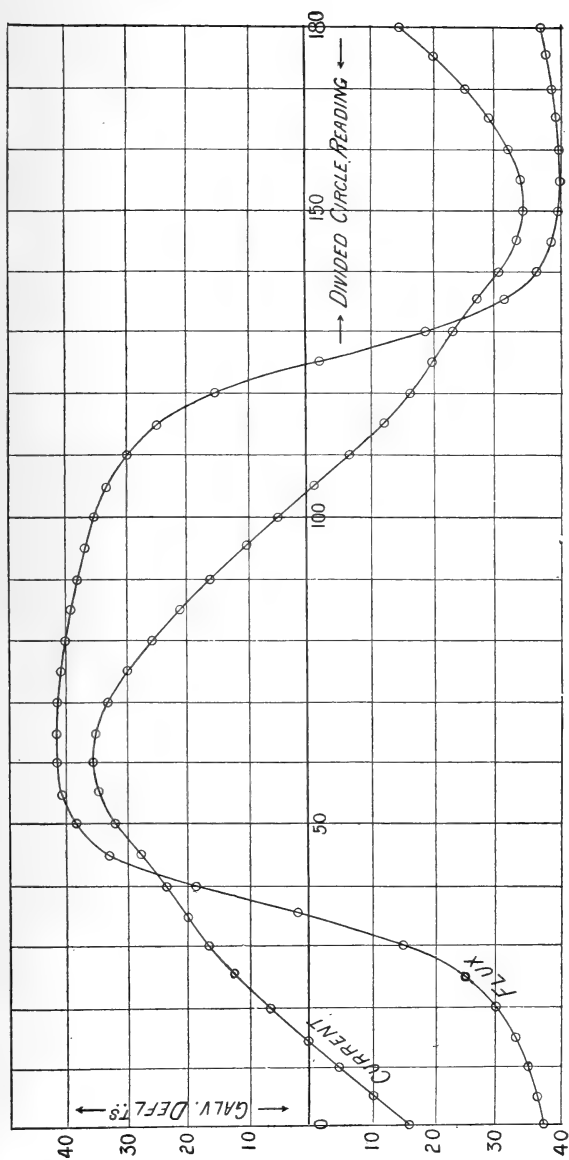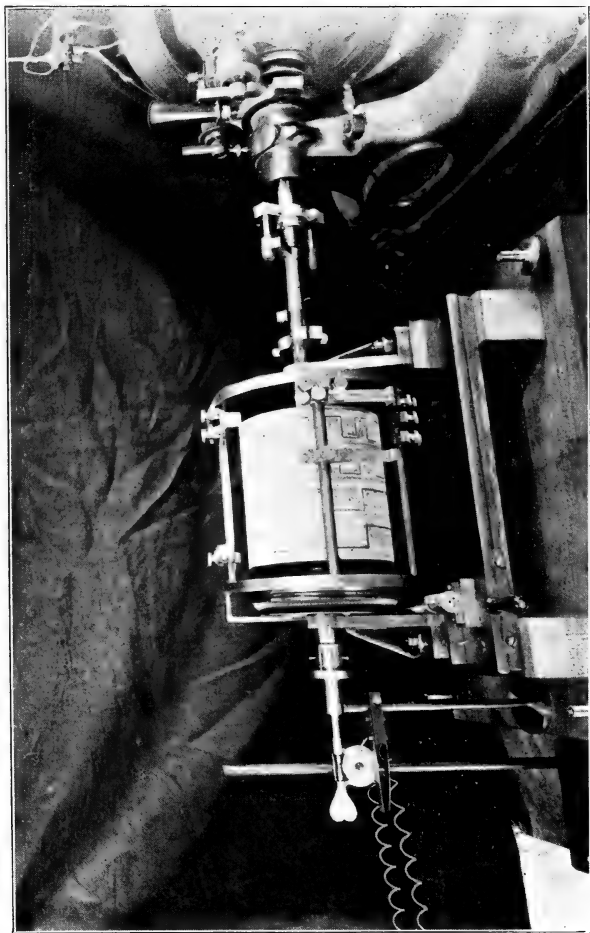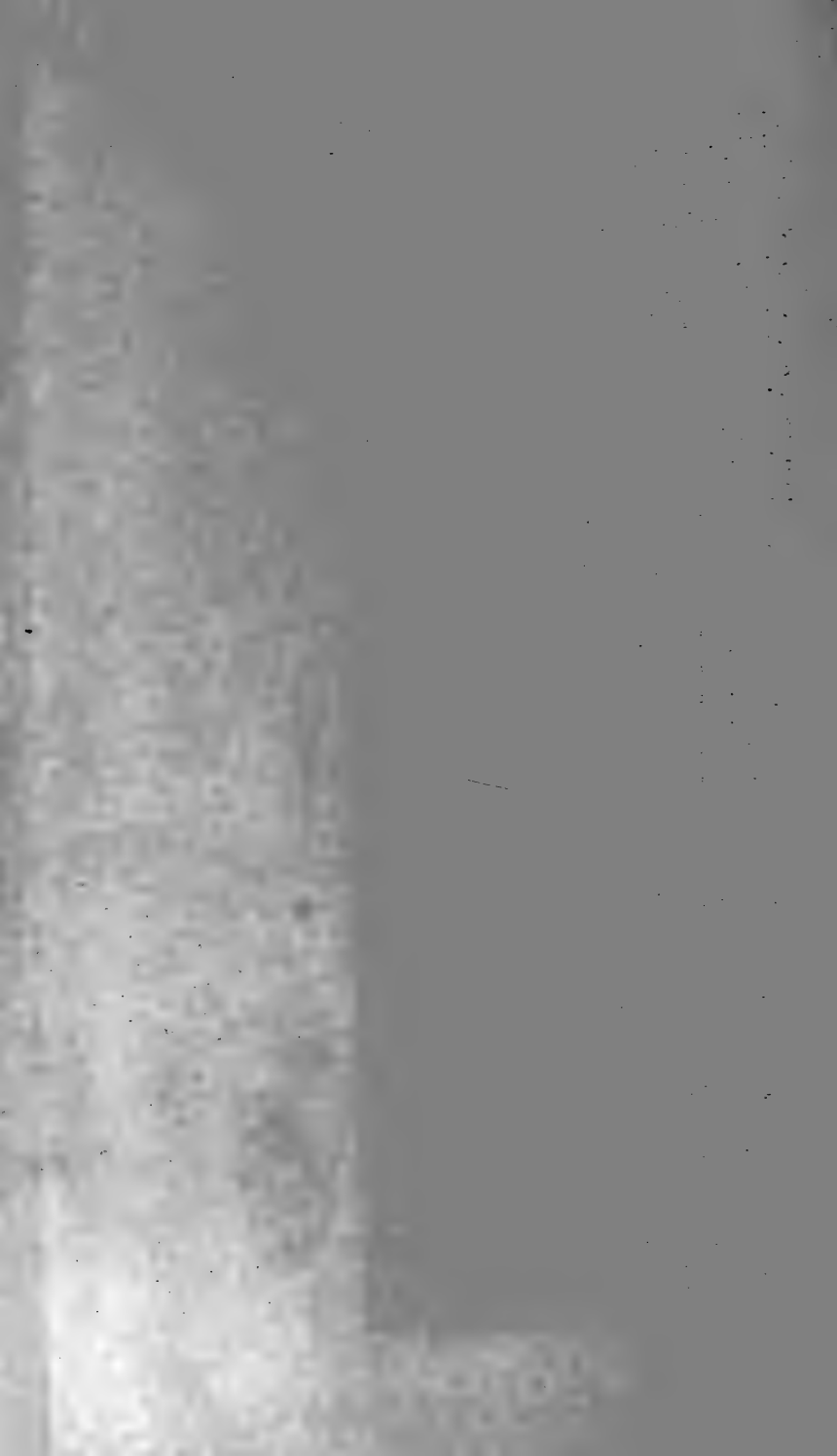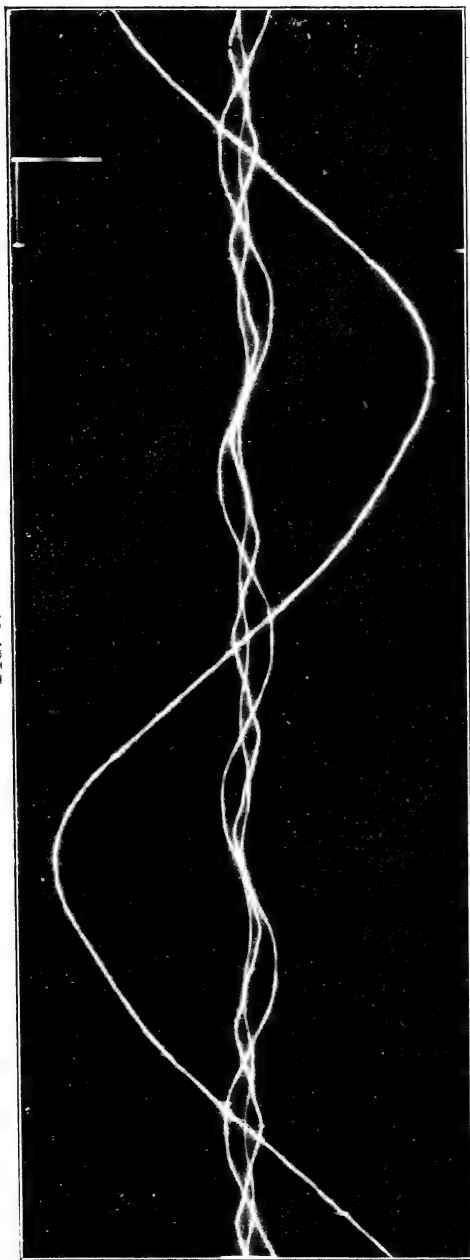
Fig. 1.

Fig. 2.

Fig. 4.

Fig. 5.

Fig. 6.
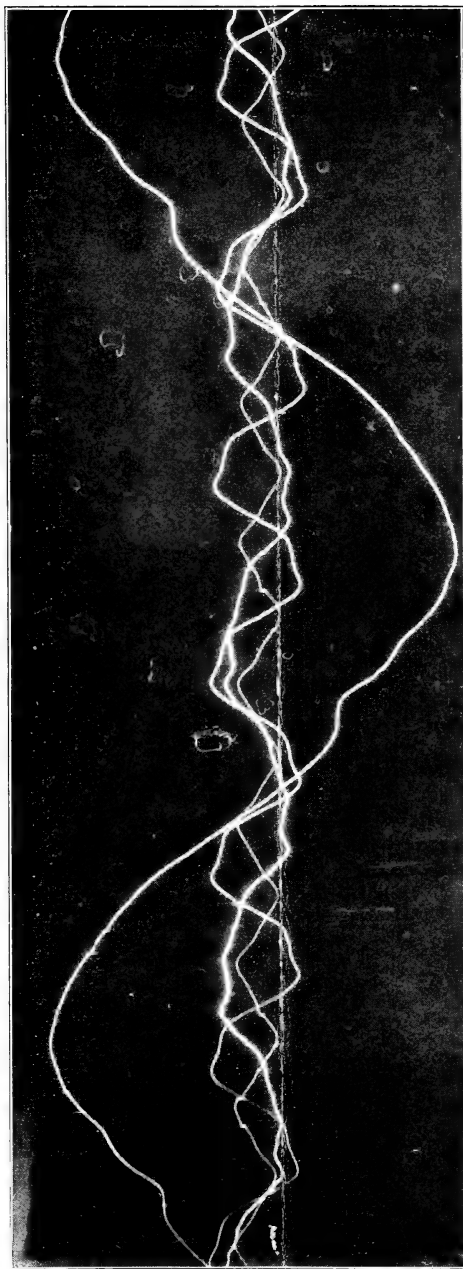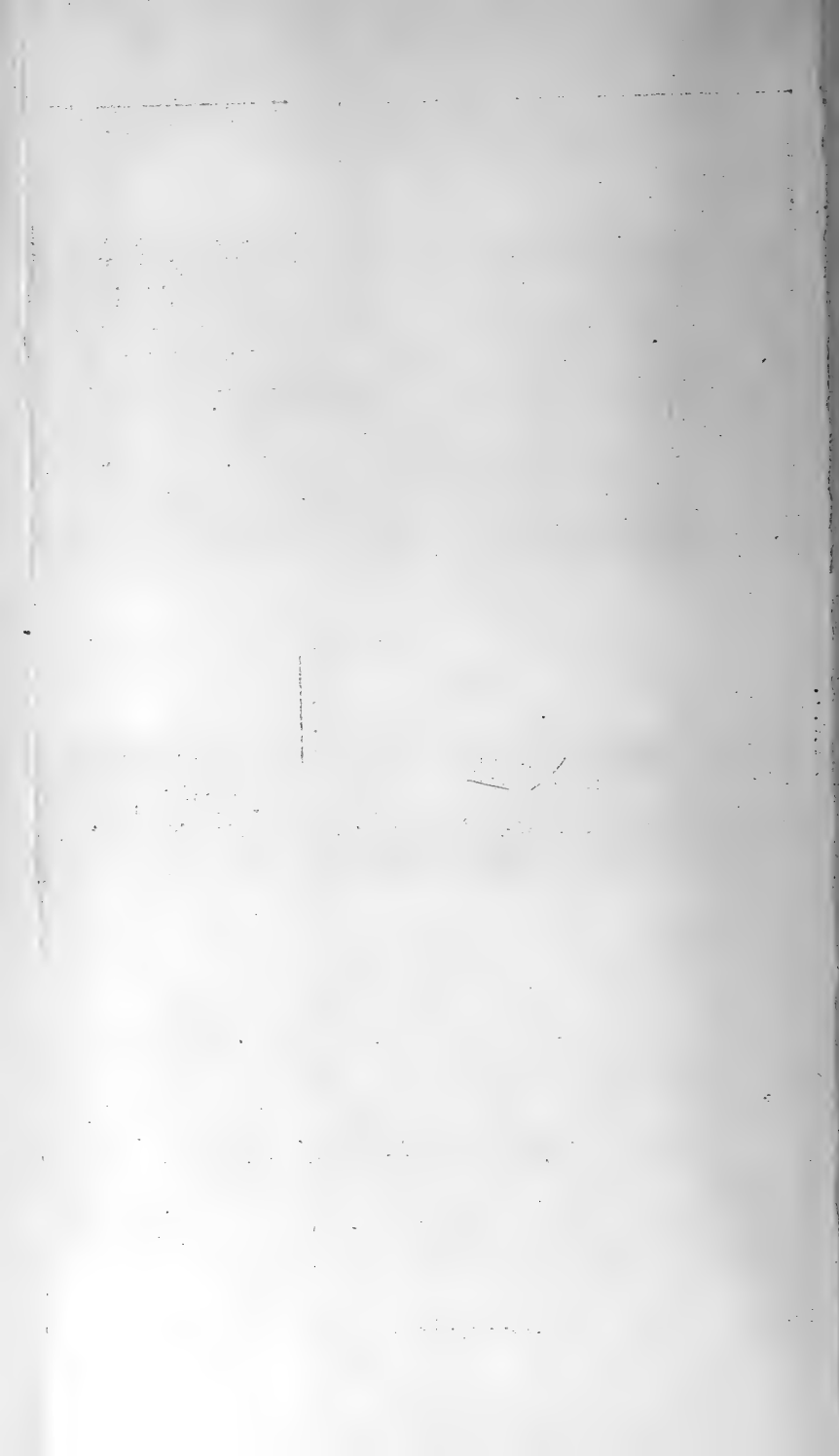
FIG. 4.
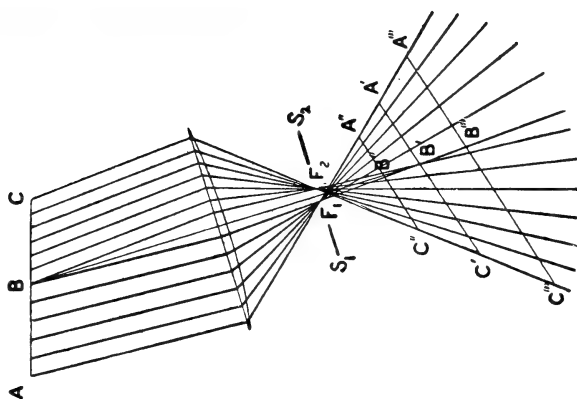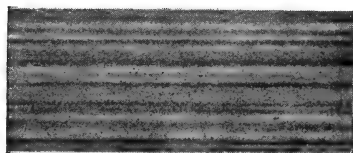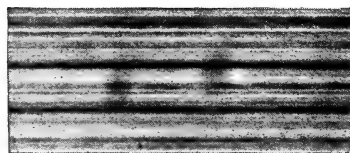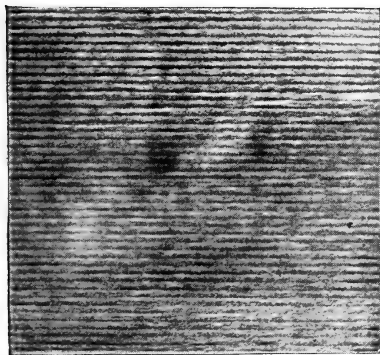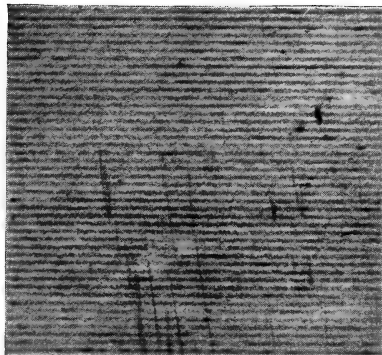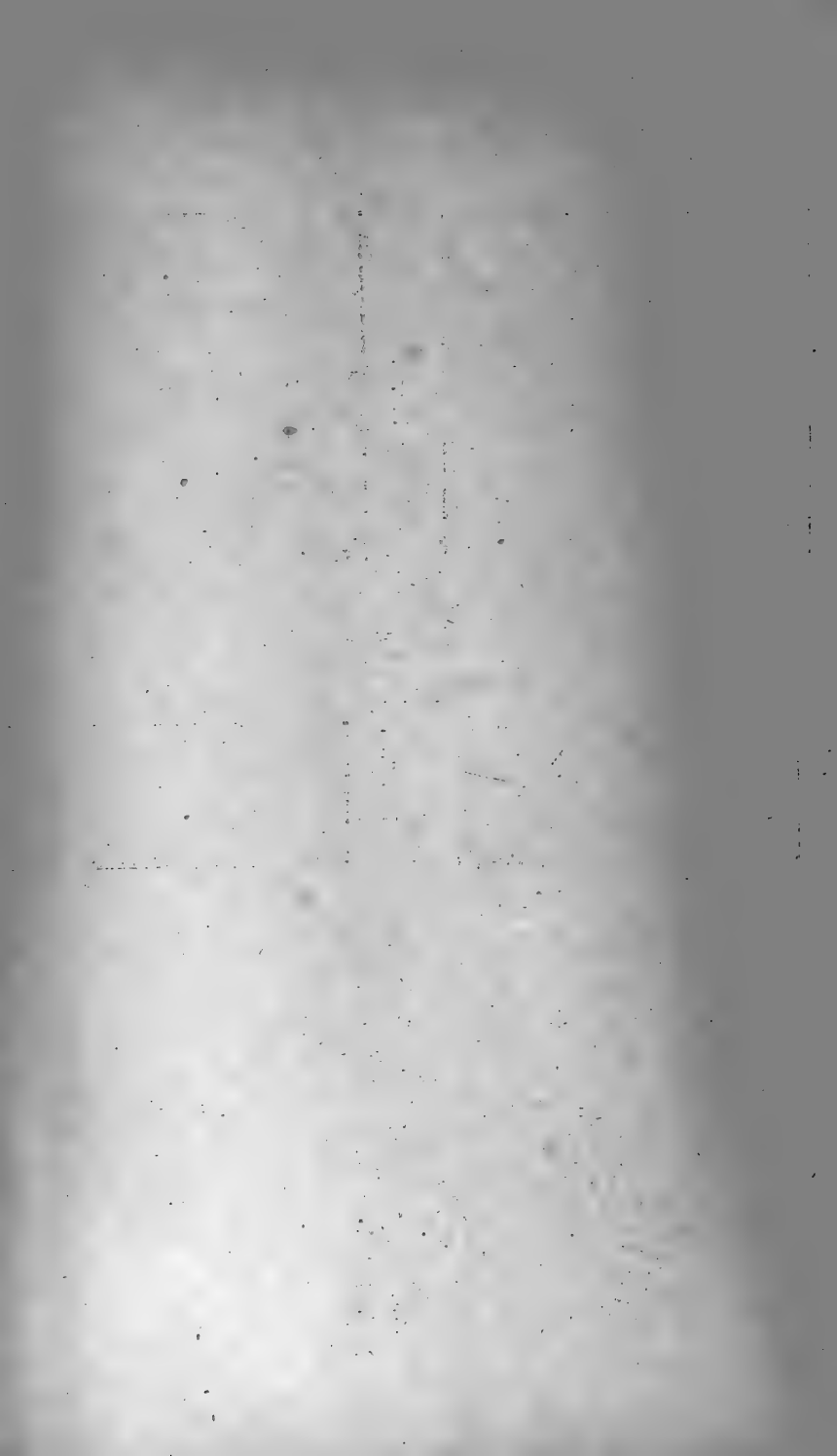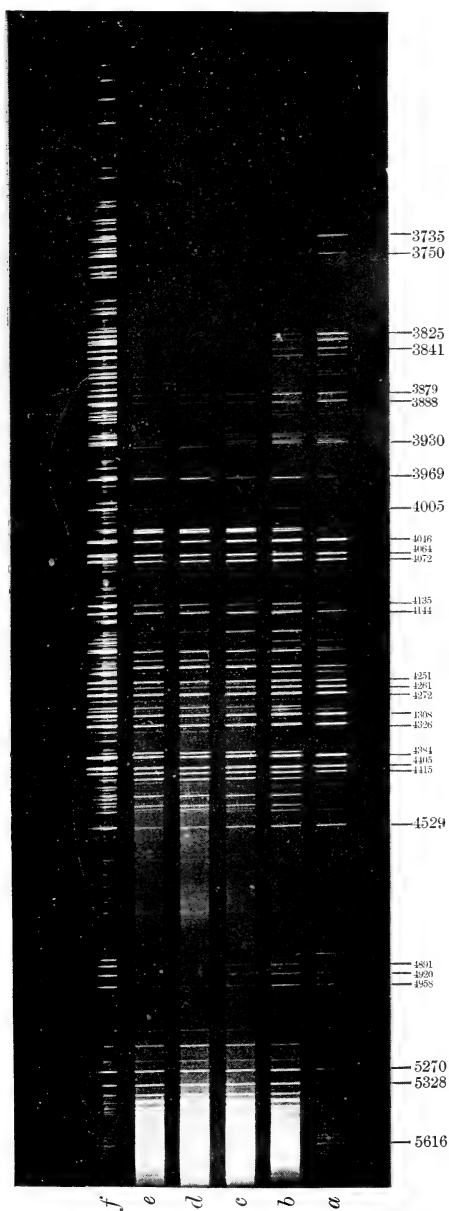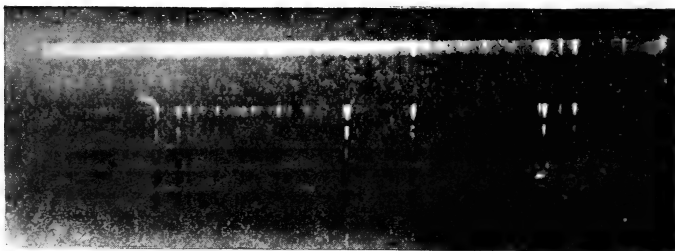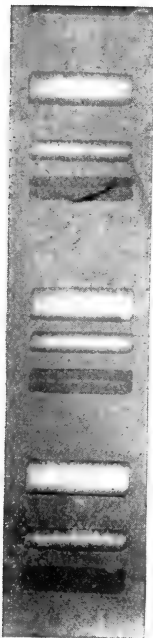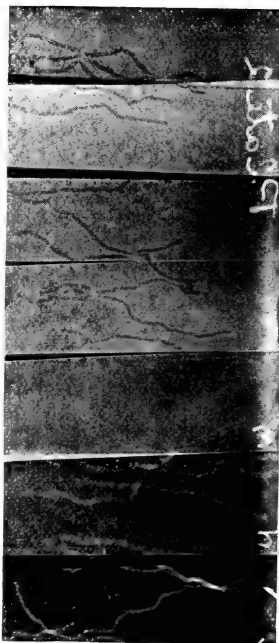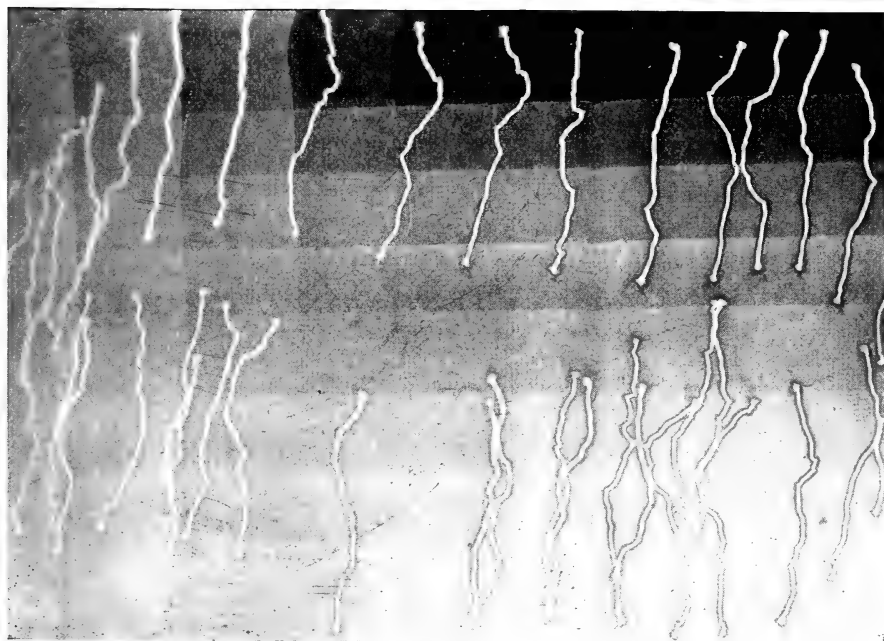
FIG. 5.

FIG. 2.

FIG. 3.

FIG. 1.

	—3735
	—3750
	—3825
	—3841
	—3879
	—3888
	—3930
	—3969
	—4005
	—4016
	—4064
	—4072
	—4135
	—4144
	—4251
	—4261
	—4272
	—4308
	—4326
	—4384
	—4405
	—4415
	—4529
	—4891
	—4920
	—4958
	—5270
	—5328
	—5616

f e d c b a

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

LXXX. *On the Mutual Solubility of Liquids.*—II.
By J. P. KUENEN [*].

THIS paper forms the continuation of a research the first part of which was published some years ago [†]. The investigation has been continued principally with further mixtures of hydrocarbons and alcohols. The advantages of this combination of substances are, (1) that the critical points of both components are not too high, and the observations may thus be carried up to the critical region ; and (2) that, as both components belong to series of substances many properties of which progress regularly as we pass from term to term, we may expect some similar progression to exist with respect to phenomena of mutual solubility.

In the former paper a definite answer was given for the first time to the question as to the effect of pressure on the miscibility of liquids. The smallness of this effect makes it advisable to test it where it is bound to be greatest, viz. in the neighbourhood of a critical point for the liquids. With the mixtures of ethane and ethyl-, propyl-, isopropyl-, and butyl-alcohol a lower critical temperature was discovered, *i. e.* a temperature below which the substances mix in all proportions and above which two liquid layers are possible ; in this case pressure was found to have the effect of making the liquids mix, when separated by a rise of temperature above the lower

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critical point, or, in other words, the critical temperature was raised by pressure. It is probably not wholly superfluous in this connexion to emphasize the fact that what is usually called critical temperature or mixing temperature for the liquids is in reality this temperature under the pressure of the saturated vapour, *i. e.* in the presence of the vapour phase ; at other pressures the critical temperature as well as the composition and the density of the critical phase must be different.

In the investigation of the influence of pressure in cases where an upper critical temperature occurs, such as that of hexane and methyl-alcohol, we were anticipated by van der Lee *, who investigated mixtures of phenol and water ; he found that with these mixtures, just above the temperature at which the two liquid layers mix, increase of pressure over and above the vapour-pressure brings the two layers back, *i. e.* the critical temperature was again raised by pressure.

As was pointed out in our previous paper †, mixtures often possess a maximum vapour-pressure (minimum boiling-point) beyond the critical point of the liquid layers. Phenol and water have this property, as shown by their boiling-point, and van der Lee, as well as Lehfeldt ‡, confirmed this by measurements of the vapour-pressures ; but it was afterwards found by Schreinemakers § that in this case the maximum already exists below the critical point amongst those mixtures which contain more water than those which take part in the three-phase equilibrium. We are not aware of other experimental contributions to the subject having appeared since our former paper.

a. *Ethane and Methyl-alcohol.*

First of all, the investigation of mixtures of ethane and methyl-alcohol had to be completed ; this combination had been found to differ from those with the higher alcohols in that no lower critical point for the liquids exists, and this difference involves the further one, that the character of one branch of the critical curve in the pressure-temperature diagram remained uncertain, and had to be separately investigated. Referring again to the mixtures with the higher

* Van der Lee, *Dissertation Amst.* 1898 ; *Zeitsch. Physik. Chemie,* vol. xxxiii. p. 622.

† Kuenen & Robson, *l. c.* p. 184. For a full discussion of this important point see van der Lee's paper, *l. c.*, and a paper by the present writer in the Proceedings of the *Kon. Ak. van Wetensch. Amsterdam,* October 1903.

‡ Lehfeldt, Phil. Mag. (5) vol. xlvii. p. 289.

§ Schreinemakers, *Zeitsch. Physik. Chemie,* vol. xxxv. p. 461.

alcohols and the p–t diagram illustrating these cases *, it will be seen that there the branch of the critical curve which has its origin at the critical point of the alcohol ends in the lower critical point of the liquids which these mixtures possess. This same branch is bound to have a different character for the mixtures with methyl-alcohol, owing to the non-existence of a similar lower critical point.

The experiments were conducted in the following manner : the compression-tube was charged with some methyl-alcohol and vapour-pressures and critical point were determined. Increasing quantities of ethane were then introduced into the tube, and each time similar measurements made ; or the tube was first filled with ethane, a certain quantity of methyl-alcohol was added, readings were taken, and small quantities of alcohol let out at the bottom, each time a set of determinations being made. The number of experiments required for an exhaustive survey of the phenomena is naturally very large, but fortunately at the lower temperatures it appeared to be possible to use one and the same mixture for the determination of a whole set of critical points, a fact which was already mentioned in our previous paper † ; properly speaking, one mixture can have one critical point only, but the composition of the critical phase in the liquid state varies so slowly with change of temperature, that a given mixture shows the critical phenomena at a whole range of temperatures, not always at the middle of the column, but at the lower temperatures in the upper half, at the higher temperatures in the lower half of the tube ; for further explanation we refer to former publications on the critical phenomena of mixtures.

We confine ourselves to the publication of the critical data, the results relating to the saturation-curves having no particular importance. Our general experience with regard to the critical mixing-points of liquids is that a great consistency in the numerical results is not easily obtained ; slight admixtures of air or other impurities which are not wholly to be excluded and naturally vary in amount appear to have a considerable influence on the critical pressure at a given temperature, or *vice versa*, much more so than in measurements of the critical point between vapour and liquid in ordinary mixtures. With other fillings than those to which the following results refer equally smooth curves of the same general character were obtained, but the pressures at the lower temperatures were sometimes as much as five atmospheres higher than those in the Table I. For publication we

* Kuenen & Robson, *l. c.* p. 199, fig. 8.
† Kuenen & Robson, *l. c.* p. 199.

selected the set which had been carried out with the mixtures in whose purity we had the greatest confidence.

TABLE I.—Ethane and Methyl-alcohol.

Critical points.

Temp.	Press.	Temp.	Press.	Temp.	Press.
241·2	80·0	78·4	135·0	18·6	114·7
241·1	80·45	74·6	133·0	13·2	118·0
240·0	81·5	70·8	131·5	12·4	118·5
219·0	100·0	66·4	129·5	— ·6	144
216·1	101·5	62·3	127·0	1·5	150
160·8	136·5	57·0	124·5	2·5	155
156·5	138·0	51·9	121·5	2·6	152
154·2	138·7	45·8	118·5	3·6	156·5
151·5	139·7	41·7	116·5	4·4	161·5
131·9	143·5	36·0	114·5	5·3	168
128·5	144·0	33·8	114·7	6·3	175
126·0	144·5	26·4	113·0		

	Temp.	Press.
Ethane	32·16	48·9
Vapour and upper ⎱ liquid............... ⎰	35·37	52·0

Fig. 1 shows the critical curve as plotted from Table I. Starting from C_2, the critical point of methyl-alcohol, the curve runs at first in a perfectly normal manner; it ascends, passes through a maximum near 120° C., and then falls as if tending towards the critical point of ethane, C_1; but, instead of completing its course in that manner, the curve passes through a minimum pressure between 25° and 30°, and then begins to rise rapidly; as far as the experiments could be carried, *i. e.* up to 275 atmospheres, the curve continues on its course upwards without further change. It is interesting to compare the completed diagram with the corresponding diagram for the higher alcohols [*].

Further light is thrown on the complicated phenomena which these mixtures display by a consideration of the saturation-curves in the volume-composition (v–x) diagram (temperature constant) and the plaits on the van der Waals' ψ surface, of which these saturation-curves form the projection [†]. Let us follow the development of these curves as the temperature is gradually lowered. The saturation-curve

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[*] Kuenen & Robson, *l. c.* p. 199, fig. 8.
[†] *Iid. l, c.* pp. 194–197, figs. 6, 7.

which appears on the alcohol side of the diagram at the moment when the temperature passes the critical temperature of methyl-alcohol gradually expands towards the ethane side;
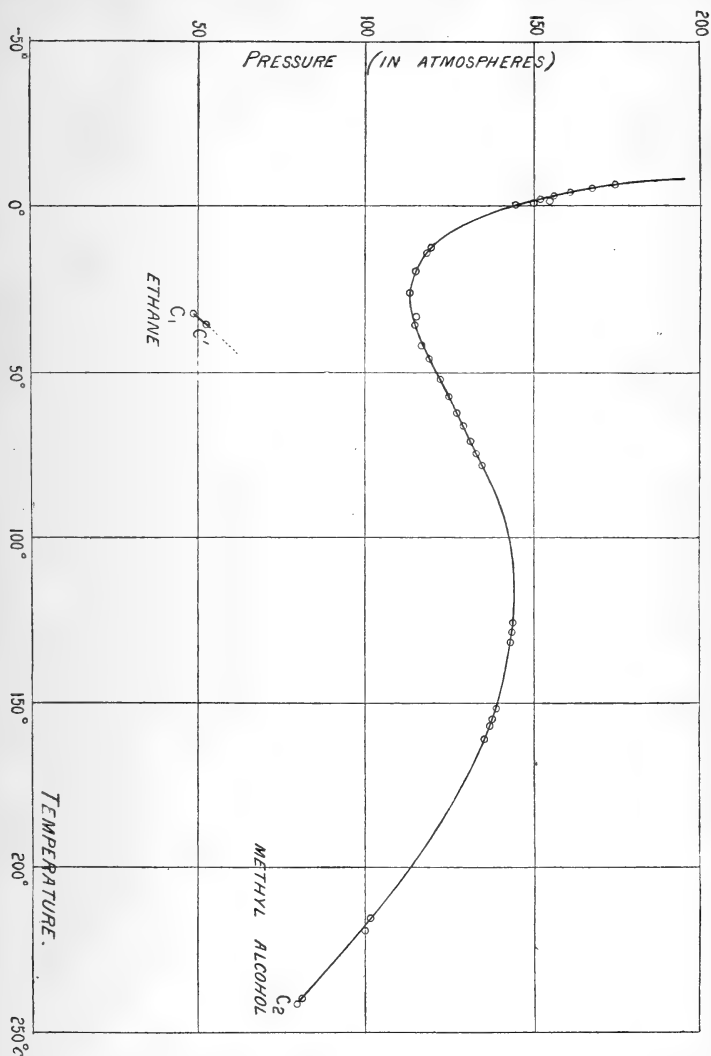
Fig. 1.—Critical curves for mixtures of Ethane and Methyl-alcohol.

its critical point, however, instead of approaching the critical point of ethane, turns away towards the *x*-axis, and so the saturation-curve gradually assumes the character of a two-liquid curve; at a certain moment a small subsidiary curve,

develops out of the main curve near the critical point of ethane; at this moment a new critical point appears (C' in the $p-t$ figure), the top layer in the tube divides into a vapour and a liquid and the three-phase equilibrium, represented in the $v-x$ diagram by a triangle, begins. The critical point of this curve approaches and finally coincides with the critical point of ethane, while the three-phase triangle continues down to the lowest temperature which was tried, viz. $-78°$ C.

With ethyl- and the other alcohols the original critical point remains near its normal position on the ethane side of the saturation-curve; after the subsidiary curve has made its appearance the two phases which lie on either side of the first critical point begin to approach each other until they coincide; at this moment the three-phase triangle and the original critical point disappear, *i. e.* the lower critical point (see above) is reached. The second critical point continues its course until it reaches the ethane axis at the critical point of this substance; below this temperature nothing remains but the ordinary vapour-liquid saturation-curve.

It follows immediately from the above results that methyl-alcohol and ethane mix completely by pressure, but at the lower temperatures the pressures required become enormous.

It is possible up to a certain point to produce the saturation-curves in the $v-x$ diagram through points of intersection with other saturation-curves into those regions where the metastable and unstable conditions are represented, and it follows that the critical curves may also be considered beyond points of intersection with three-phase curves. For a general discussion of this question and of the special features of those curves in the cases before us, we must refer to van der Waals' researches in the second volume of his treatise on the continuity of the fluid states of matter *, and to a paper by the present writer containing a criticism of some of van der Waals' views †.

In all probability mixtures like those of ether and water, sulphur dioxide and water, and others would be found to behave in a manner analogous to ethane and methyl-alcohol if it were possible to investigate them in the critical region, but the action of water on glass at high temperatures has prevented this so far.

b. *Propane and Methyl-alcohol.*

The propane was prepared from propyl iodide and alcohol by means of the zinc-copper couple; it was purified with

bromine and strong potash, collected over mercury in a Bunsen gasometer and dried over phosphorus pentoxide before being admitted into the experimental tubes. It was found to be but moderately pure, a considerable pressure being required for complete liquefaction at some distance below the critical temperature. Attempts at further purification by liquefaction and partial distillation had not the desired effect. Some of the data obtained for the purest sample are contained in the following Table II.

TABLE II.—Propane.

Temp.	p_a.	p_b.
90·35	37·4	38·8
95·1	40·9	41·9
97·0 *	42·90	critical point

Although not so pure as ethane, the propane was still sufficiently pure for an investigation of the mixtures with methyl-alcohol.

On compressing propane in the presence of methyl-alcohol a second liquid layer was formed ; the three-phase pressure appeared to differ very little from the vapour-pressure of propane, and the impurities referred to above prevented a decision of the question whether the pressures were lower or higher than for the pure hydrocarbon. Mixtures of methyl-alcohol with ethane have lower pressures than ethane, mixtures with pentane and hexane have higher pressures than the hydrocarbons themselves (see below). But the question is capable of being answered by indirect evidence. It is well known that if the three-phase pressure is above the pressures of both components, the composition of the vapour is intermediate between those of the two liquid layers ; if, on the contrary, the three-phase pressure is intermediate between the pressures of the components, the vapour composition lies outside the liquid compositions, *i. e.* the vapour contains more of the volatile component than either of the liquids. These rules need not hold if a maximum vapour-pressure exists between the three-phase pressure and either of the components, as is actually the case with phenol and water ; but we will assume that such a point does not exist for propane and methyl-alcohol.

* Olszewski (*Bulletin St. Pétersbourg,* 1889, p. xxvii) found 97° and 44 atmospheres.

The two types are represented in the v–x diagrams figs. 2 and 3 ; *a* and *b* indicate the liquids and *c* the vapour. Let us now consider the behaviour of the mixtures under isothermal compression and expansion ; suppose that the

Fig. 2.
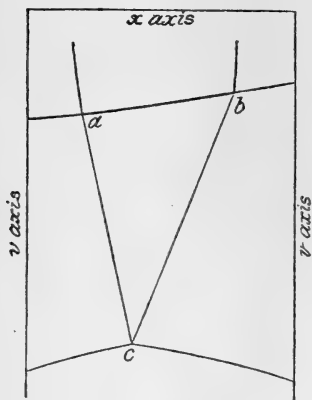
Fig. 3.
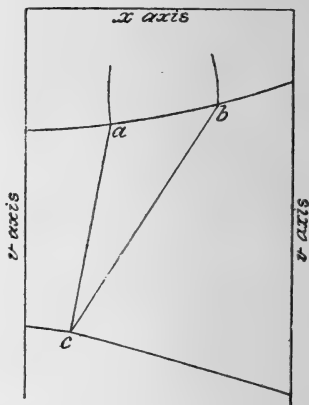
liquid *a* is present in the tube with the vapour *c*, and let the volume be changed ; this change may bring about the appearance of the second liquid *b*, and the diagrams show under what conditions. If the mixture is of the first type (fig. 2), the second liquid can only appear on compression, not on expansion ; if the mixture belongs to the second type (fig. 3), it can only appear on expansion.

Now, with a mixture containing a comparatively small amount of methyl-alcohol, it was observed that by compression the heavier liquid *b* (methyl-alcohol containing propane) separated out in small drops from the lighter liquid *a* (propane containing methyl alcohol) ; it follows that these mixtures belong to the first type, and that the three-phase pressure must be larger than the vapour-pressure of propane.

On heating, an upper critical temperature of the liquids was attained, where the two layers become identical; the temperature lies at $21°\cdot15$; at this point the three-phase equilibrium comes to an end. So far these mixtures resemble mixtures of phenol and water, and also, as we shall see, those of pentane and hexane with methyl-alcohol, but the influence of pressure is different. The approach of the two liquids and their meeting in a critical point may be brought about in two perfectly distinct ways, viz. either by the withdrawal of the saturation-curve for the liquids inside the vapour-liquid curve

(fig. 4), or by a separation of the liquid-curve from the vapour-liquid curve on the outside (fig. 5). In the former
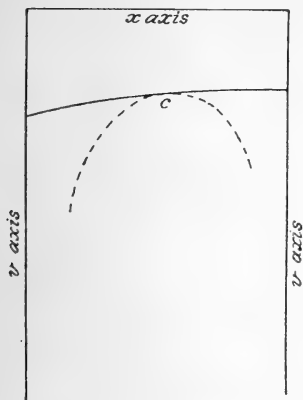
Fig. 4.   Fig. 5.
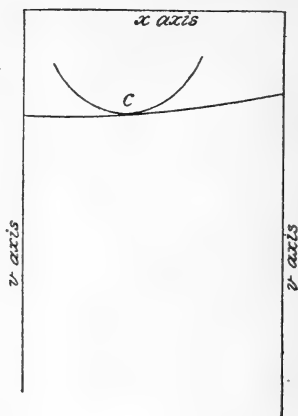
case the two branches of the liquid-curve converge towards each other on the side of the x-axis and meet in a critical point C, which disappears into the vapour-liquid curve at the critical mixing temperature of the liquid layers, and before this temperature is reached pressure will obviously make the liquids approach the critical point and ultimately mix. In the latter case the branches of the liquid-curve diverge, at least to begin with, and the curve need not be closed or have a critical point towards the small volumes; but at the moment when it separates from the main curve a critical point C, which was before hidden inside the vapour-liquid curve, makes its appearance; this critical point is thus turned towards the vapour-liquid curve, *i. e.* it lies on the side of the large volumes. Above the critical temperature, after the two curves have separated, pressure will drive the mixture into the liquid-curve, *i. e.* bring back the two layers. Now phenol and water, as shown by van der Lee *, belong to the latter type, but propane and methyl-alcohol were found to conform to the former. The effect of pressure on the critical point of the liquid phases below $21°·15$ is shown in the following Table III. The pressure considered as a function of the temperature rises much more slowly at first than later.

* Van der Lee, *Dissertation*, and *Zeitsch. Physik. Chemie*, xxxiii. p. 626.

TABLE III.—Propane and Methyl-alcohol.

Critical points of liquids.

Temp.	Press.	Temp.	Press.
21·15	10	18·6	55
20·85	11	18·05	70
20·8	13	17·85	79
20 05	23	17·8	82
19·85	26	17·65	85
19·4	34·5	17·5	93
19·2	39	17·4	95
19·0	46	17·2	(100)

Comparing this case with that of ethane we find the chief difference to be, that there the critical point between vapour and upper layer was reached before the liquid-curve had contracted into the vapour-liquid curve, here on the other hand the liquid-curve disappears far below the critical temperature of the top layers, otherwise the two cases seem very similar in character.

It was mentioned above that a consistent application of van der Waals' theory enables us to consider the character of the saturation-curves or branches of these in the non-stable conditions, *i. e.* inside other curves. This may be illustrated by some observations made with the mixtures under consideration. The liquid-curve disappears from the stable part of the diagram at 21°·15, but as explained it may still be considered to exist inside the vapour-liquid curve (fig. 4). With some difficulty I succeeded in showing this view to be legitimate. At a temperature of 21°·4, *i. e.* just above the critical point of the liquids, the mixture was completely liquefied and then made to expand very slowly. It is well known that in those circumstances, by what is called thermodynamical retardation, the vapour does not always appear when the saturation-pressure is reached, but that the mixture often remains homogeneous in metastable equilibrium. The same phenomenon took place in this instance, but it was observed that under those circumstances the liquid became and remained turbid, indicating the formation of a second liquid layer ; by a shock or by further expansion the vapour was suddenly formed, the mixture stirred up and the turbidity disappeared. The phenomenon can be explained by assuming that in the metastable condition the completely metastable liquid-curve had been reached and two metastable liquids formed, entirely in accordance with the nature of the diagram.

It follows from the fact that the three-phase pressure is

higher than the vapour-pressures of the neighbouring mixtures, that above the critical point for the two liquids there should be a maximum vapour-pressure ; this conclusion has not yet been put to the test.

c. *Butane.*

Partly with a view to the investigation of mixtures and partly for the sake of studying the properties of this substance itself, a quantity of butane was prepared by electrolysis of sodium propionate. Before we were quite satisfied with the purity of the substance and when we had only made a few preliminary experiments, the whole stock was by an accident lost during the night. We hope to resume the investigation of this substance at an early date.

d. *Pentane and Methyl-alcohol.*

The pentane used was obtained from Kahlbaum and it was considered unnecessary for the purposes of this investigation to attempt further purification.

Two layers were formed with an upper critical point at $19°\cdot4$ C. This temperature was found to rise under the influence of additional pressure (Table IV.), in other words the

TABLE IV.—Pentane and Methyl-alcohol.
Critical points of liquids.

Temp.	Press.	Temp.	Press.
19·4	0	20·25	40
19·75	15·5	21·8	91
20	(27)	22·0	100

two-liquid curve separates from the main curve as was explained above (fig. 5). The question remains, whether the liquid-curve is closed on the side of the small volumes or not. If so, the then completely closed curve must contract on heating and at last disappear ; in that case we have to expect a maximum in the critical temperature and therefore a critical curve in the p–t diagram, which is convex towards the t-axis. The figures obtained do not support this hypothesis, but the observations are not sufficiently accurate to judge of the exact relationship. Measurements of vapour-pressures or compositions have not been made yet.

e. *Hexane and Methyl-alcohol.*

The hexane was used as it was obtained from Kahlbaum. The phenomena in this case were similar to those observed with pentane. The existence of an upper critical temperature was already known *. The following Table V. shows the effect of pressure on the critical point of the liquids. There is a linear relation between temperature and pressure, as far as the accuracy of the experiments allows one to judge.

TABLE V.—Hexane and Methyl-alcohol.
Critical points of liquids.

Temp.	Press.	Temp.	Press.
37·0	0	41·4	143
37·9	33	42·4	175
38·6	55	42·6	182
39·4	81	43·8	228
40·25	105	44·8	264
41 3	141		

Above 37° C. there is a maximum vapour-pressure, as shown by the boiling-point. The mixtures also have the property of a minimum in the (ordinary) critical temperature (Table VI.); for a full discussion of the connexion between these two phenomena, viz. maximum vapour-pressure and minimum critical temperature, we refer to former papers †.

TABLE VI.—Hexane and Methyl-alcohol.
Critical points.

	Temp.	Press.
Hexane (S. Young) ...	234·8	29·6
Methyl-alcohol.........	241·2	80 0
Mixture	210·2	56·0

f. *Hexane and Ethyl-alcohol.*

An upper critical point was found at −65° C., below which two layers are possible. This point is thus very much lower than for methyl-alcohol. The influence of the alcohol on the critical point was further investigated with carbon disulphide.

* Rothmund, *Zeitsch. Physik. Chemie*, xxvi. p. 455. He found the critical point to lie at 42°·6 C., but the hexane used by him was probably purer than mine.

† Kuenen, Phil. Mag. (5) vol. xl. p. 173, vol. xliv. p. 174.

g. *Carbon disulphide and Lower Alcohols.*

The following table shows a gradual change of the critical temperature for the liquids, of the same nature as with hexane.

TABLE VII.—Carbon disulphide and Alcohol.
Critical temperatures of liquids.

Carbon disulphide and methyl-alcohol ...	+40·5 (Rothmund).
,, ,, ,, ethyl ,,	−10·6 (Guthrie −14·4 [*]).
,, ,, ,, propyl ,,	−52°
,, ,, ,, butyl ,,	−80°

h. *Triethylamine and Water.*

As discovered by Guthrie [†], triethylamine and water have a lower critical point for the liquid layers; the mixture is homogeneous at ordinary temperatures and on being warmed to 18° C. or 19° C. suddenly splits into two layers, the lower one of which is chiefly water and the upper principally triethylamine. In some unaccountable manner Guthrie obtained a completely impossible saturation-curve $(t-x)$ for these mixtures, but Rothmund [‡] repeating the experiments found a normal curve, as required by the phenomena. It seemed worth while testing the effect of pressure on the miscibility in this case. Some triethylamine was prepared from the chlorhydrate (Kahlbaum) and powdered potash, and repeatedly washed with water. The critical point was found to be at 18°·3 C. In the pressure apparatus the following readings were taken :—

TABLE VIII.—Triethylamine and Water.
Critical points of liquids.

Temp.	Press.	Temp.	Press.
18·3	0	20·0	79
18·45	9·5	21·2	140
19·63	63	21·3	144
19·95	78	21·3	146

[*] Guthrie, Phil. Mag. (5) vol. xviii. p. 504.
[†] Guthrie, Phil. Mag. (5) vol. xviii. pp. 28, 59.
[‡] Rothmund, *Zeitsch. Physik. Chemie*, vol. xxxvi. pp. 459–461.

Pressure therefore raises the critical point, and after separation of the liquids by warming, they can be made to mix by applying pressure. It follows that the liquid saturation-curve is closed on the side of the small volumes, as in fig. 4, and has a critical point on that side ; at $18°·3$ it emerges out of the vapour-liquid saturation-curve, below that temperature it is completely hidden in the non-stable part of the diagram.

In this case the two-liquid curve has the additional peculiarity of being very broad and flat, the coexisting phases dissolving each other to a very small degree only * ; the three-phase pressure may therefore be expected to be but little smaller than the sum of the vapour-pressures of the components taken separately. In order to test this conclusion the three-phase pressure was measured at $92°·9$ C. and found to be $142·6$ cms. mercury ; the pressure of triethylamine is not accurately known and I had not got a sufficient quantity of the substance left to measure it, but the boiling-point of triethylamine is about $89°$ C., and from this we may infer that at $92°·9$ C. the vapour-pressure must be about 86 cms., while the pressure of water-vapour at that temperature amounts to $58·6$ cms. Hence the sum of the pressures comes out at $144·6$ cms. as compared with $142·6$ cms. for the mixtures ; the difference as expected is very slight.

i. *Diethylamine and Water.*

Several vain attempts were made to confirm Guthrie's observation † of a lower critical temperature for these mixtures at about $120°$ C. The diethylamine which was used in the experiments had been kept for a considerable time, probably some years, and I have not been in a position to repeat the experiments with a fresh sample, but I know no reason why this fact should have completely prevented the phenomenon from showing itself. It must be added that the description by Guthrie of the behaviour of the mixture seems to make the whole phenomenon doubtful; Guthrie's words are the following : " It is to be noted that, owing in part to the refractive index of diethylamine being nearer that of water than is the refractive index of triethylamine, the above separation is not marked by the milkiness which characterizes the separation of the latter body. If it were not for accidental dust particles floating on the surface of the lower liquid, it would be, in some cases, almost impossible to distinguish the two layers." Now it seems impossible for a homogeneous liquid to separate into two layers without

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* Rothmund, *l. c.* p. 461.
† Guthrie, *l. c.* p. 500.

showing a transient turbidity preceding the settling out of the liquids, and in view of the values, given by Guthrie himself, of the refractive indices his explanation of the non-appearance of the milkiness is obviously insufficient. A more probable explanation would be the comparatively small difference in density and composition between the two phases, as shown by the narrow pointed shape of the solubility-curve $(t-x)$. Still some doubt remains, and what Guthrie saw may have been due to unequal heating; if the liquid had been hotter than the space above, distillation would have set in and recondensation, with the result of a temporary second layer settling down on top of the original liquid with a very faint line of demarcation.

General Conclusions.

It was first noticed by Lehfeldt [*] that of all partially miscible pairs of liquids at least one component is an abnormal substance ; no exceptions to this rule have yet been found. We find, therefore, that those molecular conditions—probably to be represented by a high molecular attraction—which in a liquid by itself produce certain deviations from the laws of corresponding states, characteristic of abnormal liquids, at the same time seem to make it impossible for that liquid to dissolve more than a limited amount of a second substance without entering upon an unstable condition. A molecular explanation of this fact cannot yet be given, except in so far as we can suppose the foreign molecules to be squeezed out by the high attraction of the others.

The observations recorded under e, f, and g show that, as we ascend the alcohol series, the temperature below which two layers are formed becomes lower : this is in accordance with the theory that the abnormality of the alcohols plays a part in the phenomenon of partial miscibility, for we know that the higher the alcohol the less abnormal it is.

The experiments show the influence of the hydrocarbon to be more complicated ; no doubt the phenomena depend to a large extent on the gradual rise of the vapour-pressure and the approach of the critical point of the hydrocarbon as we come down the series.

As regards the influence of pressure on the critical point of the liquid layers, this we have found to be sometimes in the direction of increasing and sometimes of diminishing the mutual solubility ; we know from thermodynamics that this difference is associated with a difference in the volume-change

* Lehfeldt, Phil. Mag. (5) vol. xlvii. p. 285.

which occurs on the liquids being mixed *, but an explanation by molecular considerations cannot be given until the phenomenon of partial miscibility itself has been more fully explained.

Assuming that the abnormality diminishes as the temperature rises, it looks at first sight as if the influence of heating ought always to be in the direction of increasing the mutual solubility, as it actually is in by far the greatest number of cases; but there are exceptions such as with water and triethylamine. Here it is that the general point of view, taken by van der Waals in his theory, allows us to bring apparent exceptions better into line with the other cases. The appearance or disappearance of the two layers does not depend on the liquid phases only, but also on the vapour in equilibrium with the liquids; if on cooling a mixture of water and triethylamine the layers mix at 18°·3 C., this does not mean that the liquids are now soluble at all densities, but it is due to the fact that the liquid which is in equilibrium with the vapour at this moment begins to be denser than the liquids of the two-liquid equilibrium; if at temperatures below 18°·3 the pressure were lowered below the vapour-pressure, it would undoubtedly be found that in the metastable region the two layers made their appearance, in the same way as we proved to be the case with mixtures of propane and methyl-alcohol under similar conditions. At higher temperatures the liquid saturation-curve will shrink and disappear in this case as in all the others. From the point of view of the theory the existence of a liquid-curve is the principal matter; and the appearance or disappearance of the two layers under the pressure of the saturated vapour is of secondary importance.

In conclusion let us classify the various cases which have been discovered with respect to the effect both of temperature and pressure, excepting those like ethane and methyl-alcohol, where the critical temperature of the upper liquid and the vapour is reached before that of the two liquids. *A priori* we find the following four combinations to be possible :—

1. Upper critical point, pressure mixes liquids. Example: propane and methyl-alcohol.
2. Upper critical point, pressure separates liquids. Example : phenol and water, hexane or pentane with methyl-alcohol, &c.
3. Lower critical point, pressure mixes liquids. Example : ethane and ethyl-, propyl-, &c. alcohol, triethylamine and water.
4. Lower critical point, pressure separates liquids.

* This question we hope to discuss in a subsequent paper.

A representative of the last class has not been found so far, and very probably will never be found; if below a lower critical point additional pressure made the homogeneous liquid split into two layers, the two saturation-curves would lie outside each other and a rise of temperature towards the critical point would make them approach each other; but this is in contradiction with the general rule * that the saturation-curves in the v–x diagram contract on heating.

University College,
    Dundee.

LXXXI. *A Method of establishing the Principles of the Calculus of Quaternions. By* CHARLES JASPER JOLY, *M.A., D.Sc., Fellow of Trinity College, Dublin, and Royal Astronomer of Ireland* †.

I DO not remember having seen the following method of establishing the principles of the calculus of quaternions. If it is new it may be of interest, as it is simple and it clearly exhibits the effect of the associative property—the property which sharply separates quaternions from other systems of geometrical analysis.

Defining $S\alpha\beta$ to be $-ab\cos\theta$ where a and b are the lengths of the vectors α and β in terms of some assumed unit, and where θ is the angle between the vectors; defining also $V\alpha\beta$ as a third vector at right angles to both and of length $ab\sin\theta$ in terms of the assumed unit; it is easy to see that $S\alpha\beta$ and $V\alpha\beta$ are both distributive with respect to the vectors. I define the product of α and β to be a linear function of $S\alpha\beta$ and $V\alpha\beta$, or

$$\alpha\beta = mS\alpha\beta + nV\dot{\alpha}\beta, \quad \ldots \quad \ldots \quad (A)$$

where m and n are certain constant numbers, characteristic of the calculus and independent of the vectors α and β. This insures the distributive property of a product of vectors.

Again, I impose the condition that the associative law is obeyed, or that $\alpha . \alpha\beta = \alpha^2 . \beta$. If the product of a vector into the sum of a scalar and a vector is distributive, we have

$$\alpha . \alpha\beta = \alpha m S\alpha\beta + \alpha . nV\alpha\beta. \quad \ldots \quad \ldots \quad (B)$$

Now $\alpha . nV\alpha\beta = n^2 V\alpha V\alpha\beta$ by (A), and by the same equation

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* Van der Waals, *Die Continuität*, vol. ii. p. 101, *sq.*
    † Communicated by the Author.

$\alpha^2 = m \mathrm{S}\alpha^2$. Thus the associative law gives

$$\alpha^2 \cdot \beta = m \mathrm{S}\alpha^2 \cdot \beta = \alpha m \mathrm{S}\alpha\beta + n^2 \mathrm{V}\alpha\mathrm{V}\alpha\beta \cdot = \alpha \cdot \alpha\beta. \quad \text{(C)}$$
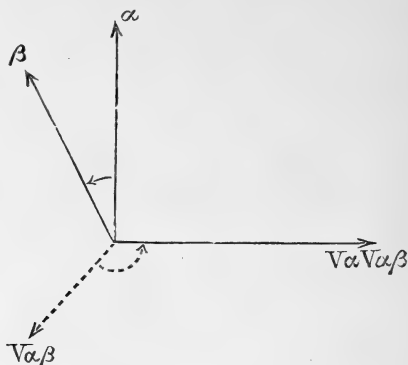
It is geometrically evident that the vector $\mathrm{V}\alpha\mathrm{V}\alpha\beta$ is coplanar with α and β and at right angles to α, while it makes the angle $\theta + \dfrac{\pi}{2}$ with β. Its length is equal to $a^2 b \sin \theta$. Resolving β along this vector and α, we have by equation (C),

$$m \mathrm{S}\alpha^2 \cdot b \cos\left(\theta + \frac{\pi}{2}\right) = n^2 a^2 b \sin \theta,$$

or simply

$$m = + n^2. \quad \text{(D)}$$

Thus m is a *positive* quantity unless we suppose n to be imaginary, and we may without loss of generality take it to be equal to unity. The number n is then ± 1, and taking $n = + 1$, we have the quaternion relation

$$\alpha\beta = \mathrm{S}\alpha\beta + \mathrm{V}\alpha\beta. \quad \text{(E)}$$

Hence follows in particular the fundamental formula

$$i^2 = j^2 = k^2 = ijk = -1.$$

The foregoing analysis establishes the proposition that *quaternions is the only system of vector analysis which is at once distributive and associative in multiplication of vectors* when we avoid introducing $\sqrt{-1}$ into a product of real vectors. I think that originators of new methods should consider very carefully whether any advantage they may gain outweighs the disadvantage of depriving their calculus of the simplicity of either of these properties.

LXXXII. *The Relation between the Rate of Recombination of Ions in Air and the Temperature of the Air.* By R. K. McClung, *M.A.* (*McGill*), *B.A.* (*Cantab.*), *Trinity College, Cambridge,* 1851 *Exhibition Scholar**.

IN a previous paper (Phil. Mag. March 1902) I gave the results of an investigation of the relation between the pressure of the air and the rate of recombination of the ions in the air which had been ionized by Röntgen rays. The experiments described in that paper were performed at the ordinary temperature of the room. It was thought that it would be of interest to pursue this investigation further, and ascertain what effect a change of temperature of the air would have upon the rate at which the ions recombine.

Prof. Rutherford investigated experimentally the rate of recombination of ions in gases at the ordinary temperature and pressure of the surrounding air (Phil. Mag. Nov. 1897), and he verified the equation

$$\frac{dn}{dt} = -\alpha n^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which represents the law of recombination of the gaseous ions.

It was shown by the author, in the paper referred to above, that over a considerable range of pressures, both above and below atmospheric pressure, the same law holds true. The present series of experiments were therefore undertaken with the object of ascertaining, in the first place, whether this same law of recombination holds at various temperatures of the air, and, secondly, what effect a change of temperature of the gas has upon the value of the coefficient of recombination α.

The method employed in the previous investigation was adopted in the present instance. It was necessary, however, to modify the apparatus somewhat in certain details, as in this case it was to be subjected to fairly high temperatures in the course of the experiments. The following apparatus was therefore employed.

Description of Apparatus.

The general arrangement of the apparatus is shown in fig. 1. The Röntgen-ray bulb and the induction-coil which run it were, as usual, inclosed in a lead-covered box as a shield, and the rays were allowed to emerge through a

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* Communicated by Prof. J. J. Thomson.

well-defined circular opening, A, in the lead.   The cylinder
in which the air was ionized is shown more in detail in

Fig. 1.
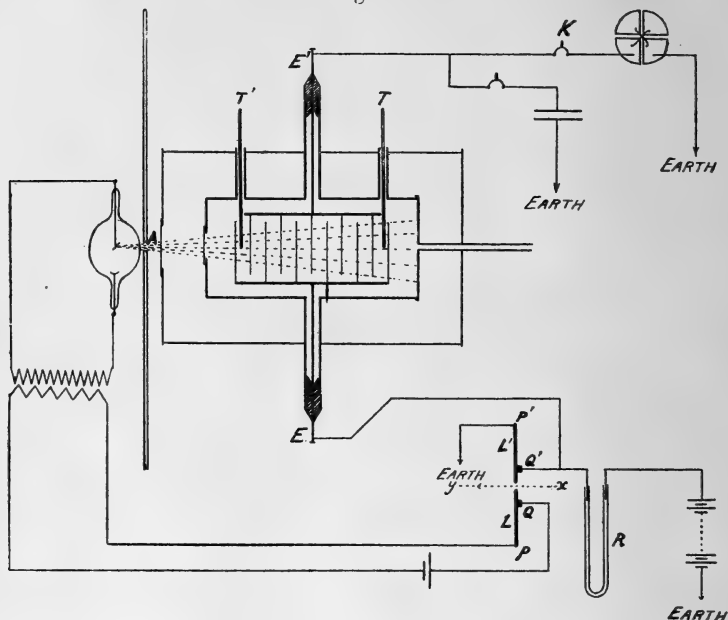
fig. 2.   XY was a circular brass cylinder about 34 cms.
long and 14 cms. diameter.   The end Y was closed by a
brass cap which screwed on to the end of the cylinder,
while at the other end was a thin aluminium window, B,
through which the rays were allowed to pass into the
cylinder.

The arrangement of electrodes inside the cylinder was
similar to that used in the previous experiments, for the
same reasons as were explained in that instance, but the
method of supports and insulation had to be modified some-
what on account of the higher temperatures which were to be
used.   The electrodes consisted of extremely thin aluminium
foil which was stretched on brass annular rings for support.
These rings were 9·1 cms. inside and 10 cms. outside
diameter, and were grooved round the outside edge, and the
aluminium foil was folded over this edge and then bound
tightly round with thin copper wire.   These rings were
eleven in number, and were placed about 2 cms. apart, so
that there was a length of about 20 cms. of air on which the
measurements were made.   Six of these alternate rings were

fastened to a heavy brass rod, *m*, thus forming one electrode, while the other five were attached to the rod *m′*, forming the

Fig. 2.
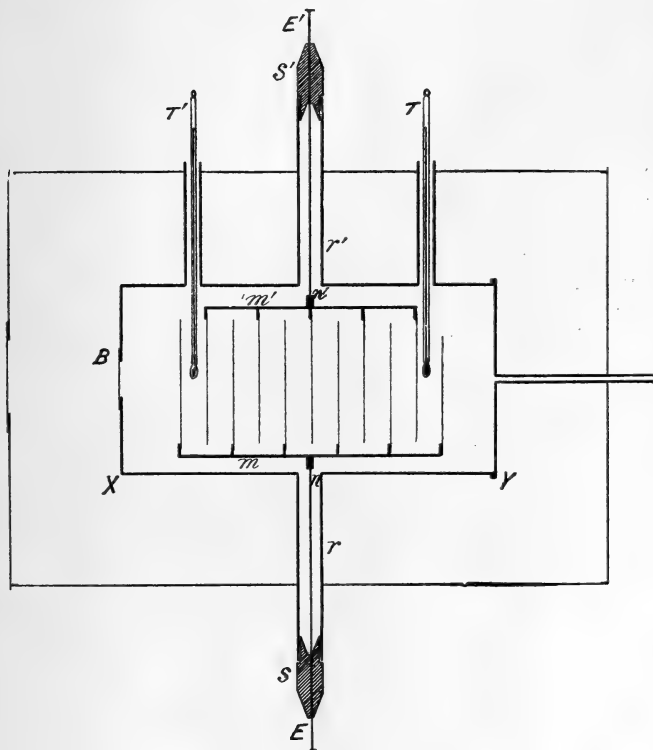
second electrode. These rods were supported by the rods *n* and *n′*, which passed out through the brass tubes *r* and *r′*, and were supported by the insulating ebonite plugs S and S′.

The brass cylinder was surrounded by a sheet-iron cylinder, separated from it by a uniform air-space of about 10 cms. The ends of this iron cylinder were also closed, as shown in the diagram. The cylinder and inclosed air were heated by means of a long bunsen-burner which was placed underneath the iron cylinder, and ran nearly the full length of the cylinder. The iron cylinder was covered with sheet asbestos to prevent as far as possible the heat from radiating. The air between the two cylinders became heated and formed a hot air-jacket round the brass cylinder, and the air inside it could thus be kept at a fairly constant temperature by regulating the supply of gas to the bunsen-burner. The

regulation of the temperature was a comparatively simple matter. The temperature of the air which was being experimented upon in the cylinder was measured by the two mercury thermometers, T and T′, shown in the diagram; and the mean of the temperatures indicated by these two thermometers was taken as the average temperature of the air which was being experimented upon.

The electrode E′ was connected to one pair of quadrants of the electrometer, the other pair being connected to earth. The electrometer used throughout these experiments was one of the Dolazalek type, and gave a deflexion of about 2000 scale-divisions for a difference of potential of one volt between the quadrants when the needle was charged to 120 volts.

The arrangement for making and breaking contacts by means of a pendulum and levers was similar to that used in the previous experiments, and has been fully described in the former paper. In the present instance the contacts were broken by means of a heavy steel-ball pendulum, the ball being suspended by two thin steel wires about three metres long. A pointer on the bottom of the ball struck the ends of the levers as it swung along the path $x\,y$. The levers L and L′ were held in contact with Q and Q′ by means of springs. The electrode E was connected through a large liquid resistance R, consisting of a solution of cadmium iodide in alcohol, to one pole of a battery of accumulators, the other pole of the battery being connected to earth.

At the beginning of the investigation a Röntgen-ray bulb was used of the ordinary form, without any vacuum regulator of any sort attached. This, however, did not prove satisfactory, for it was inclined to run hard with use, and the intensity of the rays was therefore apt to change during the course of a series of experiments, making it difficult to obtain consistent results. This bulb was therefore abandoned, and a focus-tube with an automatic vacuum-regulator attached was obtained. This bulb proved much more satisfactory, and was employed throughout the remainder of the investigation. All the results given in this paper were obtained with this bulb.

Test of the Law of Recombination at Various Temperatures.

When equation (1) is integrated it becomes

$$\frac{1}{n} - \frac{1}{N} = \alpha t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where N is the maximum number of ions per c.c. at the

instant the rays cease, and n is the number per c.c. remaining at the end of a time t after the cessation of the rays, while α is the coefficient of recombination. Now one of the objects of this investigation was to test whether this equation represents the law of recombination at various temperatures of the gas. To test this the three quantities to be measured are n, N, and t.

When the levers L and L' were placed directly opposite each other, as in the diagram (fig. 1), then the pendulum in swinging along the path $x\,y$ struck the two levers at the same instant, and thus the rays were stopped and the electromotive force applied to the electrode E at the same instant. Now if, when the ionization in the cylinder has reached a steady state, the source of ionization be suddenly cut off and a large electromotive force be applied to one of the electrodes, the other electrode will receive a charge proportional to the number of ions existing in the gas between the electrodes. Also if the electromotive force be applied to the electrode at a given interval after the cessation of the rays, the other electrode will receive a charge proportional to the number of ions which are left in the gas.

In the actual experiments, the observations were taken as follows :—When the gas had reached the temperature to be investigated, and the temperature had become steady, the rays were started and allowed to run for an interval of ten or fifteen seconds, so that the ionization might reach a steady state. During this time both the electrodes E and E', as well as the electrometer-quadrants, were connected to earth. At the end of that time the electrode E' was insulated by opening the key K, and the pendulum was released, which broke the contacts, stopping the rays and applying the electromotive force to E at the same instant. In order to avoid the violent kick of the electrometer-needle which would occur, due to the inductive effect when the large E.M.F. was applied to E, if the quadrants of the electrometer were insulated at the same time as E, the electrometer was disconnected by the opening of the key K before the pendulum was released, and was kept to earth. Immediately after the passage of the pendulum the lever L' was replaced in contact with Q', so that the induced charge was counterbalanced by putting E to earth again ; then the quadrants of the electrometer were insulated and connected to E' by closing the key K, and the deflexion of the electrometer-needle observed. This deflexion should therefore be proportional to the maximum number of ions N in the given volume of the gas. This then gave a measure of the quantity N.

The measurements of the different values of n were made in exactly the same way, except that the levers were separated at known intervals from one another along the line $x\,y$. By separating these levers at definite distances from each other, the electromotive force could be applied to E at definitely known times after the rays were stopped. By this means different values of n were obtained corresponding to different known intervals of time t. Values corresponding to N and n were thus obtained. The values of t corresponding to the different values of n were determined by calibrating the pendulum for the different distances between the two levers.

In making these measurements, as is almost always the case in working with Röntgen rays, some difficulty was experienced on account of the tendency of the rays to vary slightly in intensity. Even with an automatic regulating bulb, it is almost impossible to keep the intensity of the rays quite constant during a whole series of observations. In order to obviate this difficulty as much as possible the bulb was run at regular intervals, as far as could be done throughout the series ; and in addition, instead of depending upon a single reading for any one quantity, several readings were taken in each case and the mean of these taken. This system of means was followed throughout the whole investigation.

To test the validity of the law of recombination at the various temperatures, experimental curves were obtained showing the relation between the number of ions n and the time t. These curves were plotted, and then taking two of the experimental points on the curve with which to calculate α from the formula, several arbitrary values for t were chosen, and the corresponding theoretical values for n were calculated from the equation (2). These theoretical points were then plotted and compared with the experimental curve. Experimental curves were obtained for several different temperatures ranging between 15° C. and 300° C. Some of these curves are shown in figures 3, 4, and 5. In these curves the ordinates represent numbers corresponding to the different values of n, while the abscissæ represent the corresponding values of t in seconds. The experimental points in these diagrams are marked with crosses, while the theoretical points are marked with circles. To avoid confusion the experimental curve alone is drawn in each case.

The agreement between the theoretical and the experimental points, as shown by these diagrams, is quite close. Any slight variations which may occur can easily be accounted for by variations in the intensity of the rays, experimental

error, &c. ; for it is difficult to keep all the conditions abso-
lutely steady during a whole series of observations, such as
is represented by any one of these curves, especially at the

Fig. 3.
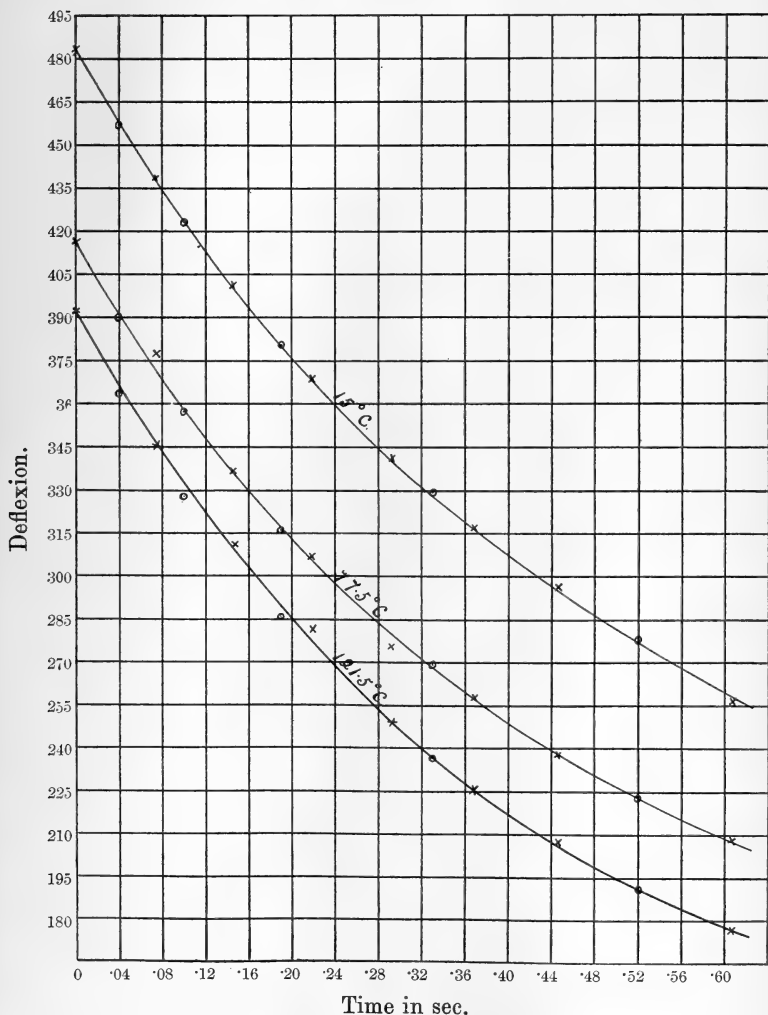
higher temperatures. The agreement is therefore quite close
enough to show that equation (2) represents the law of
recombination at these higher temperatures quite as well as

at the ordinary temperature of the air. It appears then that the rate of recombination follows the same law at different temperatures, at least over the range of temperatures investigated.

Fig. 4.
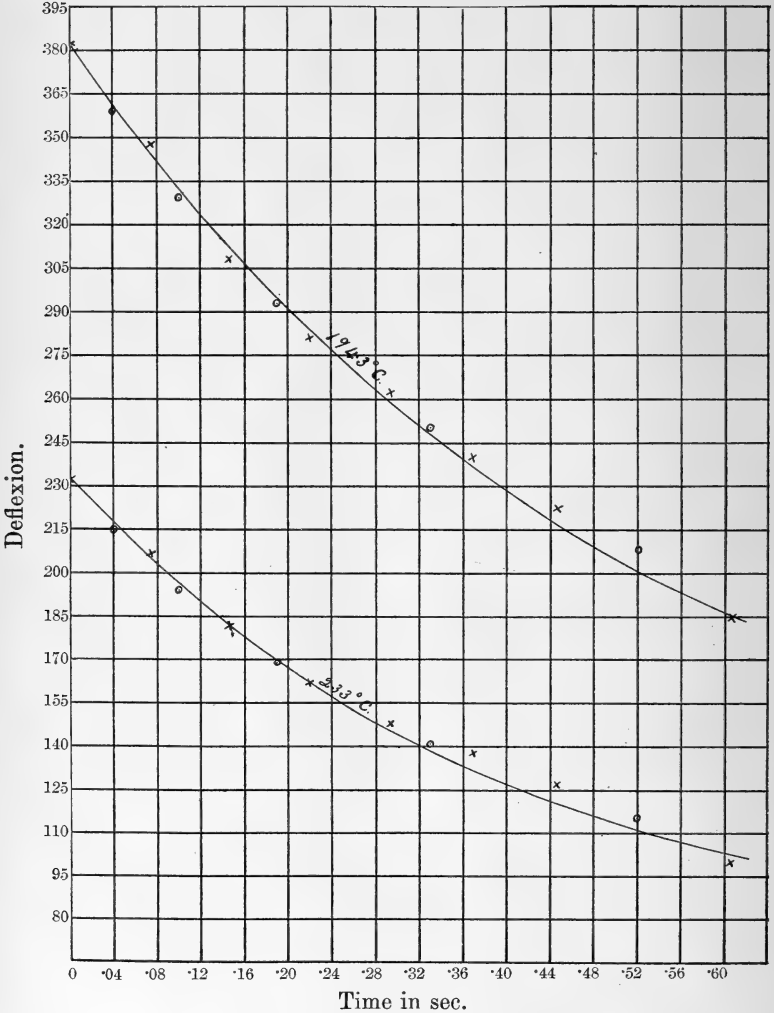
It may be just noted here that the density of the air decreased as the temperature increased, inasmuch as the air was free to expand into the outside air. It was, however, shown by the author in the former paper that within the

range of densities of the air occurring in this instance, this
same law of recombination holds true. Therefore in this
case the validity of the proof of the law at the various

Fig. 5.
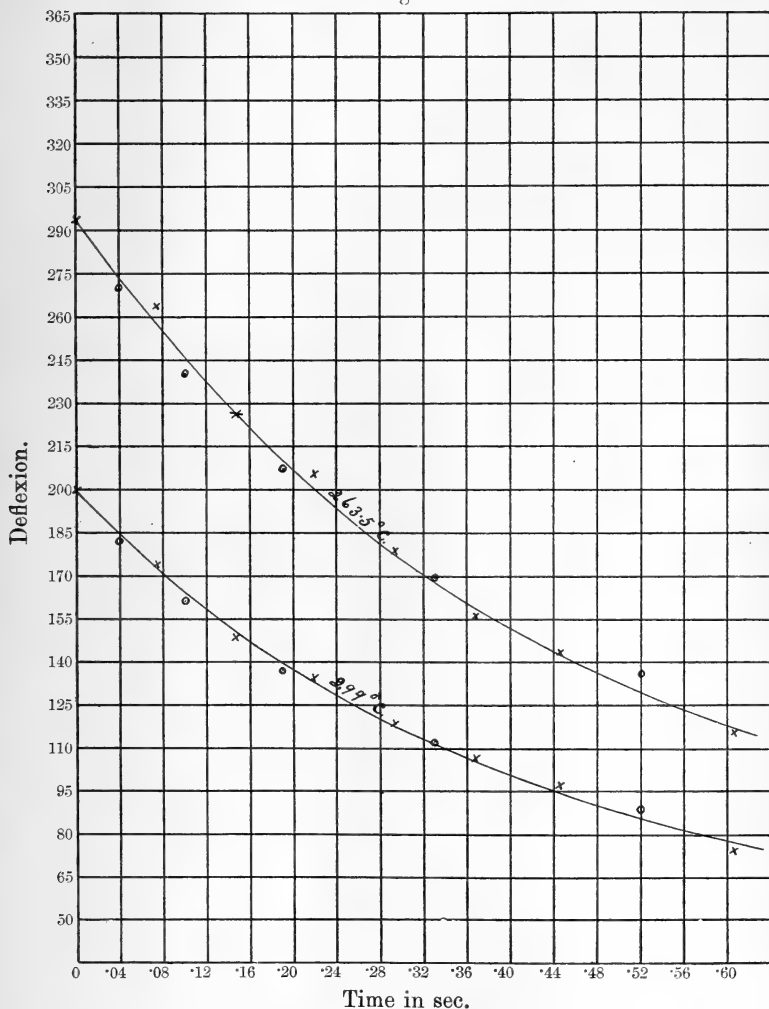
temperatures is not affected by the change of density of the
air. Taking into consideration the present experiments in
connexion with the previous investigation on pressure, this
law of recombination of ions in air appears to be a universal
one, at least within the range of temperatures and pressures
that have been investigated.

Relation between Temperature and Coefficient of Recombination.

The other question which we set out to investigate was the effect which a change of temperature of the gas produced upon the value of the coefficient of recombination α. Although it has been shown that the same law of recombination holds true at the various temperatures, we cannot say from this that the value of the coefficient α remains the same at the different temperatures. There still remains to be settled the question of the relation between the temperature of the gas and the value of α.

From the well-known equation $q = \alpha n^2$ it follows, as has been shown by the author in the paper already referred to, that the absolute value of α is given by the equation

$$\alpha = K \frac{Q}{N^2},$$

where K is a constant quantity depending entirely upon the dimensions and position of the particular apparatus used, and Q is the total number of ions produced per second in a given volume of the gas, while N is the total number existing in the same volume at the instant the rays are shut off. Now if it is only a question of comparing the value of α at one temperature of the air with the value at another temperature, it is only necessary to compare the values of the ratio $\frac{Q}{N^2}$ at the two temperatures, if the dimensions and position of the apparatus are kept unchanged, for K will be the same in both cases. In making this comparison, then, the only two quantities to be measured are Q and N.

N was measured as has been already described in testing the law of recombination. If the electrode E be kept at a steady saturation voltage, and the rate of leak per second between the electrodes be measured in the usual way, then the deflexion of the electrometer-needle, which is proportional to the charge received by the electrode E′, is proportional to the number of ions which are produced in the gas per second. Therefore Q is proportional to the deflexion of the electrometer-needle per second. If d_1 is the deflexion of the electrometer-needle per second corresponding to Q, and if d_2 is the deflexion corresponding to N, then $\frac{d_1}{d_2^2}$ is proportional to $\frac{Q}{N^2}$. Therefore, to compare the values of α at the different temperatures it is only necessary to compare the values of the

ratio $\dfrac{d_1}{d_2{}^2}$ at these temperatures, provided all the conditions are kept constant except of course the temperature.

In comparing these values, then, the observations were taken in the following manner :—The deflexions d_1 and d_2 were measured at the temperature of the room. The air in the cylinder was then heated up to a given steady temperature, and when it had become steady the deflexions d_1 and d_2 were again measured. The air was then heated to a still higher temperature, and when it reached a steady point d_1 and d_2 were measured once more. This was repeated in the same manner for several temperatures. The values of the

Fig. 6.
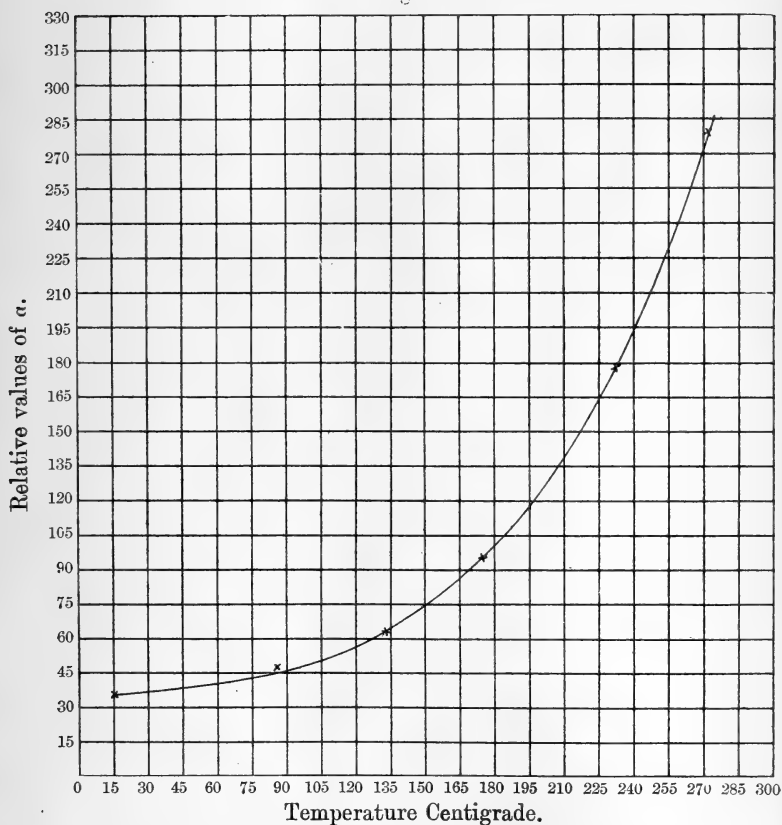
ratio $\dfrac{d_1}{d_2{}^2}$ thus found were then plotted against the corresponding temperatures. A number of these series of observations were made, and one of the curves is given in fig. 6.

In this curve the ordinates represent numbers proportional to the relative values of α, while the abscissæ represent the temperatures of the air in degrees centigrade.

As will be seen from this curve, it was found that as the temperature of the air increased the value of α also increased, at first fairly rapidly, and at the higher temperatures much more rapidly. The rise in temperature of the gas appears to cause the value of the coefficient of recombination to increase to a considerable extent.

The density of the air decreases of course with the rise of temperature, but, as in the previous case, this need not be taken into account; for it was shown in the paper on the relation between pressure and α, that over the range of densities of the air which are involved in this experiment the value of α was practically a constant, so the change of density of the air need not be taken into account in considering the relation between α and the temperature.

This relation between α and the temperature does not appear to be a very simple one, for the curve showing the connexion between them does not seem to be capable of being represented by any simple mathematical formula. Whatever the changes are which take place among the ions, either in the individual ions or in their relation to one another, when the gas is heated they would appear from this to be somewhat complicated. As seen from the curve, however, the relation between the coefficient of recombination and the temperature of the gas seems to be quite a definite one.

Summary.

The results which have been obtained during this investigation may be summed up as follows :—(1) The rate of recombination of ions in air follows the same law, namely, $\frac{dn}{dt} = -\alpha n^2$, at different temperatures, at least over the range of temperatures investigated, that is between 15° C. and 300° C. (2) A rise in the temperature of the air causes a considerable increase in the value of the coefficient of recombination, and the relation between the temperature and this coefficient appears to be of a somewhat complicated nature.

In conclusion I wish to express my appreciation of the kindly interest shown and the advice given by Professor Thomson during this investigation.

Cavendish Laboratory,
Cambridge, July 29, 1903.

LXXXIII. *The Impossibility of Undamped Vibrations in an Unbounded Dielectric.* By Prof. W. McF. ORR, *M.A.*[*]

1. IN his recent important work on "Electric Waves," Macdonald claims that there is an essential difference between a simply-connected and a multiply-connected space in respect of the propagation of electric effects, in that an indefinitely extended space of the latter description possesses modes of free oscillation which do not involve loss of energy by radiation, and are therefore absolutely permanent. As I believe this view to be erroneous, and as such authorities as Larmor[†] and Pocklington[‡] have commented on it without definitely rejecting it, no serious apology seems necessary for endeavouring to point out objections which may be urged against such a conclusion and against the arguments on which it is based.

2. Macdonald discusses in detail [§] those modes of free oscillation of the unlimited doubly-connected region bounded internally by a conductor in the form of an infinitely thin anchorring in which the wave-fronts intersect the surface orthogonally in circles whose planes contain the axis of revolution, and obtains wholly real values for the periods, in agreement with a result previously obtained by Pocklington [||] in much the same way. The fact, however, that these periods are real, in itself proves nothing as to the absence of radiation. If we take a ring whose circular axis is in a fixed position and trace the effect of continually diminishing its thickness on the free vibrations of any of the types considered, making the supposition that the maximum electric force has an assigned value at a given point, as the thickness diminishes indefinitely the energy inside any given closed surface which incloses the ring at a finite distance increases indefinitely, since the normal electric force at the surface contains a term which varies inversely as the thickness. The reality of the free period accordingly implies merely that energy is not being radiated at an infinite rate. It may plausibly be conjectured, indeed, that when the ring is infinitely thin, if the surrounding space be made *simply-connected* by cutting away a portion of the ring, in this case also there will be modes of free vibration having real periods.

3. Macdonald gives an independent proof that there is no

[*] Communicated by the Author.
[†] 'Nature,' Feb. 19, 1903.   [‡] Ibid. Mar. 26, 1903.
[§] 'Electric Waves,' p. 65.
[||] Proc. Camb. Phil. Soc. 1897. A complex value is there obtained for the fundamental period when the thickness is finite ; I make no further reference to this, through inability to understand Art. 4 of the paper. Of course if the correct value is complex, Macdonald's view is wrong, for such a conductor at all events.

damping* by evaluating the radiation across the tubular surface generated by a small sphere of radius ρ whose centre is moved round the circuit. He states that the electric force at any point on this surface consists of a part which varies as $1/\rho$, together with a part which involves ρ as a factor since it must vanish with ρ. One is, however, justified in stating only that the ratio of the second part to the first must vanish with ρ; a similar remark applies to his expression for the magnetic force. The proof is consequently invalid: such a line of argument can in fact legitimately show only that as the wire becomes infinitely thin the rate of radiation becomes indefinitely small compared with the logarithm of the ratio of the wave-length to the thickness of the wire.

4. Furthermore, it may be shown directly that any of the solutions given by Macdonald involves radiation, and that this remains true if we combine two such waves of the same period travelling in opposite directions round the ring. Whether we adopt Poynting's expression for the rate of radiation across a closed surface or Macdonald's modification of it†, the mean rate for a complete period, supposing that there is no damping (energy being supplied to the conductor if necessary)‡, is equal to the mean time value of

$$\frac{1}{4\pi}\int [\alpha(mZ-nY)+\beta(nX-lZ)+\gamma(lY-mX)]dS,$$

where α, β, γ denote the magnetic force, X, Y, Z the electric force, and l, m, n the direction cosines of the outward drawn normal, the integral being taken over the surface.

If we have any system of perfect conductors performing a simple harmonic vibration, regarded as undamped (energy being supplied to the conductors if necessary), and if S be a sphere of radius R, large compared with the dimensions of the conductors, enclosing them all, and such that the distances of its centre from them is comparable with their dimensions, the magnetic and electric forces at points on S will each contain, *primâ facie* at least, terms of order $1/R$. In such a case then we require in the above integral to take account of terms of this order only. To this order the electric and magnetic forces are at right angles to each other and to the radius of the sphere, and the integral assumes the form

$$(4\pi)^{-1}.V\int (\alpha^2+\beta^2+\gamma^2)dS,$$

where V is the velocity of transmission through the medium. Accordingly, any system whatever executing harmonic vibrations of one definite period will radiate energy unless, to the order $1/R$, we have $\alpha=0$, $\beta=0$, $\gamma=0$, at all points of the

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* *Loc. cit.* p. 83.   † *Loc. cit.* p. 72.

‡ The discussion is simplified by thus ensuring that the forces do not involve the time exponentially.

sphere and at all times: this result equally holds if the vibration be not simple but compounded of a number of simple harmonic types. Now if we have two waves of the same period travelling in opposite directions round the infinitely thin ring the expressions for α, β, γ at a point P (x_1, y_1, z_1) are of the type

$$\alpha = e^{i\kappa V t} \int (n d/dy_1 - m d/dz_1)(C_1 e^{i\kappa(s-r)} + C_2 e^{i\kappa(-s-r)}) r^{-1} ds,$$

where r is the distance of P from a point Q on the wire; s is the distance, measured along the ring, of Q from a fixed point A at which the magnetic forces due to the two waves are in the same phase; l, m, n are the direction cosines of the tangent to the ring at Q; and the integral is taken all along the ring. Let us take for the moment as axis of z the axis of revolution of the ring, and as axes of x, y, radii of the circular axis such that P lies in the plane yz. Let the angles AOX, XOQ, POZ be denoted respectively by ϕ, ψ, θ, and the radius of the ring by a; we have $\kappa a = \sigma$, where σ is some integer. The coordinates of P are then $0, R \sin \theta, R \cos \theta$, and of Q (x, y, z), $a \cos \psi$, $a \sin \psi$, 0; while $l = -\sin \psi$, $m = \cos \psi$, $n = 0$. Using Lord Kelvin's symbol $\doteqdot$ to denote equality to the order of approximation necessary we have

$$r \doteqdot R - a \sin \theta \sin \psi,$$

$$e^{i\kappa(s-r)} \doteqdot e^{i\kappa(a\phi + a\psi - R + a \sin \theta \sin \psi)},$$

$$\frac{d}{dx_1}\{r^{-1}e^{i\kappa(s-r)}\} \doteqdot -i\kappa(x_1-x)R^{-2}e^{i\kappa(s-r)} \doteqdot 0,$$

$$\frac{d}{dy_1}\{r^{-1}e^{i\kappa(s-r)}\} \doteqdot -i\kappa(y_1-y)R^{-2}e^{i\kappa(s-r)} \doteqdot -i\kappa \sin \theta . R^{-1}e^{i\kappa(s-r)},$$

$$\frac{d}{dz_1}\{r^{-1}e^{i\kappa(s-r)}\} \doteqdot -i\kappa(z_1-z)R^{-2}e^{i\kappa(s-r)} \doteqdot -i\kappa \cos \theta . R^{-1}e^{i\kappa(s-r)},$$

Therefore, omitting the time factor, the coefficient of C_1 in α

$$\doteqdot i\kappa a R^{-1} \cos \theta . e^{i\kappa(a\phi - R)}\int_{-\pi}^{+\pi} \cos \psi . e^{i\sigma(\psi + \sin \theta \sin \psi)}d\psi,$$

$$\doteqdot i\sigma R^{-1} \cos \theta . e^{i\kappa(a\phi - R)}\int_{-\pi}^{+\pi} \cos \psi \cos \sigma(\psi + \sin \theta \sin \psi)d\psi,$$

$$\doteqdot i\sigma R^{-1} \cos \theta . e^{i\kappa(a\phi - R)}F_1(\theta),$$

where $F_1(\theta)$ denotes the integral above.

Similarly the coefficient of C_2 in α

$$\doteqdot i\sigma R^{-1} \cos \theta . e^{i\kappa(-a\phi - R)}\int_{-\pi}^{+\pi} \cos \psi \cos \sigma(-\psi + \sin \theta \sin \psi)d\psi,$$

$$\doteqdot i\sigma R^{-1} \cos \theta . e^{i\kappa(-a\phi - R)}F_2(\theta).$$

It may be shown that $F_1(\theta) \pm F_2(\theta) = 0$ according as σ is even or odd.

The coefficient of C_1 in β

$$\doteq i\sigma R^{-1} \cos\theta . e^{i\kappa(a\Phi - R)} \int_{-\pi}^{+\pi} \sin\psi . e^{i\sigma(\psi + \sin\theta \sin\psi)} d\psi,$$

$$\doteq -\sigma R^{-1} \cos\theta . e^{i\kappa(a\Phi - R)} \int_{-\pi}^{+\pi} \sin\psi \sin\sigma(\psi + \sin\theta \sin\psi) d\psi,$$

$$\doteq -\sigma R^{-1} \cos\theta . e^{i\kappa(a\Phi - R)} f_1(\theta),$$

and the coefficient of C_2

$$\doteq -\sigma R^{-1} \cos\theta . e^{i\kappa(-a\Phi - R)} \int_{-\pi}^{+\pi} \sin\psi \sin\sigma(-\psi + \sin\theta \sin\psi) d\psi,$$

$$\doteq -\sigma R^{-1} \cos\theta . e^{i\kappa(-a\Phi - R)} f_2(\theta).$$

It may be shown that $f_1(\theta) \pm f_2(\theta) = 0$ according as σ is odd or even.

The coefficient of C_1 in $\gamma \doteq \sigma R^{-1} \sin\theta . e^{i\kappa(a\Phi - R)} f_1(\theta)$,

and that of C_2 $\qquad \doteq \sigma R^{-1} \sin\theta . e^{i\kappa(-a\Phi - R)} f_2(\theta)$.

Inserting the time factor and selecting the real parts, we have

$$\alpha \doteq -\sigma R^{-1} \cos\theta \{ C_1 F_1(\theta) \sin(a\phi + Vt - R) + C_2 F_2(\theta) \sin(-a\phi + Vt - R) \},$$

$$\beta \doteq -\sigma R^{-1} \cos\theta \{ C_1 f_1(\theta) \cos(a\phi + Vt - R) + C_2 f_2(\theta) \cos(-a\phi + Vt - R) \},$$

$$\gamma \doteq \sigma R^{-1} \sin\theta \{ C_1 f_1(\theta) \cos(a\phi + Vt - R) + C_2 f_2(\theta) \cos(-a\phi + Vt - R) \}.$$

It is evidently impossible that any one of these components, far less all three, should vanish for all values of θ, ϕ, t. There must then be radiation.

5. This result may be extended; it may be proved that permanent free oscillations in any indefinitely extended dielectric space are impossible. For let us suppose that the vibrations are undamped, being maintained by energy supplied to the conductors if, as is here held, this be necessary, and describe any sphere inclosing the latter. Outside this sphere the components of the vector-potential F, G, H can be written in forms of the type

$$F = \Sigma\Sigma E_n(\kappa r) . F_{n,\kappa} . e^{i\kappa Vt},$$

where $E_n(\rho)$ denotes $(d/\rho d\rho)^n(\rho^{-1} e^{-i\rho})$, and $F_{n,\kappa}$, $G_{n,\kappa}$, $H_{n,\kappa}$ are any solid harmonics of degree n, the summation being extended over all positive integral values of n, including zero, and over values of κ corresponding to the periods of all the constituent simple vibrations [*]. The components of magnetic

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[*] *See* J. J. Thomson, Proc. Lond. Math. Soc. vol. xv. 'Recent Researches in Electricity and Magnetism,' pp. 361 *et seq.*

force are of the type

$$\alpha = dH/dy - dG/dz$$
$$= \Sigma\Sigma\kappa(yH_{n,\kappa} - zG_{n,\kappa})E_{n+1}(\kappa r)e^{i\kappa Vt}$$
$$+ \Sigma\Sigma(dH_{n,\kappa}/dy - dG_{n,\kappa}/dz)E_n(\kappa r)e^{i\kappa Vt}.$$

At a great distance from the centre, and selecting the real parts, these last take the forms

$$\alpha \doteq \Sigma\Sigma\kappa(yH_{n,\kappa} - zG_{n,\kappa})\cos(\kappa Vt - \alpha_{n,\kappa} - \kappa r + n\pi/2)(\kappa r)^{-n-2},$$

the approximations being to the order $1/r$. Evidently the right-hand members of these equations cannot vanish for all values of the time unless the coefficient of each cosine vanishes separately; therefore, if there is to be no radiation, for each value of κ we must have three equations of the type

$$\Sigma(yH_{n,\kappa} - zG_{n,\kappa}) = 0,$$

the summation extending over all the values of n which occur. It is readily seen then that the only harmonics which can occur are of the first order, and that for each value of κ

$$F_{1,\kappa} = \lambda x, \quad G_{1,\kappa} = \lambda y, \quad H_{1,\kappa} = \lambda z,$$

where λ is a constant. Accordingly the values of the magnetic and electric forces outside the sphere vanish, not merely to the order $1/r$, but absolutely. Again, in periodic motion these forces and all their spacial differential coefficients are continuous throughout all space except at sources. Hence we see that at any time they must absolutely vanish in any space which can at that time be reached from the outside without passing through a source. Accordingly, in order that there may be no radiation in periodic motion, the sources must include a continuous distribution over a closed surface or surfaces, though there may be other sources inside them; and the distribution must be such that the dielectric outside these surfaces is absolutely free from disturbance; in other words, the vibrating medium must be finite and bounded externally. This seems indeed to be the proof indicated, too briefly, by Pocklington *.

It thus appears that, if either Poynting's expression for the rate of radiation, or Macdonald's modification of it, is to be accepted, it is impossible for any combination of electron-sources executing periodic motions to constitute a molecule which, when isolated, will not lose energy by radiation. This does not contradict Larmor's † result that if the period of the slowest constituent vibration is small compared with the time which radiation would take to travel across the molecule, the arrangement may be such that the radiation is very small compared with the sum of the radiations from each electron

* Nature, March 26, 1903.   † 'Æther and Matter,' p. 232.

separately, moving in the same way, but at an infinite distance from the others.

6. In the second paragraph of Pocklington's letter to 'Nature' it is pointed out that a system of sources can be arranged on the surface of a sphere so that there is no external oscillating field. We have, in fact, to deal with such a system in any of the free oscillations of the dielectric inside a perfectly conducting hollow surface. But this is quite different from a system in which there is an actual oscillating field external to the conductors and at the same time no radiation.

7. Considering the matter without regard to mathematical analysis, I submit that it is difficult to imagine how a system of sources can maintain an infinite medium in vibration without losing energy. To take what seems a parallel illustration from another subject, can we conceive of a sonorous body emitting waves of sound into an infinite atmosphere, even if it be doubly-connected, which do not carry off energy?

And if Macdonald's claim be tenable, what distinction is to be drawn between divergent and convergent waves, between those which involve $e^{i\kappa(Vt-r)}$ and those which involve $e^{i\kappa(Vt+r)}$, between a system of sources and a system of sinks?

8. A few *errata* in 'Electric Waves' may be here pointed out. In the discussion of the vibrations in the space between two concentric spherical surfaces (p. 49) the solutions involving harmonics of tesseral type * have been overlooked; it also seems doubtful whether or not it is intended to include that corresponding to an harmonic of order zero, this being a possible solution and the one cited by Pocklington. In the investigation of the modes of oscillation of a dielectric sphere surrounded by a different dielectric which extends to infinity (p. 59), the solution given corresponds to absorption instead of radiation, and at the surface of separation the tangential electric displacement instead of the tangential electric force is made continuous. In the expression for the rate of increase of the energy inside a surface (l. 6, p. 70) there appears to be an omission from the bracket of the terms

$$-f\frac{d^2\phi}{dxdt} - g\frac{d^2\phi}{dydt} - h\frac{d^2\phi}{dzdt};$$

this, however, does not affect any result.

Royal College of Science, Dublin.
October 14, 1903.

[*Note by Mr. Macdonald.*—Prof. Orr's consideration in sending me a proof of his paper affords me an opportunity of withdrawing the hasty generalization referred to, which, as he remarks, had been already challenged by Prof. Larmor and

* J. J. Thomson, *loc. cit.*

by Mr. Pocklington. For given electrical force on the wire the vibrations become more persistent as the wire becomes thinner, the energy of the wave system increasing rapidly while the radiation is not much altered ; but it was a mistake to express the result, which is sensibly though not rigorously true for thin wire rings, as if it were true for rings of large cross-section. The other corrections pointed out by Prof. Orr deserve my best thanks and afford gratifying evidence of the kindly interest he has taken in my work.]

LXXXIV. *The Magnetic Properties of Systems of Corpuscles describing Circular Orbits. By* J. J. Thomson, *M.A., F.R.S., Cavendish Professor of Experimental Physics, Cambridge*[*].

THE problems discussed in this paper are :—(1) The magnetic field due to a number of negatively electrified corpuscles situated at equal intervals round the circumference of a circle and rotating in one plane with uniform velocity round its centre ; and (2) The effect of an external magnetic field on the motion and periods of vibration of such a system. These problems are met with when we attempt to develop the theory that the atoms of the chemical elements are built up of large numbers of negatively electrified corpuscles revolving around the centre of a sphere filled with uniform positive electrification.

If an electrified particle is moving uniformly with a velocity small compared with that of light, it produces a magnetic field whose components α, β, γ at a distance r from the particle are given by the equations

$$\alpha = e\left(v\frac{d}{dz} - w\frac{d}{dy}\right)\frac{1}{r}, \quad \beta = e\left(w\frac{d}{dx} - u\frac{d}{dz}\right)\frac{1}{r}, \quad \gamma = e\left(u\frac{d}{dy} - v\frac{d}{dx}\right)\frac{1}{r}, \quad (1)$$

where e is the charge on the particle and u, v, w the components of its velocity. These expressions will not give the magnetic force when the motion is variable ; when, however, the velocity is periodic, proportional say to $\epsilon^{\iota p t}$, the equations for the components of the magnetic force can be derived at once from equations (1) by writing $\epsilon^{-\iota\frac{p}{V}r}\Big/r$ for $1/r$ in those equations: V being the velocity of light through the medium. We can see this at once by noticing that the modified equations satisfy differential equations of the type

$$\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2} = \frac{1}{V^2}\frac{d^2\alpha}{dt^2};$$

and for values of r small compared with the wave-length, $2\pi V/p$, of the vibrations, they give the same magnetic force as that which would be produced if the particle was moving uniformly with the velocity it possessed at the instant under consideration.

This derivation of the magnetic force when the velocity is variable from the solution obtained on the assumption that the velocity is uniform may be extended to cases more complicated than that of a single particle; if we have a collection of particles separated from each other by distances small compared with the wave-lengths of their vibrations, and if in the expression for the magnetic force due to these particles, calculated on the assumption that the velocity is uniform, there is the term

$$\phi(t)\frac{d^l}{dx^l}\frac{d^m}{dy^m}\frac{d^n}{dz^n}\frac{1}{r};$$

r being the distance from a point in the midst of the particles, then if $\phi(t)$ is proportional to $\epsilon^{\iota s t}$, the solution, when we take into account the variability of the motion, will have for the corresponding term

$$\phi(t)\frac{d^l}{dx^l}\frac{d^m}{dy^m}\frac{d^n}{dz^n}\frac{\epsilon^{-\frac{\iota s}{V}r}}{r}.$$

2. We shall first find the magnetic field due to a particle with charge e describing with uniform velocity a circular orbit in the plane xy. Let O be the centre of the orbit, Q the position of the particle, P the point, not necessarily in the plane of xy, at which the magnetic force is to be determined; then if a is the radius of the orbit, ω the angular

Fig. 1.
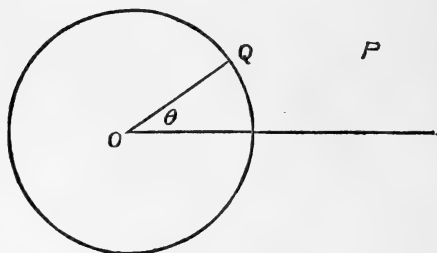
velocity of rotation, θ the angle OQ makes with the axis of x,

$$u=-a\omega\sin\theta,\quad v=a\omega\cos\theta,\quad w=0.$$

Hence γ the z component of the magnetic force at P, calculated on the assumption that the velocity of the particle is uniform, is given by the equation

$$\gamma = -ea\omega\left\{\sin\theta\frac{d}{dy} + \cos\theta\frac{d}{dx}\right\}\frac{1}{r'},$$

r' being written for PQ.

Writing r for OP we have

$$\frac{1}{r'} = \frac{1}{r} - a\left(\cos\theta\frac{d}{dx} + \sin\theta\frac{d}{dy}\right)\frac{1}{r} + \frac{a^2}{1.2}\left(\cos\theta\frac{d}{dx} + \sin\theta\frac{d}{dy}\right)^2\frac{1}{r} - \ldots;$$

hence, writing ϑ for $\cos\theta\dfrac{d}{dx} + \sin\theta\dfrac{d}{dy}$, we have

$$\gamma = -ea\omega\left\{\vartheta\frac{1}{r} - a\vartheta^2\frac{1}{r} + \frac{a^2}{1.2}\vartheta^3\frac{1}{r} - \ldots\right\} \quad . \quad . \quad (2)$$

To pass to the solution in which the acceleration of the particle is taken into account we must express γ in terms of the time.

Now $\theta = \omega t$ when t the time is measured from the instant when the particle is on the axis of x; hence, substituting the exponential values for the cosine and sine, we have

$$\vartheta = \tfrac{1}{2}\left\{\epsilon^{\iota\omega t}\left(\frac{d}{dx} - \iota\frac{d}{dy}\right) + \epsilon^{-\iota\omega t}\left(\frac{d}{dx} + \iota\frac{d}{dy}\right)\right\}.$$

If we introduce two new variables ξ and η, defined by the equations

$$\xi = x + \iota y, \quad \eta = x - \iota y,$$

then

$$\frac{d}{dx} - \iota\frac{d}{dy} = 2\frac{d}{d\xi} \text{ and } \frac{d}{dx} + \iota\frac{d}{dy} = 2\frac{d}{d\eta};$$

thus

$$\vartheta = \left\{\epsilon^{\iota\omega t}\frac{d}{d\xi} + \epsilon^{-\iota\omega t}\frac{d}{d\eta}\right\}$$

and

$$\vartheta^n = \left\{\epsilon^{\iota n\omega t}\left(\frac{d}{d\xi}\right)^n + n\epsilon^{\iota(n-2)\omega t}\left(\frac{d}{d\xi}\right)^{n-1}\frac{d}{d\eta}\right.$$
$$\left. + \frac{n.n-1}{1.2}\epsilon^{\iota(n-4)\omega t}\left(\frac{d}{d\xi}\right)^{n-2}\left(\frac{d}{d\eta}\right)^2 + \ldots\right\}.$$

Hence to deduce from (2) the value of the magnetic force when we take into account the acceleration of the particle, we must write in that expression instead of $\vartheta^n\dfrac{1}{r}$

$$\epsilon^{\iota n \omega t}\left(\frac{d}{d\xi}\right)^n \frac{\epsilon^{\frac{-\iota n \omega r}{V}}}{r} + n\epsilon^{\iota(n-2)\omega t}\left(\frac{d}{d\xi}\right)^{n-1}\frac{d}{d\eta}\frac{\epsilon^{\frac{-\iota(n-2)\omega r}{V}}}{r}$$

$$+ \frac{n \cdot n-1}{2}\epsilon^{\iota(n-4)\omega t}\left(\frac{d}{d\xi}\right)^{n-2}\left(\frac{d}{d\eta}\right)^2\frac{\epsilon^{\frac{-\iota(n-4)\omega r}{V}}}{r} + \ldots$$

when this substitution is made (2) will give the z component of the magnetic force.

The x component α is, on the assumption that the motion is steady, given by the equation

$$\alpha = ea\omega \cos\theta \frac{d}{dz}\frac{1}{r'}$$

$$= \tfrac{1}{2}ea\omega\left\{\epsilon^{\iota\theta} + \epsilon^{-\iota\theta}\right\}\frac{d}{dz}\left\{\frac{1}{r} - a\vartheta\frac{1}{r} + \frac{a^2\vartheta^2}{1.2}\frac{1}{r} - \ldots\right\};$$

hence writing ωt for θ and proceeding as before, we find that when the acceleration of the particle is taken into account

$$\alpha = \tfrac{1}{2}ea\omega\frac{d}{dz}\left\{\frac{\epsilon^{\iota\left(\omega t - \frac{\omega r}{V}\right)}}{r} + \frac{\epsilon^{-\iota\left(\omega t - \frac{\omega r}{V}\right)}}{r} - a\left(\epsilon^{2\iota\omega t}\frac{d}{d\xi}\frac{\epsilon^{\frac{-2\iota\omega r}{V}}}{r}\right.\right.$$

$$\left.+ \left(\frac{d}{d\xi} + \frac{d}{d\eta}\right)\frac{1}{r} + \epsilon^{-2\iota\omega t}\frac{d}{d\eta}\frac{\epsilon^{\frac{2\iota\omega r}{V}}}{r}\right)$$

$$+ \frac{(-a)^n}{n!}\left(\epsilon^{(n+1)\iota\omega t}\left(\frac{d}{d\xi}\right)^n\frac{\epsilon^{-(n+1)\frac{\iota\omega r}{V}}}{r}\right.$$

$$+ \epsilon^{(n-1)\iota\omega t}\left(\frac{d}{d\xi}\right)^{n-1}\left(\frac{d}{d\xi} + n\frac{d}{d\eta}\right)\frac{\epsilon^{-\frac{(n-1)\omega r}{V}}}{r}$$

$$\left.\left. + n\epsilon^{n-3\iota\omega t}\left(\frac{d}{d\xi}\right)^{n-2}\frac{d}{d\eta}\left(\frac{d}{d\xi} + \frac{n-1}{2}\frac{d}{d\eta}\right)\frac{\epsilon^{-\frac{(n-1)\iota\omega r}{V}}}{r}\right)\ldots\right\}.$$

The value of $\iota\beta$ can be got in a similar way by multiplying $\frac{d}{dz}\frac{1}{r'}$ by $\frac{1}{2}(\epsilon^{\iota\theta} - \epsilon^{-\iota\theta})$.

3. These expressions consist of two parts : (1) periodic terms proportional to some power of $\epsilon^{\iota\omega t}$, and (2) terms independent of the time; in considering the magnetic force due to the particles we shall take these terms separately, retaining in each only the lowest power of a.

Let us begin with the single particle and take first the terms independent of the time. These terms are given by the equation

$$\gamma = ea^2\omega\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)\frac{1}{r} = -ea^2\omega\frac{d^2}{dz^2}\frac{1}{r};$$

$$\alpha = -ea^2\omega\frac{d}{dx}\frac{d}{dz}\frac{1}{r};$$

$$\beta = -ea^2\omega\frac{d}{dy}\frac{d}{dz}\frac{1}{r}.$$

i. e., the part of the magnetic force which is independent of the time is that due to a small magnet magnetized in the direction of the axis of z and whose moment is $ea^2\omega$.

For the periodic terms we have, retaining only the lowest powers of a,

$$\gamma = ea\omega\left\{\frac{x}{r^3}\cos\left(\omega t - \frac{\omega r}{V}\right) + \frac{y}{r^3}\sin\left(\omega t - \frac{\omega r}{V}\right) - \frac{x\omega}{Vr^2}\sin\left(\omega t - \frac{\omega r}{V}\right)\right.$$
$$\left. + \frac{y\omega}{Vr^2}\cos\left(\omega t - \frac{\omega r}{V}\right)\right\};$$

$$\alpha = ea\omega\left\{-\frac{z}{r^3}\cos\left(\omega t - \frac{\omega r}{V}\right) + \frac{z\omega}{Vr^2}\sin\left(\omega t - \frac{\omega r}{V}\right)\right\};$$

$$\beta = ea\omega\left\{-\frac{z}{r^3}\sin\left(\omega t - \frac{\omega r}{V}\right) - \frac{z\omega}{Vr^2}\cos\left(\omega t - \frac{\omega r}{V}\right)\right\}.$$

When r the distance from the centre of the orbit considerably exceeds V/ω, the terms involving the lowest powers of $1/r$ are the most important; confining our attention to these we have

$$\gamma = \frac{ea\omega^2}{Vr^2}\left\{y\cos\left(\omega t - \frac{\omega r}{V}\right) - x\sin\left(\omega t - \frac{\omega r}{V}\right)\right\};$$

$$\alpha = \frac{ea\omega^2}{Vr^2}z\sin\left(\omega t - \frac{\omega r}{V}\right);$$

$$\beta = -\frac{ea\omega^2}{Vr^2}z\cos\left(\omega t - \frac{\omega r}{V}\right).$$

Thus if O is the centre of the orbit, Oz the normal to its plane, then the magnetic force at the point P is equivalent to a magnetic force

$$-\frac{ea\omega^2}{Vr}\sin\left(\omega t - \frac{\omega r}{V} - \phi\right)$$

in the plane POz at right-angles to OP, together with a force, at right-angles to the plane POz, equal to

$$\frac{ea\omega^2}{Vr}\cos\theta\cos\left(\omega t - \frac{\omega r}{V} - \phi\right).$$

Here θ is the angle OP makes with Oz, and ϕ the angle the plane POz makes with the plane of xz.

Using the language of the Electromagnetic Theory of Light, we may say that the rotating particle produces a wave of elliptically polarized light, the ratio of the axes of the ellipse being $\cos\theta$; thus along the normal to the orbit we have circular polarized light, while in the plane of the orbit the light is plane polarized.

Along with these magnetic forces we have in the plane POz at right-angles to OP an electric force equal to

$$\frac{ea\omega^2}{r}\cos\theta\cos\left(\omega t - \frac{\omega r}{V} - \phi\right),$$

and at right-angles to the plane POz another electric force equal to

$$\frac{ea\omega^2}{r}\sin\left(\omega t - \frac{\omega r}{V} - \phi\right);$$

applying Poynting's theorem we see that the rate at which energy is streaming through unit area at P is

$$\frac{1}{4\pi}\frac{e^2a^2\omega^4}{Vr^2}\left\{\cos^2\theta\cos^2\left(\omega t - \frac{\omega r}{V} - \phi\right) + \sin^2\left(\omega t - \frac{\omega r}{V} - \phi\right)\right\},$$

the mean value of this is

$$\frac{1}{8\pi}\frac{e^2a^2\omega^4}{Vr^2}(1 + \cos^2\theta);$$

integrating this over the sphere through P we find that the mean rate at which the rotating corpuscle is emitting energy is

$$\frac{2}{3V}e^2a^2\omega^4 = \frac{2}{3}\frac{e^2}{V}(\text{acceleration of the particle})^2.$$

4. Case of p particles separated by equal angular intervals rotating with uniform velocity ω round a circle.

Suppose that the particle we call (1) makes at the time t an angle ωt with the axis of x, the particle (2) will make an angle $\omega t + \frac{2\pi}{p}$, the particle (3) an angle $\omega t + 2.\frac{2\pi}{p}$, and so on; hence if $\gamma_1, \gamma_2, \gamma_3, \ldots$ are the magnetic forces parallel to z

due to the particles (1) (2) (3) respectively, γ_1 will be given by the expression already found, γ_2 will be got from γ_1 by writing $\omega t + \dfrac{2\pi}{p}$ for ωt, γ_3 by writing $\omega t + 2 \cdot \dfrac{2\pi}{p}$ for ωt, and so on. The total magnetic force parallel to z due to the p particles will be

$$\gamma_1 + \gamma_2 + \gamma_3 + \ldots \gamma_p.$$

The term independent of the time will be the same in γ_1, γ_2, ..., hence for the p particles it will be

$$-pea^2\omega \frac{d^2}{dz^2} \frac{1}{r}.$$

5. Let us now consider the periodic terms. The term corresponding to $\epsilon^{\iota n \omega t}$ in γ_1 will in γ_2 be $\epsilon^{\iota n\left(\omega t + \frac{2\pi}{p}\right)}$, in γ_3 $\epsilon^{\iota n\left(\omega t + 2 \cdot \frac{2\pi}{p}\right)}$, and so on, hence the corresponding term in $\gamma_1 + \gamma_2 + \gamma_3 + \ldots$ will be

$$\epsilon^{\iota n \omega t} + \epsilon^{\iota n\left(\omega t + \frac{2\pi}{p}\right)} + \epsilon^{\iota n\left(\omega t + 2 \cdot \frac{2\pi}{p}\right)} + \epsilon^{\iota n\left(\omega t + 3 \cdot \frac{2\pi}{p}\right)} + \ldots;$$

now this expression vanishes unless n is a multiple of p, when it equals $p\epsilon^{\iota n \omega t}$.

Hence the largest periodic term in the expression for the magnetic force due to the p corpuscles will be that corresponding to the term $\epsilon^{\iota p \omega t}$ in γ_1, and its magnitude will be p times this term; referring to the expression for γ_1 we see that the largest periodic term in the magnetic force due to the p particles will be

$$(-1)^p \frac{e\omega a^p}{1 \cdot 2 \cdot 3 \ldots p-1} \times p \left\{ \epsilon^{\iota p \omega t}\left(\frac{d}{d\xi}\right)^p \frac{\epsilon^{-\frac{\iota p \omega r}{V}}}{r} + \epsilon^{-\iota p \omega t}\left(\frac{d}{d\eta}\right)^p \frac{\epsilon^{\frac{\iota p \omega r}{V}}}{r} \right\}$$

$$= (-1)^p \frac{e\omega a^p \, p}{1 \cdot 2 \cdot 3 \cdot (p-1)} \cdot \frac{1}{2^p} \left\{ (x - \iota y)^p \left(\frac{d}{r\,dr}\right)^p \frac{\epsilon^{\iota\left(p\omega t - \frac{p\omega r}{V}\right)}}{r} \right.$$

$$\left. + (x + \iota y)^p \left(\frac{d}{r\,dr}\right)^p \frac{\epsilon^{-\iota\left(p\omega t - \frac{p\omega r}{V}\right)}}{r} \right\}.$$

Similarly the x and y components of the force due to the p particles are given by the equations

$$a = -(-1)^p \frac{1}{2 \cdot 1 \cdot 2 \cdot 3 \cdots (p-1)} \frac{e\omega a^p p}{dz} \frac{d}{dz} \left\{ \epsilon^{\iota p\omega t} \left(\frac{d}{d\xi}\right)^{p-1} \frac{\epsilon^{-\frac{\iota p\omega r}{V}}}{r} \right.$$

$$\left. + \epsilon^{-\iota p\omega t} \left(\frac{d}{d\eta}\right)^{p-1} \frac{\epsilon^{\frac{\iota p\omega r}{V}}}{r} \right\}$$

$$= -(-1)^p \frac{e\omega a^p p}{1 \cdot 2 \cdot 3 \cdots p-1} \frac{1}{2p} \frac{d}{dz} \left\{ (x-\iota y)^{p-1} \left(\frac{d}{rdr}\right)^{p-1} \frac{\epsilon^{\iota\left(pt - \frac{\omega r}{V}\right)}}{r} \right.$$

$$\left. + (x+\iota y)^{p-1} \left(\frac{d}{rdr}\right)^{p-1} \frac{\epsilon^{-\iota\left(pt - \frac{\omega r}{V}\right)}}{r} \right\}$$

$$\iota\beta = (-1)^{p+1} \frac{e\omega a^p p}{1 \cdot 2 \cdot 3 \cdot p-1} \frac{1}{2p} \frac{d}{dz} \left\{ (x-\iota y)^{p-1} \left(\frac{d}{rdr}\right)^{p-1} \frac{\epsilon^{\iota\left(pt - \frac{\omega r}{V}\right)}}{r} \right.$$

$$\left. - (x+\iota y)^{p-1} \left(\frac{d}{rdr}\right)^{p-1} \frac{\epsilon^{-\iota\left(pt - \frac{\omega r}{V}\right)}}{r} \right\}.$$

These expressions give the intensity of the magnetic force at any distance from the rotating system; the terms in these expressions most important for the study of the physical properties of the system are those proportional to $1/r$, as at a distance from the system large compared with the wavelength of the vibration the other terms become insignificant. Confining ourselves to these terms we find after a little reduction that the magnetic force at a point P is equivalent to (1) a component M in the plane POz at right-angles to OP given by the equation

$$M = (-1)^p \frac{2e\omega p}{1 \cdot 2 \cdot 3 \cdots (p-1)} \left(\frac{ap\omega}{2V}\right)^p$$
$$\frac{\sin^{p-1}\theta}{r} \cos p\left\{\left(\omega t - \frac{\omega r}{V}\right) - \phi - \frac{\pi}{2}\right\},$$

and to a component L at right-angles to this plane given by the equation

$$L = (-1)^p \frac{2e\omega p}{1 \cdot 2 \cdot 3 \cdots (p-1)} \left(\frac{ap\omega}{2V}\right)^p$$
$$\frac{\sin^{p-1}\theta \cos\theta}{r} \sin p\left\{\left(\omega t - \frac{\omega r}{V}\right) - \phi - \frac{\pi}{2}\right\},$$

where, as before, θ is the angle between OP and Oz and ϕ the angle the plane POz makes with the plane of xz.

The components of the electric force are VL in the meridian plane and VM at right-angles to it.

We see that these expressions represent elliptically polarized light, the ratio of the axes being, as in the case when there is only one particle, $\cos\theta$; in this case, however, since L and M vanish when $\theta = 0$, the intensity of the light vanishes along the normal to the plane of the orbit; thus a system of two or more particles rotating uniformly in a circular orbit does not give out any light along the axis of that orbit, a point which has to be kept in mind when considering the interpretation of the Zeeman effect. We see that as p increases the vibrations tend to become confined to the plane of the orbit.

6. The rate at which energy is streaming through unit area of the surface at P is by Poynting's theorem equal to

$$\frac{V}{4\pi}(L^2 + M^2).$$

Substituting for L and M the values given above we find that the average rate at which energy is streaming through the surface of a sphere with its centre at O is

$$\frac{V}{4}\left(\frac{2e\omega p}{1.2.3\ldots p-1}\right)^2 \left(\frac{ap\omega}{2V}\right)^{2p} \int_0^\pi \sin^{2p-2}\theta(1+\cos^2\theta)\sin\theta\,d\theta,$$

$$= \frac{V}{4}\left(\frac{2e\omega p}{1.2.3\ldots p-1}\right)^2 \left(\frac{ap\omega}{2V}\right)^{2p} \left\{\frac{2^2.4\ldots(2p-2)}{1.3\ldots2p-1} - \frac{2.4\ldots2p}{1.3\ldots2p+1}\right\}$$

$$= \frac{V}{4}\left(\frac{2e\omega p}{1.2.3\ldots p-1}\right)^2 \left(\frac{ap\omega}{2V}\right)^{2p} \frac{2.4\ldots2p-2.2p+2}{1.3\ldots2p+1}.$$

7. From this expression we see that when $a\omega$ the velocity of the particle is small compared with V the velocity of light, the rate at which energy radiates diminishes very rapidly as the number of particles increases. This is brought out by the following table, which gives the average radiation *per particle* for various groups of particles in terms of the radiation from a single particle moving with the same velocity in the same orbit; the table gives the results for two cases, in the first case the velocity of the particle is 1/10 that of light, in the second it is 1/100 of the same velocity. The radiation from a single particle is in each case taken as unity.

Number of Particles.	Radiation from each particle.	
	$a\omega = V/10.$	$a\omega = V/10^2.$
1	1	1
2	$9 \cdot 6 \times 10^{-2}$	$9 \cdot 6 \times 10^{-4}$
3	$4 \cdot 6 \times 10^{-3}$	$4 \cdot 6 \times 10^{-7}$
4	$1 \cdot 7 \times 10^{-4}$	$1 \cdot 7 \times 10^{-10}$
5	$5 \cdot 6 \times 10^{-5}$	$5 \cdot 6 \times 10^{-13}$
6	$1 \cdot 6 \times 10^{-7}$	$1 \cdot 6 \times 10^{-17}$

From this we see that the radiation from each of a group of 6 particles moving with one tenth the velocity of light is considerably less than one five-millionth part of the radiation from a single particle describing the same orbit with the same velocity, while when the particles are moving with only 1/100 of the velocity of light the radiation falls to $1 \cdot 6 \times 10^{-17}$ of that from the single particle.

8. Action of an external magnetic field on a ring of rotating particles.

Since a ring of particles produces a magnetic field similar to that due to a current flowing round the orbit of the particles, it might seem at first sight as if a body whose atoms contained systems of such rotating particles ought to be strongly magnetic, the following investigation of the action of an external field on such a system shows that this is not the case.

Let us suppose that the external magnetic force H is uniform and parallel to the axis of X ; and let the moving electrified particle be describing its orbit under a radial attractive force proportional to its distance from a fixed point. If m is the mass of the particle, e its electric charge, μ the force at unit distance, the equations of motion are

$$m \frac{d^2x}{dt^2} = -\mu x \quad \cdots \cdots \quad (1)$$

$$m \frac{d^2y}{dt^2} = -\mu y - \mathrm{H} e \frac{dz}{dt} \quad \cdots \cdots \quad (2)$$

$$m \frac{d^2z}{dt^2} = -\mu z + \mathrm{H} e \frac{dy}{dt} \quad \cdots \cdots \quad (3)$$

If η and ζ are the coordinates of the point referred to axes in the plane of yz rotating with the angular velocity $p = \frac{1}{2} \mathrm{H} e / m$ so that

$$y = \eta \cos pt - \zeta \sin pt, \qquad z = \eta \sin pt + \zeta \cos pt,$$

equations (2) and (3) become, if we neglect terms in H^2,

$$m \frac{d^2\eta}{dt^2} = -\mu \eta, \qquad m \frac{d^2\zeta}{dt^2} = -\mu \zeta.$$

The solutions of these equations are if $\omega^2 = \mu / m$

$$x = \mathrm{A} \cos \omega t + \mathrm{B} \sin \omega t,$$
$$\eta = \mathrm{C} \cos \omega t + \mathrm{D} \sin \omega t,$$
$$\zeta = \mathrm{E} \cos \omega t + \mathrm{F} \sin \omega t.$$

Let the time be measured from the instant when the

magnetic force was applied. Let $y_0 z_0$, $\dot{y}_0$, $\dot{z}_0$ be the values of y, z, $\dfrac{dy}{dt}$, $\dfrac{dz}{dt}$ at this time, then when $t = 0$ we have

$$\eta = y_0, \qquad \zeta = z_0,$$

$$\frac{d\eta}{dt} = \dot{y}_0 + pz_0,$$

$$\frac{d\zeta}{dt} = \dot{z}_0 - py_0 ;$$

hence

$$\eta = \zeta_0 \cos \omega t + \frac{1}{\omega}(\dot{y}_0 + pz_0) \sin \omega t,$$

$$\zeta = z_0 \cos \omega t + \frac{1}{\omega}(\dot{z}_0 - py_0) \sin \omega t ;$$

and similarly

$$x = x_0 \cos \omega t + \frac{1}{\omega} \dot{x}_0 \cos \omega t ;$$

the motion parallel to x is not affected by the magnetic force.

The values of y and z are given by

$$y = \eta \cos pt - \zeta \sin pt, \quad z = \eta \sin pt + \zeta \cos pt.$$

The magnetic force α parallel to x calculated on the assumption that the motion is steady is given by the equation

$$\alpha = -e \left\{ \frac{dz}{dt} \frac{d}{dy'} \frac{1}{r'} - \frac{dy}{dt} \frac{d}{dz'} \frac{1}{r'} \right\}$$

where x', y', z' are the coordinates of the point P at which the magnetic force is calculated.

Now,

$$\frac{1}{r'} = \frac{1}{r} - \left(x \frac{d}{dx'} + y \frac{d}{dy'} + z \frac{d}{dz'} \right) \frac{1}{r} + . \quad . .$$

Substituting this value of $1/r'$ in the expression for α, and expressing the coefficients of the differential coefficients of $1/r$ as periodic functions of the time, we can pass to the solution when we take the acceleration of the particles into account by the method explained at the beginning of this paper.

For the purpose of discussing the magnetic properties of the system under a steady magnetic field, the only terms which are of importance are the non-periodic ones. Substituting the values of x, y, z, $\dfrac{dy}{dt}$, $\dfrac{dz}{dt}$ in the expression for α,

and picking out the non-periodic terms, we find, neglecting terms in p^2,

$$\alpha = e(y_0\dot{z}_0 - \dot{y}_0 z_0)\left(\frac{d^2}{dy'^2} + \frac{d^2}{dz'^2}\right)\frac{1}{r} + \frac{ep}{4}\left\{\frac{\dot{y}_0^2 + \dot{z}_0^2}{\omega^2}\right.$$
$$\left. - (y_0^2 + z_0^2)\right\}\left(\frac{d^2}{dy'^2} + \frac{d^2}{dz'^2}\right)\frac{1}{r},$$

or

$$\alpha = -e(y_0\dot{z}_0 - \dot{y}_0 z_0)\frac{d^2}{dx'^2}\frac{1}{r} - \frac{He^2}{8m}\left\{\frac{\dot{y}_0^2 + \dot{z}_0^2}{\omega^2} - (y_0^2 + z_0^2)\right\}\frac{d^2}{dx'^2}\frac{1}{r}.$$

The first term represents the force due to the particle in its undisturbed orbit, and may for our present purpose be neglected. At the time $t=0$ the particle was describing a circular orbit with angular velocity ω, so that

$$\dot{y}_0^2 + \dot{z}_0^2 + \dot{x}_0^2 = a^2\omega^2, \qquad x_0^2 + y_0^2 + z_0^2 = a^2,$$

where a is the radius of the orbit: hence, as far as the term involving H is concerned,

$$\alpha = \frac{He^2}{8m}\left(\frac{\dot{x}_0^2}{\omega^2} - x_0^2\right)\frac{d^2}{dx_0^2}\frac{1}{r}.$$

Thus the particle produces the same effect as a little magnet whose moment is parallel to the axis of x and equal to

$$\frac{He^2}{8m}\left(\frac{\dot{x}_0^2}{\omega^2} - x_0^2\right).$$

10. Suppose that the plane of the orbit makes an angle ϕ with the plane xz, the line of intersection of the orbit with xz making an angle ψ with the axis of x, then if θ_1 is the angle

Fig. 2.
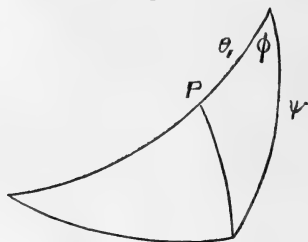
the radius to the particle makes with the line of intersection at the time $t=0$, we have

$$x_0 = a\,(\cos\theta_1\cos\psi + \sin\theta_1\sin\psi\cos\phi),$$
$$\dot{x}_0 = a\omega\,(-\sin\theta_1\cos\psi + \cos\theta_1\sin\psi\cos\phi)\,;$$

so that

$$\frac{\dot{x}_0{}^2}{\omega^2} - x_0{}^2 = a^2\{\cos 2\theta_1 (\sin^2\psi \cos^2\phi - \cos^2\psi) - \sin 2\theta_1 \sin 2\psi \cos\phi\}.$$

Thus each particle produces the same effect as a magnet magnetized parallel to the direction of H and having a moment equal to

$$\frac{\mathrm{H}\,e}{8m} a^2\{\cos 2\theta_1(\sin^2\psi \cos^2\phi - \cos^2\psi) - \sin 2\theta_1 \sin 2\psi \cos\phi\}.$$

The coefficient of magnetization of any substance is the sum of these moments for the unit of volume when the external force H is unity. We see from the preceding expression for the magnetic moment due to a single orbit that if the particles and their orbits are uniformly distributed this sum must be zero. For consider orbits whose planes are all in any given direction : for such orbits ψ and ϕ will have constant values ; the phase θ_1 at the time $t=0$ will, however, vary from orbit to orbit, and if these phases are equally distributed the mean values of $\cos 2\theta_1$ and $\sin 2\theta_1$ will be zero. Thus the coefficient of magnetization of the system will be zero; so that we cannot explain the magnetic or diamagnetic properties of bodies by the supposition that the atoms consist of charged particles describing closed periodic orbits under the action of a force proportional to the distance from a fixed point. I find that Professor Voigt has already come to this conclusion by a different method.

11. I find that the same absence of diamagnetic or magnetic properties will persist whatever be the law of force, provided the force is central and there is no dissipation of energy.

To prove this, let x, y, z be the coordinates of the particle ; then α, the x-component of the magnetic force at the point (x', y', z'), is given by the equation

$$\alpha = -e\left\{\frac{dz}{dt}\frac{d}{dy'}\frac{1}{r'} - \frac{dy}{dt}\frac{d}{dz'}\frac{1}{r'}\right\},$$

where r' is the distance between the points (x, y, z) and (x', y', z'). If r is the distance of (x', y', z') from the centre of the orbit taken as the origin, then

$$\frac{1}{r'} = \frac{1}{r} - \left(x\frac{d}{dx'} + y\frac{d}{dy} + z\frac{d}{dz'}\right)\frac{1}{r} + \ldots;$$

so that

$$\alpha = -e\left\{\frac{dz}{dt}\frac{d}{dy'}\frac{1}{r} - \frac{dy}{dt}\frac{d}{dz'}\frac{1}{r} - \frac{dz}{dt}\left(x\frac{d^2}{dx'\,dy'} + y\frac{d^2}{dy'^2} + z\frac{d^2}{dy'\,dz'}\right)\frac{1}{r}\right.$$
$$\left. + \frac{dy}{dt}\left(x\frac{d^2}{dx'\,dz'} + y\frac{d^2}{dy'\,dz'} + z\frac{d^2}{dz'^2}\right)\frac{1}{r} + \dots\right\}.$$

Now if the motion is periodic, the mean values of $\frac{dz}{dt}$, $\frac{dy}{dt}$, $z\frac{dz}{dt}$, and $y\frac{dy}{dt}$ are evidently zero.

Since

$$y\frac{dz}{dt} = \frac{1}{2}\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + \frac{1}{2}\frac{d}{dt}(yz),$$
$$z\frac{dy}{dt} = -\frac{1}{2}\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + \frac{1}{2}\frac{d}{dt}(yz),$$

the mean values of $y\frac{dz}{dt}$ and $-z\frac{dy}{dt}$ are one-half the mean value of $y\frac{dz}{dt} - z\frac{dy}{dt}$. We shall proceed to find these values and also to show that the mean values of $x\frac{dz}{dt}$ and $y\frac{dz}{dt}$ are zero.

The equations of motion are

$$m\frac{d^2x}{dt^2} = \mathrm{X}, \quad \dots \dots \dots \quad (1)$$

$$m\frac{d^2y}{dt^2} = \mathrm{Y} - \mathrm{H}e\frac{dz}{dt}, \quad \dots \dots \quad (2)$$

$$m\frac{d^2z}{dt^2} = \mathrm{Z} + \mathrm{H}e\frac{dy}{dt}. \quad \dots \dots \quad (3)$$

The force whose components are $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ is supposed to be central; so that

$$\frac{\mathrm{X}}{x} = \frac{\mathrm{Y}}{y} = \frac{\mathrm{Z}}{z}.$$

From (1) and (2) we see

$$m\frac{d}{dt}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right) = \mathrm{H}e\,x\frac{dz}{dt};$$

so that the mean value of $x\frac{dz}{dt}$ vanishes. Similarly, we may show that the mean value of $y\frac{dz}{dt}$ vanishes.

From (2) and (3) we have

$$m \frac{d}{dt}\left(y \frac{dz}{dt} - z \frac{dy}{dt}\right) = \tfrac{1}{2}\text{H}e \frac{d}{dt}(y^2 + z^2)$$

or

$$y \frac{dz}{dt} - z \frac{dy}{dt} = \frac{1}{2} \frac{\text{H}e}{m}\{(y^2 + z^2) - (y_0^2 + z_0^2)\} + y_0 \frac{dz_0}{dt} - z_0 \frac{dy_0}{dt},$$

where $y_0, z_0, \dfrac{dy_0}{dt}, \dfrac{dz_0}{dt}$ are the values of y, z, and their differential coefficients when $t=0$. Hence, if we retain in the expression for α only those terms whose mean values do not vanish, we have

$$\alpha = \left\{\tfrac{1}{2}e\left(y_0 \frac{dz_0}{dt} - z_0 \frac{dy_0}{dt}\right) + \frac{1}{4} \frac{\text{H}e^2}{m}\left((y^2 + z^2) - (\overline{y_0^2 + z_0^2})\right)\right\}\left(\frac{d^2}{dy'} + \frac{d^2}{dz'^2}\right)\frac{1}{r}.$$

Since

$$\left(\frac{d^2}{dy'^2} + \frac{d^2}{dz'^2}\right)\frac{1}{r} = -\frac{d^2}{dx'^2}\frac{1}{r},$$

we see that the orbit is equivalent to a little magnet whose moment is

$$\tfrac{1}{2}e\left(z_0 \frac{dy_0}{dt} - y_0 \frac{dz_0}{dt}\right) + \frac{1}{4} \frac{\text{H}e^2}{m}\left(y_0^2 + z_0^2 - (y^2 + z^2)\right).$$

The part of the moment which depends upon the external magnetic force is

$$\frac{1}{4} \frac{\text{H}e^2}{m}\{y_0^2 + z_0^2 - (y^2 + z^2)\},$$

and the mean value of this is

$$\frac{1}{4} \frac{\text{H}e^2}{m}\{y_0^2 + z_0^2 - (\overline{y}^2 + \overline{z}^2)\},$$

where $\overline{y}^2$ and $\overline{z}^2$ are the mean values of y^2 and z^2 round the orbit.

If we consider a large number of particles describing orbits, the phases of the particles in their orbits being uniformly distributed, we can see that the total moment of their equivalent magnets will be zero; for consider a number of similar orbits which only differ from each other by the position of the particle when the magnetic force is first applied. For all these orbits $\overline{y}^2 + \overline{z}^2$ is the same; so that if there are n orbits the sum of the second term inside

the bracket will be $n(\bar{y}^2 + \bar{z}^2)$; if the number of particles which at any time are situated between two neighbouring points P and Q on the orbit is proportional to the time taken by the particle to pass through PQ, then the sum of $y_0^2 + z_0^2$ for the n orbits will be $n(\bar{y}^2 + \bar{z}^2)$, and thus the two terms inside the bracket will balance each other and the resultant magnetic moment of the system will be zero. We thus see that we cannot explain the magnetic properties of bodies by means of charged particles describing without dissipation of energy closed orbits.

12. When there is dissipation of energy the collection of moving charged particles may, I think, possess magnetic properties. Let us represent the dissipation of energy by supposing that the motion of the particles is resisted by a force proportional to the velocity : the equations of motion are

$$m \frac{d^2 y}{dt^2} = Y - k \frac{dy}{dt} - He \frac{dz}{dt},$$

$$m \frac{d^2 z}{dt^2} = Z - k \frac{dz}{dt} + He \frac{dy}{dt},$$

where $k \frac{dy}{dt}$, $k \frac{dz}{dt}$ represent the forces due to the viscous resistance. Then, if the components Y and Z are those of a central force, we have

$$m \frac{d}{dt}\left(y \frac{dz}{dt} - z \frac{dy}{dt}\right) + k \left(y \frac{dz}{dt} - z \frac{dy}{dt}\right) = \tfrac{1}{2} He \frac{d}{dt}(y^2 + z^2).$$

The solution of this equation is

$$y \frac{dz}{dt} - z \frac{dy}{dt} = \epsilon^{-\frac{k}{m} t}\left(y_0 \frac{dz_0}{dt} - z_0 \frac{dy_0}{dt}\right) + \frac{1}{2} \frac{He}{m} \epsilon^{-\frac{k}{m} t} \int_0^t \epsilon^{\frac{k}{m} t} \frac{d}{dt}(y^2 + z^2)\, dt.$$

Thus, corresponding to the term $-\tfrac{1}{2}e\left(y \frac{dz}{dt} - z \frac{dy}{dt}\right)\frac{d^2}{dx^2}\frac{1}{r}$ in the expression for α, the magnetic force parallel to x, we have as the coefficient of $\frac{d^2}{dx^2}\frac{1}{r}$, and therefore as the moment of the equivalent magnet,

$$\tfrac{1}{2}e\left(z_0 \frac{dy_0}{dt} - y_0 \frac{dz_0}{dt}\right)\epsilon^{-\frac{k}{m} t} - \frac{1}{4} \frac{He^2}{m} \epsilon^{-\frac{k}{m} t} \int_0^t \epsilon^{\frac{k}{m} t} \frac{d}{dt}(y^2 + z^2)\, dt.$$

The contribution of this particle to the coefficient of magnetization is the mean value of the coefficient of H in

the preceding expression, *i. e.* the mean value of

$$-\frac{1}{4}\frac{e^2}{m}\,\epsilon^{-\frac{k}{m}t}\int_0^t \epsilon^{\frac{k}{m}t}\frac{d}{dt}(y^2+z^2)\,dt. \quad . \quad . \quad (1)$$

If the motion of the particle is strongly retarded, the particle will tend to fall in to the centre of the orbit and $\frac{d}{dt}(y^2+z^2)$ therefore be negative ; thus the mean value of the integral will be negative and the mean value of (1) positive. The contribution of this particle to the coefficient of magnetization will therefore be positive, and a collection of such particles will be paramagnetic.

13. Thus, to take a definite case, suppose the atoms of a substance, like the atoms of radio-active substances, were continually emitting corpuscles ; the velocity of projection of the corpuscles under consideration being, however, insufficient to carry them clear of the atom, so that the corpuscles describe orbits round the centre of the atom : then, if the motion of the corpuscles were not accompanied by dissipation of energy, the corpuscles would not endow the body with either magnetic or diamagnetic properties ; if, however, the energy of the corpuscles was dissipated during their motion outside the atom, so that they ultimately fell with but little energy into the atom, a system consisting of such atoms would be paramagnetic. If the energy of projection were derived from the internal energy of the atom, there would thus be a continual transference of energy from the atom to the surrounding systems ; this would tend to raise the temperature of the system. I am not aware that any experiments have been made to find whether the temperature in the middle of a mass of a magnetic substance like iron whose surface is kept at a constant temperature differs from the temperature inside a mass of a non-magnetic substance like brass whose surface is kept at the same temperature. I hope, however, soon to be able to test this point.

14. The origin of the difference between the effects produced by charged particles describing free orbits, and those produced by constant electric currents describing circular circuits, as in Ampère's theory of magnetism, is that in the case of the particles describing their orbits we get, in addition to the effects due to the constant electric currents, effects of the same character as those due to the induction of currents in conductors from the variation of the magnetic field ; these induced currents tend to make the body diamagnetic, while the Ampèrean currents tend to make it magnetic : in the case of

the particles describing free orbits these tendencies balance each other.

15. With a system of particles whose energy is being dissipated in the way described in § 13 we should get magnetic properties, these properties would be properties inherent in the atom, and would not explain the magnetic properties of iron for example, where the magnetization is so much affected by gross mechanical strain as to indicate that it depends largely if not entirely on the properties possessed by aggregations of large numbers of molecules. In the case of such aggregations, however, we may easily conceive that the orbits of charged bodies moving within them may not be free, but that in consequence of the forces exerted by the molecules in the aggregate, the orbit may be constrained to occupy an invariable position with respect to the aggregate —as if, to take a rough analogy, the orbit was a tube bored through the aggregate, so that the orbit and aggregate move like a rigid body, and in order to deflect the orbit it is necessary to deflect the aggregate. Under these conditions it is easy to see that the orbits would experience forces equivalent on the average to those acting on a continuous current flowing round the orbit; the aggregate and its orbit would under these forces act like a system of little magnets; and the body would exhibit magnetic properties quite analogous to those possessed by a system of Ampèrean circuits.

16. On the effect of a magnetic field on the frequency of the vibrations emitted by a system of n corpuscles rotating in a circular orbit.

The results obtained on pages (679) and (680) give the solution of this problem. We shall for the sake of brevity confine our attention (1) to the vibrations emitted along the axis of x, which is parallel to the magnetic force; for points along this axis $y = z = 0$; and (2) to the vibrations emitted in a direction at right-angles to the magnetic force, say along the axis of y, so that $x = 0$, $z = 0$.

The components of the magnetic force are expressed by terms of the form

$$\frac{d^l}{dx^l}\frac{d^m}{dy^m}\frac{d^n}{dz^n}\frac{e^{\iota\left(pt - \frac{pr}{V}\right)}}{r} \quad . \quad . \quad . \quad (1)$$

When the distance from the vibrating system is a large number of wave-lengths, the most important term in the expression (1) is proportional to $1/r$, and this term can be got by writing

$$-\frac{\iota p x}{\mathrm{V} r} \text{ for } \frac{d}{d x}, \quad -\frac{\iota p y}{\mathrm{V} r} \text{ for } \frac{d}{d y}, \text{ and} -\frac{\iota p z}{\mathrm{V} r} \text{ for } \frac{d}{d x}$$

in that expression, so that as far as this term is concerned, (1) reduces to

$$\left(-\frac{\iota p}{\mathrm{V}}\right)^{l+m+n}\left(\frac{x}{r}\right)^{l}\left(\frac{y}{r}\right)^{m}\left(\frac{z}{r}\right)^{n}\frac{\epsilon^{\iota\left(pt-\frac{pr}{\mathrm{V}}\right)}}{r};$$

thus along the axis of x where y and z both vanish we need only consider terms for which $m=n=0$, and along the axis of y those for which $l=n=0$.

17. Let x, y, z denote the coordinates of one of the corpuscles, the y component of the magnetic force at the point (x', y', z') is given by the equation

$$\beta = \Sigma e\left(\frac{dz}{dt}\frac{d}{dx'}\frac{1}{r'} - \frac{dx}{dt}\frac{d}{dz'}\frac{1}{r'}\right) \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

where r' is the distance between (x, y, z) and (x', y', z'), and Σ denotes that the sum of the corresponding expressions for all the corpuscles is to be taken. If r is the distance of $x' y' z'$ from the origin of coordinates (the centre of the orbit) then

$$\frac{1}{r'} = \frac{1}{r} - \left(x\frac{d}{dx'} + y\frac{d}{dy'} + z\frac{d}{dz'}\right)\frac{1}{r} + \frac{1}{1.2}\left(x\frac{d}{dx'} + y\frac{d}{dy'} + z\frac{d}{dz'}\right)^{2}\frac{1}{r} - \ldots$$

Along the axis of x the only terms we need consider are those in which the differentiation is entirely with regard to x'; hence, confining ourselves to these terms, we see from § 5 that if there are n corpuscles regularly spaced round the orbit, the term in β with which we are concerned will be of the form

$$e\frac{dz}{dt} \cdot \frac{x^{n-1}}{1.2.3.n-1}\left(\frac{d}{dx'}\right)^{n-1}\frac{\epsilon^{\iota\left(pt-\frac{pr}{\mathrm{V}}\right)}}{r} \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

Now x is of the form $\quad \mathrm{A}\cos \omega t + \mathrm{B}\sin \omega t$,

while if $$\delta = \frac{1}{2}\frac{\mathrm{H}e}{m},$$

$$z = \zeta \cos \delta t + \eta \sin \delta t,$$

where $$\zeta = \mathrm{A}'\cos \omega t + \mathrm{B}'\sin \omega t,$$

$$\eta = \mathrm{A}''\cos \omega t + \mathrm{B}''\sin \omega t.$$

Substituting these values for x and z in $x^{n-1}\dfrac{dz}{dt}$, we see that

(2) takes the form

$$\beta = \mathrm{A}\cos\left(n\omega t \pm \delta t\right)\epsilon^{-\iota\left(\frac{n\omega\pm\delta}{\mathrm{V}}\right)r}\frac{}{r}\ ;$$

thus, to use the language of spectrum analysis, instead of a single line of frequency $n\omega$ we have two lines whose frequencies are $n\omega + \delta$ and $n\omega - \delta$ respectively: it is easy to show that these lines are circularly polarized in opposite senses.

18. Let us now consider the effect at right-angles to the magnetic force, and first take the component of the magnetic force parallel to x; we have

$$\alpha = \Sigma\epsilon\left(\frac{dy}{dt}\frac{d}{dz'}\frac{1}{r'} - \frac{dz}{dt}\frac{d}{dy'}\frac{1}{r'}\right).$$

Along the axis of y the only terms we need consider are those in which the differentiation is entirely with regard to y'; picking out this term we have, when there are n corpuscles,

$$\alpha = e\frac{dz}{dt}y^{n-1}\frac{\epsilon^{\iota\left(pt - \frac{pr}{\mathrm{V}}\right)}}{r}\ ;$$

in this equation some constants have been omitted which do not affect the argument.

$$y = \eta\cos\delta t - \zeta\sin\delta t.$$

Substituting the values for z, η, and ζ given above, we find that α contains terms of the following types :—

$$\cos\left(n\omega t \pm n\delta t\right),\quad \cos\left(n\omega t \pm (n-2)\delta t\right),\quad \cos\left(n\omega t \pm (n-4)\delta t\right),$$

and so on, the last term being

$$\cos\left(n\omega t \pm 2\delta t\right)\text{ if }n\text{ is even,}$$
$$\cos\left(n\omega t \pm \delta t\right)\text{ if }n\text{ is odd.}$$

The equation for the magnetic force parallel to z is

$$\gamma = \Sigma e\left(\frac{dx}{dt}\frac{d}{dy'}\frac{1}{r'} - \frac{dy}{dt}\frac{d}{dz'}\frac{1}{r'}\right);$$

along the axis of y this reduces to terms proportional to

$$\frac{dx}{dt}\cdot y^{n-1}\frac{\epsilon^{\iota\left(pt - \frac{pr}{(\mathrm{V})}\right)}}{r}.$$

Substituting for x and y their values, we get in the expression for γ terms of the types

$$\cos\left(n\omega t \pm (n-1)\delta t\right), \qquad \cos\left(n\omega t \pm (n-3)\delta t\right), \ldots$$

the last term is $\cos n\omega t$ if n is odd, $\cos\left(n\omega t \pm \delta t\right)$ if n is even. Thus at right-angles to the direction of the magnetic force the effect is much more complicated than along that direction. At right-angles to the magnetic force the line whose frequency is $n\omega$ is split up into lines whose frequencies are $n\omega \pm n\delta$, $n\omega \pm (n-2)\delta$, $n\omega \pm (n-4)\delta$ .. : the last term being $n\omega \pm 2\delta$, or $n\omega \pm \delta$, as n is even or odd; all these lines are polarized in the plane parallel to the magnetic force : besides these we have lines whose frequencies are $n\omega \pm (n-1)\delta$, $n\omega \pm (n-3)\delta$, the last term being $n\omega$ or $n\omega \pm \delta$ as n is odd or even ; all these lines are polarized in the plane at right-angles to the magnetic force. Thus while the original line is, looking in the direction of the magnetic force, only split into two whose frequencies are $n\omega \pm \delta$; in the direction at right-angles to the force it is split into $2n$ or $2n+1$ lines according as n is even or odd, while the maximum change of frequency is n times that for the radiation along the lines of force.

LXXXV. *The Influence of Stress and of Temperature on the Magnetic Change of Resistance in Iron, Nickel, and Nickel-Steel.* *By* W. E. WILLIAMS, *B.Sc.**

THE influence of stress on the magnetic change of resistance in iron and nickel has been investigated by Tomlinson †; and in a paper previously communicated to the Philosophical Magazine ‡ the present writer described some experiments on the same subject. The results obtained seemed to indicate that a further investigation of this effect both in nickel and in iron would be desirable ; and the experiments described below were undertaken with that object.

The present paper also contains an account of experiments on the effects of variations of temperature on the magnetic change of resistance. This effect has been investigated for the case of nickel by Dr. Knott §. As, however, his experiments were confined to fields below 60 C.G.S. units, while

the apparatus at my disposal enabled fields of 800 units to be obtained, it is thought that an account of these experiments will not be without interest.

Apparatus.

The magnetizing field was obtained by means of a solenoid 1 metre long, having a resistance of about 3 ohms and giving a field of 40 c.g.s. units per ampere. The highest field used was 800 units.

The coil was provided with a water-jacket through which flowed a current of cold water from the mains. Inside this jacket a glass tube was placed, and the space between the two was filled with cotton. The coil was fixed vertically in a wooden framework.

The ends of each of the wires experimented upon were soldered to two thick copper rods, one of which was fixed at the top of the glass tube so that the wire hung vertically inside it. The wires were shorter than the coil by about 20 cms., and being placed centrally in the coil were in an approximately uniform field. All the wires on which experiments were made were less than $\frac{1}{2}$ mm. in diameter, so that the demagnetizing factor may in all cases be neglected.

The wires were stretched by placing weights in a pan attached to the lower copper rod. The change of resistance was measured in the same way as described in the previous paper. The curves for the temporary and for the residual effect were found separately, and the ordinates added together to construct a curve for the total effect.

Experiment I.—*The Influence of Stress on the Magnetic Change of Resistance in Nickel.*

The nickel wire used was ·35 mm. in diameter, and was annealed by heating in the flame of a Bunsen burner.

The wire was suspended as described above and change-of-resistance curves were taken for a number of different loads up to the breaking-load. The results obtained are shown by the curves in Pl. XXVII. fig. 1. It will be seen that the effect of loading is to diminish the change of resistance at low fields and to increase it at high fields. The residual change of resistance is diminished by loading at all the fields tried.

It is well known that the magnetic properties of nickel and iron are altered by stretching even after the load has been removed. Some measurements were made to investigate this effect for the magnetic change of resistance, and the results obtained are shown in fig. 2. The curve A in this figure is the original curve taken before loading the wire, the curve B was taken after the load of 10·9 kilos had been

applied and removed, and the curve C after the highest load 32·7 kilos had been removed. It will be noticed that stretching has the effect of increasing the residual change of resistance while, as shown in fig. 1, the residual change is diminished when the load is on the wire.

In the course of these experiments it was noticed that the magnetic change of resistance was very different in different specimens of wire. Three specimens of different diameters were tried, the diameters being ·35 mm., ·21 mm., and ·14 mm. In the finer wires the change of resistance is smaller at low fields and greater at high fields than in the thicker wire.

In the wire of ·14 mm. diameter the change of resistance at the highest field is over 3 per cent. A considerable portion of this difference is removed by heating the wire in the flame. These results are shown by the curves in fig. 3, which give the change of resistance for the three specimens.

These curves were all taken with the wires unloaded and at a temperature of 15° C. The difference is perhaps due to the strain which the wire undergoes in drawing.

It will be noticed that the curves for the finer wires are somewhat similar to those obtained by loading the thicker wire, as shown in fig. 1.

Experiment II.—*The Influence of Stress on the Magnetic Change of Resistance in Iron.*

The iron wire experimented upon was ·32 mm. in diameter, and was connected up as previously described. The results of the measurements are shown by the curves in fig. 4. It will be noticed that the effect of stress on the change of resistance is much smaller in iron than in nickel, and that there is hardly any effect at the highest fields.

The dotted curve in the diagram represents the after-effect of stress, being taken after the greatest load (24·5 kilos.) had been removed. This effect is in the opposite direction to the temporary effect of stress, being an increase at all the fields used ; a similar and smaller effect was observed after removing the other loads. In the case of iron there is very little difference between the value of the change of resistance in specimens of different diameters.

Experiment III.—*The Influence of Stress on the Magnetic Change of Resistance in Platinite.*

This was a specimen of nickel-steel kindly supplied by Mr. T. Agar Baugh. The wire was ·35 mm. in diameter, and was quite soft. The effect of stress on the change of resistance of this specimen is shown by the curves in fig. 5. The highest value of the change of resistance is rather

greater than in the case of iron, the curve being, however, of a very different shape. The effect of stress is to diminish the value of the change at all fields. The after-effect of stress on the change of resistance was very small and appeared to be in the same direction as the temporary effect.

The residual change of resistance was also very small, and was inappreciable when the wire was loaded.

Some measurements were carried out on the effect of stress on the magnetization of this specimen; but, owing to the small diameter of the wire, only approximate results could be obtained. The effect of stress is to increase the magnetization at low fields, and to diminish it at high fields. The residual magnetism was inappreciable. The change of resistance in two other specimens of nickel-steel was measured. The first was a sample obtained from Messrs. Felten & Guillaume. The change of resistance of this specimen was very small. The resistance was increased at low fields, but the increase reached a maximum at a field of 80, and then diminished until it became zero at a field of 400. At higher fields the resistance of the wire was diminished by magnetization. The maximum increase of resistance was $\frac{d\mathrm{R}}{\mathrm{R}} = 4\cdot5 \times 10^{-5}$, and the decrease at a field of 800 was $4\cdot9 \times 10^{-5}$.

The other specimen was obtained from Messrs. Hadfield of Sheffield, and is manufactured by them under the name of Rheostene. The change of resistance in this specimen is still smaller. The resistance is increased at all fields, and the curve is similar in form to that for Platinite. The maximum value of the change is $\frac{d\mathrm{R}}{\mathrm{R}} = 1\cdot8 \times 10^{-5}$.

Experiment IV.—*The Effect of Variations of Temperature on the Change of Resistance in Nickel.*

In all the experiments on the change of resistance at various temperatures the ends of each of the wires experimented upon were soldered to two thick copper wires which were used for connecting to the bridge. The wire was then fastened to a wooden lath, and was doubled up so as to bring the ends as close together as possible in order to avoid thermoelectric effects. The wire was heated up to a temperature of 100° C. by passing a current of steam through the jacket of the coil.

The curves in fig. 6 give the results obtained with nickel wire ·35 mm. diameter. The wire was from the same reel as the piece used in the stress experiments, but was not annealed. It will be noticed that there is a slight difference between the change of resistance at 15° C. before and after heating the wire to 100° C. Very similar results were obtained by heating a wire which had previously been annealed. The dotted curves

in fig. 6 show the results obtained by heating a nickel wire ·14 mm. diameter which had not been annealed.

Experiment V.—*The Effect of Variations of Temperature on the Change of Resistance in Iron.*

Two specimens of iron wire were tried. They were both of ·32 mm. diameter and taken from the same reel; but one had been hardened by stretching nearly to the breaking-point. The results obtained are shown in fig. 7.

Experiment VI.—*The Effect of Variations of Temperature on the Change of Resistance in Platinite.*

The wire used was the same as that used in the stress experiments, but was annealed by heating in the flame. The results obtained are shown by the dotted curve in fig. 5. The change of resistance at 100° C. exhibits a decided maximum at a field of about 350 c.g.s., the effect being smaller at higher fields.

All the experiments described above were carried out in the Physical Laboratory of the University College of North Wales; and I desire to express my best thanks to Prof. Taylor Jones for placing the necessary apparatus at my disposal, and also for much valuable help and advice given in the course of the work.

Bangor, Sept. 1903.

LXXXVI. *" Minimum Deviation through a Prism."*
By T. L. Bennett [*].

$$\text{Sin } \theta = \mu \sin \theta'. \quad \text{Sin } \phi = \mu \sin \phi'.$$
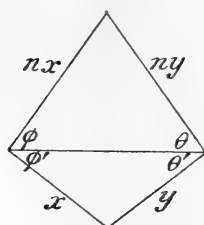
IN a note under the above title in the Philosophical Magazine for October, it is tacitly assumed that if μ and $(\theta' + \phi')$ be constants, then n is constant. This is not the case, for

$$n = \frac{\mu \sin (\theta' + \phi')}{\sin (\theta + \phi)}.$$

Thus the deduction $xy =$ maximum fails.

LXXXVII. *Relationship between Spectra and Atomic Weights.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN his papers on a relationship between spectra and atomic weights, and on the atomic weight of radium, Dr. Marshall Watts* has tried to show how it is possible to calculate the atomic weight of an element from the atomic weights of chemically related elements by means of their spectra. He first makes use of a remark due to Ramage. Ramage draws a diagram of the spectra of chemically-related elements with oscillation-frequencies as abscissæ and the squares of the atomic weights as ordinates. In this way each spectrum is represented by certain points on a parallel to the axis of abscissæ, the distance from this axis being proportional to the square of the atomic weight. Ramage now observes that in many cases the points representing *homologous* spectral lines *approximately* lie in a straight line. This law if true may of course be used to determine the atomic weight of an element from the atomic weight of two other elements as soon as three homologous lines are known in their spectra. Ramage is very careful to define what is meant by homologous†. Two spectral lines belonging to different spectra are called homologous when they show the same characteristic features. Thus we may say that the calcium line 4226 is homologous to the strontium line 4607 and to the barium line 5536. When a salt of one of these substances is brought into the flame of a Bunsen burner this is the principal line. In the arc-spectrum the three lines behave in a similar manner, they are broadened and easily reversed. In the magnetic field they are split into three rather broad components and the spacing measured in the scale of oscillation-frequencies is the same in all three cases.

It is to *homologous* lines like these that we must look in order to show the connexion between the spectra and the atomic weights.

It appears that Dr. Watts in making use of Ramage's remark for the calculation of atomic weights has not paid attention to the definition of homologous lines. He assumes lines as homologous which by their components in the magnetic field we know to be not homologous (for instance Ba 4554 and Sr 3706, or Ba 4131 and Sr 3400). It is evident that when the lines are densely distributed over some part of the spectrum any straight line drawn through this part of the spectrum in Ramage's diagram is sure to pass

* Phil. Mag. [6] vol. v. p. 203 (1903); vol. vi. p. 64 (1903).
† Proc. of the Roy. Soc. vol. lxx. p. 3 (1901).

very close to some point representing a spectral line. Therefore if no account is taken whether or no this spectral line is homologous to the spectral lines corresponding to the points where the same straight line intersects the other spectra, the whole argument falls to the ground.

Dr. Watts has secondly availed himself of a remark made by Rydberg, and by Kayser and the writer, that the differences of oscillation-frequencies of homologous doublets are approximately proportional to the square of the atomic weight. But here, as in the first case, he does not keep to homologous lines.

By neglecting this restriction, duly insisted on by Ramage and by the other authors, it is possible to make the atomic weight of a substance come out anything we please. There is no difficulty for instance in making the atomic weight of radium come out equal to 100. We need only draw the spectrum of radium in Ramage's diagram in the parallel corresponding to the atomic weight 100, then choose a point in the radium spectrum and a point in the strontium spectrum such that the straight line connecting them when prolonged intersects the spectrum of barium in some part where the spectral lines lie close. Then we take the nearest point representing a spectral line of barium. In this manner we get three lines and have only to assume that they are homologous to make out 100 as the atomic weight of radium. In a similar way doublets may be found in the spectrum of radium and the spectrum of, say, mercury, from which we make out 100 as the atomic weight of radium.

Take for instance the following lines :—

	Sr.		Ra.		Ba.	
	λ.	1/λ.	λ.	1/λ.	λ.	1/λ.
(1)	4892·20	20440·7	5081·20	19680·4	5965·06	16764
(2)	4784·43	20901·1	4971·88	20113·1	5853·91	17083
(3)	4729·93	21142·0	4918·61	20330·9	5805·86	17224
(4)	4678·39	21374·9	4826·12	20720·6	5473·94	18268
(5)	4338·00	23052·1	4533·33	22058·9	5473·94	18268
(6)	4729·93	21142·0	4837·59	20671·4	5294·40	18888

From the atomic weight 137·4 of barium and 87·6 of strontium we find for the atomic weight of radium according to Dr. Watts' first method of calculation :—

$$
\begin{aligned}
(1) &\quad 99{\cdot}95 \\
(2) &\quad 99{\cdot}93 \\
(3) &\quad 99{\cdot}97 \\
(4) &\quad 100{\cdot}17 \\
(5) &\quad 100{\cdot}00 \\
(6) &\quad 100{\cdot}07
\end{aligned}
$$

Mean .... 100·015

Again, if we assume the following doublets as homologous :

	Ra.		Hg.	
	λ.	1/λ.	λ.	1/λ.
(1)	4826	20720·6	5819	17184·9
	4436	22540·4	4078	24521·5
(2)	4826	20720·6	5819	17184·9
	4341	23037·1	3790	26382·7
(3)	3815	26215·2	3544	28219·5
	3650	27399·2	3039	32908·9
(4)	4682	21356 8	4916	20340·0
	4436	22540·4	3984	25099·9

We get by Dr. Watts' second method :—

$$
\begin{array}{ll}
(1) & 99\cdot6 \\
(2) & 100\cdot4 \\
(3) & 100\cdot5 \\
(4) & 99\cdot7 \\
\hline
\end{array}
$$

Mean .... 100·05

Instead of 100 I might just as well have chosen any other value between say 0 and 300, showing the method to be entirely futile.

Had Dr. Watts confined his calculation of the atomic weight of radium to lines that we know to be homologous *, he would have found that the two laws enunciated by him are far from being fulfilled in the case of radium and the related elements. The first law cannot even be said to be approximately true for the following homologous lines :—

Mg.	Ca.	Sr.	Ba.	Ra.
2803	3969	4216	4934	4682
2796	3934	4078	4554	3815
2791	3159	3381	3892	3650

For if it were true the radium lines ought surely to lie on the less refrangible side of the barium lines, while in reality one of the radium lines is even more refrangible than the homologous line of calcium. As to the second law the differences of the oscillation-frequencies of homologous doublets are :—

* See the paper by J. Precht and the writer, *Physikal. Zeitschrift,* 4 Jahrgang, p. 285 (1903) ; Phil. Mag. [6] vol. v. p. 476 (1903).

	Atomic Weight.	Difference of Oscillation-Frequencies.
Mg	24·36	91·7
Ca	40·1	223
Sr	87·6	801
Ba	137·4	1691
Ra	..	4858

Dr. Watts' law makes the atomic weight of radium equal to

177 if calculated from Mg,
187   ,,   ,,   Ca,
216   ,,   ,,   Sr,
233   ,,   ,,   Ba.

The law found by Precht and the writer, that the differences of the oscillation-frequencies of homologous doublets are proportional to some power of the atomic weight, in other words that the logarithm of the difference is a linear function of the logarithm of the atomic weight, seems to furnish a better means to calculate the atomic weight of radium from its spectrum. It makes the atomic weight of radium 258. Whether this value or the one found by Madame Curie, 225, be the right one, is as yet open to discussion.

Hannover Technische Hochschule,   C. Runge.
        October 1903.

LXXXVIII. *Radium Radiation and Contact Electricity.* By Lord Blythswood, *LL.D.,* and H. S. Allen, *M.A., B.Sc.*[*]

WHEN the air between two insulated plates, of different metals, is subject to the influence of a radioactive substance, similar wires connected with the plates acquire a difference of potential. This difference may be measured by joining the plates to the quadrants of a sensitive electrometer. It is of the same order of magnitude as that which would be obtained by connecting the metal plates with a drop of water.

This effect was first observed in air under the radioactive influence of a disk of the metal uranium by Lord Kelvin,

J. C. Beattie, and M. S. de Smolan *. It has also been investigated by E. Rutherford †, who found that when the plates were charged from an external source " in consequence of this action, for small electromotive forces the rates of leak are different for positive and negative."

An exactly similar effect produced by the action of X-rays has been studied by Erskine Murray ‡ and Perrin §.

The object of the experiments described in the present paper was to measure this potential-difference for various metallic couples, employing a radium salt as the source of radiation.

Preliminary Experiments.

Some preliminary experiments were made with the apparatus illustrated in fig. 1. A is a horizontal brass disk 5 cms.

Fig. 1.
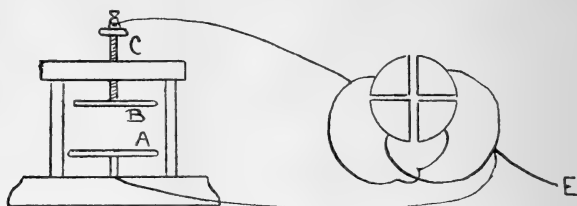
in diameter connected to one pair of quadrants of a Kelvin electrometer. B is a similar disk whose distance from A can be adjusted by the screw C. It is supported by varnished glass rods, and is in metallic connexion with the insulated quadrants of the electrometer. This plate was coated with a solution of radium salt of no great activity, and the solution was allowed to evaporate.

In making an experiment the metal plate to be tested was carefully polished and laid upon the lower brass plate. The apparatus was then inclosed in a case of lead. This cover and all the quadrants were connected to " earth." An observation was made of the deflexion of the electrometer obtained by disconnecting the upper plate from " earth." The results for various metals are given in Table I.

  * Phil. Mag. vol. xlv. p. 277 (1898), vol. xlvi. p. 115 (1898).
  † Phil. Mag. vol. xlvii. p. 155 (1899).
  ‡ Proc. Roy. Soc. March 1896.
  § C. R. vol. cxxiii. p. 496.

TABLE I.

Deflexion for Clark Cell $+145, -160$.

Metal on Lower Plate.	Potential of Brass Plate.	
	Deflexion.	Volts.
Aluminium	$+40$	$+\cdot40$
Zinc	$+35$	$+\cdot35$
Lead	$+32$	$+\cdot32$
Copper	-2	$-\cdot02$
Silver (five-shilling piece)	-7	$-\cdot06$
Gold	-13	$-\cdot12$
Mercury (in a watch-glass)	-20	$-\cdot18$
Platinum	-23	$-\cdot21$

As was to be expected, it was found that the deflexion observed was dependent on the state of the metal surface. Thus an aluminium plate showed a deflexion of 10 after polishing with the "buff," which would leave traces of grease on the surface, but a deflexion of 40 after it was cleaned with fine emery-paper.

Potential-Differences obtained with a Lead Plate.

Several observations made during our preliminary experiments showed the importance of limiting the effects to the two metals it was proposed to examine. As it was necessary to inclose the apparatus and the radium salt in a lead case to prevent interference from external sources, it was decided to make a new apparatus to consist wholly of lead. In this way all the subsequent determinations would measure the potential-difference between lead and any one metal. This had a further advantage from the fact that lead lies at one end of the voltaic series, so that by employing it as a standard larger deflexions of the electrometer could be obtained. The new apparatus is shown in fig. 2 (p. 704).

The lead plates were circular disks with faces 5 cms. in diameter. They were put in connexion with the electrometer-quadrants by lead rods passing through glass tubes. The distance between them was 2 cms. The radium was contained in a small lead box which could be slipped into a tube at the side of the case. The sample used was obtained through Isenthal and Co. It was considerably more active than that used in the preliminary experiments. For further security

the whole apparatus was placed in a large box lined with lead. A long wooden lever was arranged so that contact between

Fig. 2.
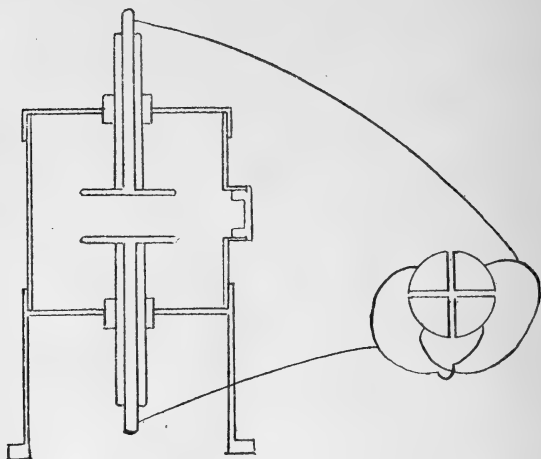
the insulated quadrant and "earth" might be made or broken without the observer approaching the apparatus or electrometer.

In the first experiments made with this apparatus the radium was protected by a loosely fitting cover of thin mica. This allowed the "emanation" from the radium to enter the testing-vessel, so that a potential-difference continued to show itself after the radium was removed. In order to do away with any uncertainty in the measurements arising from the presence of the emanation with the accompanying excited activity, the box containing the radium was hermetically sealed. The cover was made of aluminium-foil, through which the radiation (β and γ rays) could penetrate. In this way a constant source of radiation was secured.

When making an observation a plate of the metal to be tested was placed upon the lower electrode. Two readings of the electrometer were taken, the first giving the potential of the upper electrode when the lower was put to earth, the second that of the lower electrode when the upper was put to earth. The second reading was always about twice as great as the first, the sign of the deflexion being, of course, reversed. The smaller reading in the first case is due to the fact that the testing-vessel (which was always connected to earth) being of the same material as the electrode receives part of the ionization current.

The results of these experiments are given in Table II. When careful attention was paid to polishing the metal surfaces, readings taken on different days only differed by two or three divisions.

TABLE II.

Deflexion for Clark Cell, +125, −125.

Metal on Lower Plate.	Potential of Upper Plate.		Potential of Lower Plate.	
	Deflexion.	Volts.	Deflexion.	Volts.
Zinc	+ 5	+·05	−13	−·15
Aluminium	+ 5	+·05	−12	−·14
Tin............................	− 5	−·05	+ 6	+·07
Bismuth	− 6	−·07	+10	+·11
Antimony	− 8	−·09	+14	+·16
Brass	−11	−·12	+20	+·23
Iron	−12	−·14	+20	+·23
Copper	−13	−·15	+22	+·25
Silver.........................	−17	−·20	+29	+·33
Gold	−17	−·20	+31	+·36
Platinum	−21	−·24	+35	+·40

It would doubtless have been better if the lower electrode had been composed wholly of the metal to be examined, but this would have restricted the number of metals that could be used. The simpler plan of laying the metal plate on the lead electrode gives a smaller potential-difference, but could scarcely affect the relative order in the voltaic series. In order to observe the increased potential in one case an electrode was prepared of the same size as the lead electrode, but consisting wholly of brass. The potential-difference in this case amounted to ·32 volt, as compared with ·23 volt when a brass plate was placed on the lead electrode.

The plates used in these experiments were of commercial metal, and therefore not chemically pure.

An inspection of the results in Table II. will show that the metals fall into their places in the voltaic series, as given for example by Ayrton and Perry (Phil. Trans. Roy. Soc. 1880, p. 15). The only difference is in the case of brass, which we find between iron and antimony, but this may well be due to a difference in composition.

Potential-Difference in an Exhausted Vessel.

It appeared to be of interest to determine whether there would be any alteration in the potential-difference between

two metals under the influence of radium radiation when they were placed in an exhausted vessel.

Measurements of contact difference of potential in high vacua have been made by Dr. J. T. Bottomley * and also by Mr. F. S. Spiers †.

Fig. 3.

The former measured the potential-difference between zinc and copper at a pressure of less than 1/523 mm. of mercury, and found that when air was admitted to atmospheric pressure the potential-difference did not alter by so much as 1 per cent. Mr. Spiers, after a series of careful experiments, in which the pressure was reduced to 1/10000 mm. of mercury, came to the conclusion that no mere mechanical pumping is capable of removing "the insoaked condensed air-films from the surfaces of such highly oxidizable metals as zinc and aluminium," to the presence of which he attributes the volta-effect ‡.

In view of the results obtained by these experimenters it is scarcely surprising that the potential-difference under the influence of radium radiation showed no variation as great as 5 per cent. when air to atmospheric pressure was admitted to a vessel previously exhausted to 1/7000 mm. of mercury.

The experiment was carried out with a "vacuum tube," designed for the purpose, and constructed by Casella. One electrode was of platinum, the other of aluminium. It was found that the heat required to seal the aluminium electrode into the tube was sufficient to oxidize the surface and completely destroy the volta effect. The tube was therefore made as shown in fig. 3, with the aluminium electrode attached to a ground-in stopper, so that it could be removed and cleaned before use. The joint
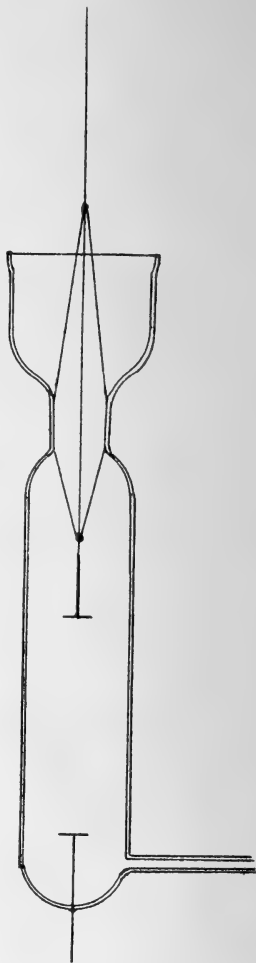
* B. A. Report, 1885.
† Phil. Mag. vol. xlix. p. 40 (1900).
‡ The volta-effect would be diminished by "insoaked condensed air-films" if present on zinc or aluminium surfaces.—KELVIN.

was made tight by a mercury seal. The tube, with the radium beside it, was placed in a large box lined with lead.

In making the experiments the tube was sealed to a three-fall Sprengel pump and exhausted, being gently warmed at intervals during the process. When the McLeod gauge showed a high vacuum had been obtained dry air was slowly admitted through a series of drying-tubes until atmospheric pressure was reached. The glass tube giving connexion with the external air was then sealed off and observations of the potential-difference were commenced. The pump was started and kept working for about six hours, readings of the electrometer being taken from time to time. When the highest vacuum attainable had been reached dry air was once more admitted to the testing-vessel, but no alteration in the deflexion could be perceived. The series of operations was repeated with the same result. It was estimated that an alteration of 5 per cent. could be detected.

In conclusion we must express our great indebtedness to Lord Kelvin and Dr. J. T. Bottomley for their interest and advice in the course of these experiments.

Blythswood Laboratory,
Renfrew, N.B.

LXXXIX. *On the Application of Alternating Currents to the Calibration of Capacity-boxes, and to the Comparison of Capacities and Inductances.* By W. STROUD, *M.A.*, *D.Sc.*, *Cavendish Professor of Physics*, and J. H. OATES, *B.Sc.*, *Assistant-Demonstrator in the Physical Laboratory at the Yorkshire College, Leeds*[*].

IN the May number of the Philosophical Magazine for the current year there appeared a paper by Prof. Fleming and Mr. Clinton upon the measurement of small capacities and inductances. By means of a substantially built and rapidly revolving commutator and a special form of movable double-coil galvanometer, capacities amounting to only a few micro-microfarads and inductances amounting to only a few millihenries are measured. For some time prior to the publication of this paper we had been engaged upon the solution of a similar problem, viz. the accurate comparison of a capacity with either an inductance or a second capacity; we had not, however, any idea of measuring such very small capacities as those dealt with by Prof. Fleming's method, neither had we any idea of the practical importance of such

measurements. Within the last few months, however, we have succeeded in increasing the sensitiveness of our apparatus to such an extent that we can detect a capacity of say one-third of a micro-microfarad, and we can measure with certainty a small inductance to at least two microhenries, *i. e.* one five-hundredth of a millihenry.

In connexion with this question of the detection of a capacity amounting to a fraction of a M.M.F. (1 M.M.F.= 1 micro-microfarad=0·9 electrostatic unit), it should be remembered that J. J. Thomson and Searle* and E. Rosa† succeeded in detecting very small capacities in their respective determinations of "*v.*" These experimenters were, however, using very sensitive high resistance galvanometers.

Our solution of the problem is in some respects simpler than that of Prof. Fleming. As to sensitiveness our apparatus is probably not quite so sensitive for the measurement of very small capacities, whereas for the measurement of small inductances it is, we think, superior. On the other hand, our apparatus is not so suitable, in its present form at all events, for effecting the accurate measurement of "*v.*"

The essential feature of our method consists in the employment of what is substantially an electrodynamometer with laminated iron cores, or it may be briefly described as a movable-coil D'Arsonval galvanometer, in which the permanent magnet is replaced by an electromagnet, with a laminated iron core, actuated by a 100-volt alternating current. The solid iron core inside the moving coil of the ordinary D'Arsonval is in our instrument also replaced by a laminated one.

Donle ‡ has used a Bellati-Giltay electrodynamometer for the measurement of specific inductive capacities in an entirely different way. The oscillating current from an induction-coil (without iron core) was passed through a condenser (with adjustable plates) joined in series with the electrodynamometer and the deflexion was observed. On inserting a parallel slab of the dielectric between the plates and readjusting their positions this deflexion was approximately reproduced.

Our instrument is used just like an electrodynamometer when it is arranged for measuring the conductivity of electrolytes, *i. e.* the field-magnet is placed across the mains, and the highly insulated movable coil is used to replace the galvanometer in the bridge. We have found this piece of

* Phil. Trans. vol. clxxxi. A (1890).   † Phil. Mag. Oct. 1889.
‡ Wied. *Ann.* xl. p. 307 (1890).

apparatus of great utility (1) where there is no difference in phase between the currents in the two arms of the bridge, and (2) where the phases of the current in the moving coil and of the magnetic field are far removed from quadrature.

It is very important in connexion with the use of this instrument that the ends of the movable coil should be permanently attached to the bridge—to use Wheatstone bridge terminology the key must be in the battery-circuit and not in the galvanometer circuit, in fact it is absolutely essential that there should be no key in the galvanometer circuit. The reason for this is as follows :—The " false-zero " position of the movable coil, *i. e.* the position it takes up when the alternating current is magnetizing the field, is a function of several things, (1) its zero position for no alternating field, (2) the intensity of the alternating field, (3) the shape, size, and conductivity of the metal frame on which the coil is wound, (4) the resistance and capacity in circuit as a shunt on the movable coil. Thus, for example, suppose the field is magnetized by an alternating current and that we join up the terminals of the movable coil to a resistance-box, we shall probably find that the " false-zero " position depends on the resistance ; we shall certainly find that this is the case if we twist the head, which supports the strip carrying the coil, so that the latter is in an asymmetrical position in the field. If the coil be short-circuited we find that it takes up a position of great stability in what would be magnetically a symmetrical position were it not for the torsional effect of the strip ; as it is, the position taken up is intermediate between its original zero position before the magnetization of the field and the position it would take up if there were no torsional rigidity in the strip. This being so it is clear that any increase in resistance in the short-circuiting box will have the effect of diminishing the deflexion due to the alternating field, and so the " false-zero " position of the coil is a function of the resistances when that coil is used as a galvanometer in an ordinary Wheatstone's bridge circuit. The apparent vagaries due to this " false-zero " bothered us a good deal at one time, although no doubt we ought to have foreseen all these points.

A number of experiments were made upon the positions of equilibrium of copper frames of various shapes and sizes when suspended in the alternating field, and although there is nothing new in any of these results (all of them being entirely in conformity with Elihu Thomson's experiments on the repulsion produced by alternating currents), yet it may be well to reproduce some of them so that if the instrument

we are describing ever comes into use in physical laboratories its behaviour may be better understood. In describing these experiments we shall speak of a frame which sets itself so as to include the maximum or minimum number of lines of forces as being equatorially or axially stable respectively.

Experiment 1.—A thin copper wire (No. 24) with its ends joined so as to form a rectangle was axially very stable as we expected. The equatorial position was also one of stability, though to a much less extent.

Experiment 2.—A similar wire with the ends insulated from one another was equatorially stable to a small extent, axially unstable.

Experiments 3 & 4.—Precisely similar results were obtained with a thicker copper wire (No. 16) bent into a similar shape.

Experiment 5.—A rectangular frame was made out of a strip of thin sheet copper 16 cms. long and 2 cms. wide with the ends soldered together. This proved to be axially very unstable, equatorially very stable.

Experiment 6.—This frame behaved in just the same manner when the ends were insulated from one another.

Experiment 7.—A D'Arsonval coil with the ends joined showed great axial stability (as of course was expected).

The explanation of the behaviour of the frames in these experiments is very simple. As is well known a coil of wire, with the ends joined, in an alternating field sets itself so that there is minimum magnetic induction through it, *i. e.* axially (Expt. 7). Continuous metal sheets in which the induced currents can freely circulate are vigorously repelled (Expts. 5 & 6), *i. e.* set equatorially. Thus it will be seen that the one effect or the other may predominate according to the shape of the frame.

When we originally devised the method here described, as it happened we had in our possession a D'Arsonval galvanometer with a field produced by an electromagnet. It was easy to try the method with this instrument (afterwards referred to as A) by simply substituting alternating for continuous currents in the coils surrounding the field-magnets. We were very well pleased with the results, especially considering the very feeble magnetization of the solid soft-iron core. We next had an instrument made with laminated iron cores. The electromagnet was built up of soft charcoal-iron stampings, 0·3 mm. thick, in the shape of a hollow rectangle—one short side being missing—21·6 cms. long by 8·9 cms. wide. The limbs were 2·5 cms. broad, and the stampings were first varnished and then piled up to a

thickness of 2·5 cms. The lower part of each long limb was surrounded by a wooden frame 12·7 cms. long, on which were wound 370 turns of No. 20 copper wire. A brass cross-piece supported from the poles of the electromagnet carried an insulated torsion-head, from which was suspended the moving coil, which swung between the poles of the electromagnet.

We fully anticipated that the movable coil (especially if wound on a copper frame) would exhibit great axial stability, and consequently very little sensitiveness. To overcome this defect we originally tried supporting the coil by a crossed bifilar so as to partially neutralize the axial stability by super-posing gravitational instability, but the result was far from successful. We then tried the effect of increasing the accuracy of reading without attempting to reduce the axial stability by attaching a pointer to the coil. Horizontal rigidity was imparted to the pointer by making it of a light aluminium wire 30 cms. long, bent in the middle at a very acute angle so as to form a V. This was lightly latticed across the middle to increase the rigidity. The two free ends of the V were attached in a horizontal plane to the upper part of the coil, and the weight suitably counterpoised. The acute end of the V, to which was attached a suitable mark, could be examined by a microscope, magnifying 16 diameters, placed with its axis horizontal, although we should have preferred a vertical arrangement of axis if that had been equally easy to attain. After considerable practice with this method of reading the position of a movable coil, we much prefer it for laboratory purposes to the ordinary mirror-and-scale arrange-ment. For rough work the pointer can be viewed directly in full daylight without the microscope, the latter only being used for the final adjustment of the balancing resistances. However, the necessity for using this method of indication has now disappeared, for we have by accident discovered a method of adjusting the sensitiveness with the utmost ease and convenience. If the moving coil is wrapped on a closed copper frame of appropriate breadth, this frame will possess axial instability in the alternating field (see expt. 5 above), whereas the movable coil, if short-circuited, would possess immense axial stability (see expt. 7); so that if we shunt the movable coil through a variable resistance, in all cases many times its own, we can get any degree of stability desired. We are thus in a position to employ the mirror method of reading the deflexions if we so desire. We are, however, ourselves convinced that the mirror method is a mistake in many cases where the movable system is heavy

or of a large size.   We are, of course, aware of the evil effects of convection-currents upon such long pointers ; in fact, in one form of movable coil (which we shall refer to as B), consisting of 211 turns of No. 38 copper wire—resistance = 30 ohms—wound on an aluminium rectangular frame 5·5 cms. × 2·5 cms. and 0·6 cm. broad, these convection-currents proved a great nuisance.   This effect, however, was almost entirely eliminated by surrounding the pointer by a fixed brass tube which had been flattened very considerably. The mark on the pointer only just protruded through the open end of this tube.   In a later form of coil (referred to as C), consisting of 1508 turns of No. 42 copper wire—resistance = 485 ohms—wound on a copper frame 5·5 cms. × 2·5 cms. and 2 cms. broad, we can observe very well without this protecting tube.   However, in a final form of our instrument we hope to have the magnetizing coils entirely outside the case supporting the movable parts.   It would, moreover, be desirable to have the torsion-head which supports the coil readily accessible from the outside.

We shall consider (i.) the comparison of capacities of the order of from 10^{-3} to 1 microfarad ; (ii.) the comparison of capacities of the order of a few M.M.F.'s ; (iii.) the comparison of an inductance of the order of a henry with a capacity ; and (iv.) the measurement of an inductance of the order of a few microhenries.

Comparison of Capacities and Calibration of Capacity-box.— The method we have used is the well-known one of De Sauty, the alternating current at 100 volts, from the Leeds

Fig. 1.
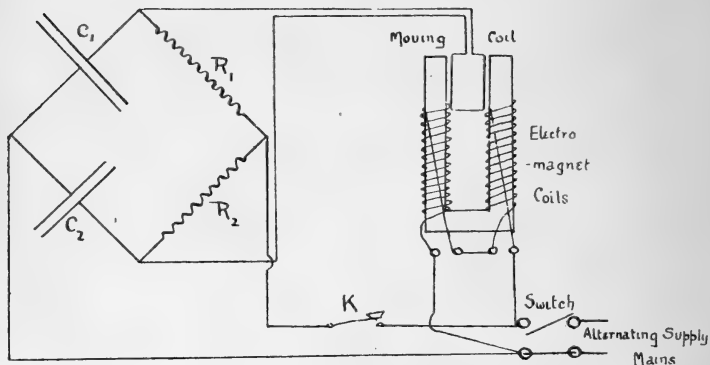
mains, being supplied to the bridge as shown in fig. 1.   It will be seen that as there is a capacity in each limb of the bridge, the only currents traversing the high resistances R_1

and R_2 are those required for charging the condensers ; the heating-effects of these currents are comparatively small, especially as the key, K, supplying the bridge is only momentarily depressed. The condition for a balance in De Sauty's method is that $C_1 R_1 = C_2 R_2$. Now the tangents of the angle of the lag in the two limbs of the bridge are $\left(L_1 p - \dfrac{1}{C_1 p}\right)\Big/ R_1$ and $\left(L_2 p - \dfrac{1}{C_2 p}\right)\Big/ R_2$, and since L_1 and L_2 are both approximately zero, the lags become $\tan^{-1}\left(-\dfrac{1}{C_1 R_1 p}\right)$ and $\tan^{-1}\left(-\dfrac{1}{C_2 R_2 p}\right)$, so that when the condition $C_1 R_1 = C_2 R_2$ is satisfied the lags are precisely the same.

We had in our possession a capacity-box, Elliott No. 65, consisting of five capacities labelled ·5, ·2 A, ·2 B, ·05 A, and ·05 B. These capacities could be divided at any junction into two groups, the two groups being in series and the members of each group being joined in parallel ; they were also capable of being used singly or together. We took the capacity labelled ·5 as having a true capacity of 0·5 M.F., and compared each of the other capacities and various combinations of them with it, and obtained the results given in the following table.

TABLE I.

Results obtained using Coil B. $C_1 = 0.5$ M.F.

Capacity tested.	R_1.	R_2.	$C_2 = C_1 \cdot \dfrac{R_1}{R_2}$ M.F.'s.	Calculated values of C_2.
·05 B	1970·3*	19738	·049912	Assumed
·05 A	1970·3	19722	·049952	,,
·2 B	985·2	2483·6	·19834	,,
·2 A	985·2	2492·7	·19762	,,
·05 B + ·05 A	1970·3	9869·6	·09982	·09986
·2 B + ·05 B	985·2	1984·1	·24827	·24825
·2 B + ·05 A	985·2	1983·9	·24830	·24829
·2 A + ·05 B	985·2	1989·6	·24759	·24753
·2 A + ·05 A	985·2	1989·1	·24765	·24757
·2 B + ·05 A + ·05 B	1970·3	3304·2	·29816	·29820
·2 A + ·05 A + ·05 B	1970·3	3310·9	·29755	·29748
·2 A + ·2 B	1970·3	2487·0	·39613	·39596
·2 A + ·2 B + ·05 A + ·05 B	1970·3	1986·2	·49601	·49582

* The decimal points are due to the fact that we have been using a B.A. box. This box has been carefully calibrated, and the results expressed in legal ohms.

The sensitiveness of adjustment in these observations amounts to 5 M.M.F.'s, nevertheless we find that the ratio of two capacities obtained on different days is nothing like so consistent as the sensitiveness would lead us to infer that it

should be.   We have been driven to the conclusion that the
ratio of certain pairs of our capacities is a function of the
temperature.   This may well be the case when we consider
that the capacities are finally adjusted by squeezing.   The
result of a change in temperature may thus produce much
greater effects due to stress in the case of one capacity than
in that of another.   This notion is confirmed by the following
very rough experiments on the effect of change of tem-
perature on the ratio of two capacities, the results of which
are given in Table II.   The experiments would have been
done much more carefully if it had been possible to have
measured the temperature of the condensers, but no provision
had been made in the box for the insertion of a thermometer.
We propose to investigate this point later.

TABLE II.

Remarks.	C_1.	C_2.	R_2.
	0·5	·05 B	100374
Left current on for five minutes and tested..................................	...	...	100400
Left and tested again	...	...	100360
Warmed the box up in front of a fire and tested again	...	...	100370
Warmed again and tested	...	...	100390
,, ,, ...	...	...	100420
,, ,, ...	...	...	100450
,, ,, ...	...	...	100450
Tested again next morning	...	...	100394

These results seem to show that the ratio of the two
capacities compared decreases as the temperature rises.   If
it be true that the ratio of two capacities varies with tem-
parature, it follows that a capacity must have quite a
sensible temperature-coefficient.

We also had in our possession a Kelvin air-leyden made
by White and engraved as ·0025 M.F.   This we compared
with the Elliott-box with the following results.

TABLE III.

C_1.	C_2.	R_2.	R_1.	$C_1 = C_2 \cdot \dfrac{R_2}{R_1}$ M.F.'s.
The Kelvin air-leyden.	The Elliott Box totalling 1 M.F.	197·05	79703	·0024722
		206·90	83689	·0024723
		216·74	87660	·0024725
		226·58	91681	·0024714
		236·42	95654	·0024716
		246·27	99595	·0024727
		256·11	103624	·0024715
		265·95	107625	·0024711
		275·80	111636	·0024706

An ordinary "quart" leyden-jar was compared with the air-leyden with the following results.

TABLE IV.

C_1.	C_2.	R_2.	R_1.	$C_1 = C_2 \cdot \dfrac{R_2}{R_1}$ M.F.'s.
A "Quart" leyden-jar.	The Kelvin air-leyden $= \frac{1}{400}$ M.F.	5030	9000	·001119 M.F.'s.

It will thus be seen that two leyden-jars can be compared with an accuracy approaching $\frac{1}{10}$ per cent.

By constructing an air-condenser, preferably in the form of two concentric tubes, as recommended by Prof. Fleming, and comparing its capacity with that of a standard condenser, first with air as the dielectric, and secondly with another medium replacing the air in the condenser, it is possible to find the specific inductive capacity of that medium.

It will, we think, be admitted that the method of comparing capacities described is one of great convenience and accuracy. It is as easy to carry out as an ordinary Wheatstone's-bridge measurement of resistance, and now that alternating currents are so easily procurable, we hope that the method will be found useful in the workshop as well as in the laboratory.

Comparison of Small Capacities.—For capacities of the order of a few M.M.F.'s the Wheatstone's bridge is not so suitable. In this case it is preferable to join the capacity up in series with the moving coil, and to put the whole pressure on the coil. To get increased sensitiveness we constructed the coil C, described above. A very rough air-condenser was made of two sheets of tin plate 10 cms. × 10 cms., separated by three small pieces of ebonite 0·5 cm. thick. The capacity of this is approximately 18·8 M.M.F.'s. The coil was deflected through an angle equivalent to 6° after magnification in the microscope. The same tin plates separated by ebonite 1·3 cm. thick (capacity = 7·1 M.M.F.'s) gave a deflexion equivalent to 2°·4. It was estimated that $\frac{1}{24}$ of this, *i. e.* 0·3 M.M.F., would have produced an observable effect. In this case the lag in the current expressed by $\tan^{-1}\left(Lp - \dfrac{1}{Cp}\right)\Big/R$, is almost exactly $-\pi/2$, for though L, the self-induction of the moving coil, is quite appreciable, yet its effect is completely drowned by the exceedingly large value of $\dfrac{1}{C}$.

Up to the present we have carried out only the roughest measurements of these very small capacities. For quantitative experiments we propose to construct a small air-condenser of variable capacity, so as to reproduce the deflexion.

Comparison of Capacities with Inductances of the Order of a few millihenries and upwards.—The arrangement we adopt is founded on that described by Prof. Anderson[*].

On the publication of Prof. Anderson's paper one of us saw that the method could be improved by placing the high resistance " r "[†] in the battery-circuit instead of in the galvanometer-circuit. This resistance " r " is frequently of the order of 2000 or 3000 ohms; and when joined in series with an ordinary high-resistance D'Arsonval in the galvanometer-circuit, it reduces its sensitiveness very considerably; whereas when placed as shown in fig. 2, the effect of its resistance can be annulled by increasing the potential of the battery, the currents in the arms of the bridge remaining the same as before. This modification of Anderson's method has been in use at the Yorkshire College for ten years. The relation between L and C may be proved to be

$$ \mathrm{L} = \mathrm{CP} \left\{ \frac{r \cdot \mathrm{S}}{\mathrm{Q}} + \mathrm{S} + r \right\}. $$

The old method of working an experiment with a battery is as follows:—The bridge is balanced for steady currents, then the galvanometer-key is closed and the battery-key is made or broken, whilst " r " is adjusted till there is no induction-throw. The galvanometer we use is a 6600-ohm Elliott pattern astatic instrument. The pressure is about 17 volts, of which from 10 to 50 per cent. are applied to the terminals d and e of the bridge.

While getting the balance for steady currents it is convenient to join a carbon resistance-box in series with the inductance. This carbon rheostat consists of 20 or 30 pieces of ordinary battery-carbon which can be compressed by a screw. It is preferable to put this adjustable resistance in the arm $e\,g$ and not in the arm $e\,f$, as then no measurement of this carbon resistance is required. As the inductances are usually made of copper with a considerable temperature-coefficient, it is continually necessary to verify the steady current-balance. If much difficulty is experienced in getting the steady balance, it is frequently well to bank up the arms P and R with non-inductive resistances.

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* Phil. Mag. vol xxi. p. 329 (1891), or Electrician, vol. xxvii. p. 10.
† Fleming and Clinton, Phil. Mag. May 1903, Plate I.

The following results were obtained by this modification of Anderson's method for a copper coil which we have been using throughout these experiments.

<div align="center">TABLE V.</div>

	C.	r.	L.
S=Q=1000, and P=130 B.A. ohms, throughout. C is the nominal value of the capacity. The value of L is given corrected for the true value of C and for S, Q, r, and P being in B.A. ohms.	$\cdot1 \times 10^{-6}$ farads	17000	$\cdot442$ henries
	$\cdot2 \times 10^{-6}$,,	8300	$\cdot441$,,
	$\cdot3 \times 10^{-6}$,,	5340	$\cdot440$,,
	$\cdot4 \times 10^{-6}$,,	3900	$\cdot440$,,
	$\cdot5 \times 10^{-6}$,,	2960	$\cdot439$,,
	$\cdot6 \times 10^{-6}$,,	2400	$\cdot440$,,
	$\cdot7 \times 10^{-6}$,,	1990	$\cdot441$,,
	$\cdot8 \times 10^{-6}$,,	1680	$\cdot441$,,
	$\cdot9 \times 10^{-6}$,,	1440	$\cdot440$,,
	$1\cdot0 \times 10^{-6}$,,	1240	$\cdot439$,,

The sensitiveness of adjustment is about 1 in 250, or ·4 per cent., in these experiments.

In what follows it is necessary to remember that we are no longer using a sensitive high-resistance galvanometer, but a moving coil instrument of from 30 to 500 ohms' resistance. On the other hand, we are using a much higher pressure, viz. 100 volts., of which from 10 to 50 per cent. is applied to the high-resistance coils forming the bridge. This pressure is only applied for a very small fraction of a second at a time during the act of testing. It is of course necessary at first to balance the bridge for direct currents as before, and for this purpose the magnetic field in the galvanometer must be generated by a direct current. Thus it is necessary to supply both direct and alternating currents to both the bridge and the field-magnet coils. Arrangements were made as shown diagrammatically in fig. 2.

A series of experiments were performed with instrument A, the sensitiveness of adjustment being approximately 1 in 4000, or ·025 per cent.

A second series of experiments were performed, using the coil B on the new instrument, the results of which are given below in Table VI. (p. 718). The sensitiveness here was about 1 in 8000, or ·012 per cent.

Measurement of very small Inductances.—The foregoing experiments show that it is possible to measure an inductance with certainty to something well under $\frac{1}{10}$ of a millihenry.

TABLE VI.

	Capacity.	r.	L.
	·05 B	33678	·44355 henries
S=985·2 ohms.	·05 A	33648	·44354 ,,
Q=984·7 ohms and	·05 B + ·05 A	16563	·44275 ,,
P=130 ohms.	·2 B	8066	·44146 ,,
All values of resist-	·2 A	8091	·44113 ,,
ances are given in	·2 A + ·05 A + ·05 B	5209·0	·44116 ,,
legal ohms.	·2 B + ·05 A + ·05 B	5197·0	·44118 ,,
The capacity used	·2 A + ·2 B	3788·5	·44101 ,,
was the Elliott box	·5	2911·8	·44266 ,,
and the names of	·2 A + ·2 B + ·05 A + ·05 B	2930·0	·44137 ,,
the various plugs	·5 + ·05 A + ·05 B	2337·0	·44146 ,,
are used to denote	·5 + ·2 A	1941·1	·44157 ,,
the capacities.	·5 + ·2 B	1938·6	·44154 ,,
	·5 + ·2 A + ·05 A + ·05 B	1635·9	·44152 ,,
	·5 + ·2 B + ·05 A + ·05 B	1634·1	·44149 ,,
	·5 + ·2 A + ·2 B	1400·6	·44125 ,,
	·5 + ·2 A + ·2 B + ·05 A + ·05 B	1211·2	·44139 ,,

We therefore decided to see what could be done in the way of measuring a very small inductance indeed. We took a coil of copper wire, having 4 layers of 80 turns each, the resistance being 0·2 ohm. The length of the coil was 16·6 cms. and the mean diameter 2·1 cms. We can roughly calculate a superior value for the inductance of this, by assuming the field uniform and neglecting the effect of the ends. We have $L = 4\pi N^2 A/l$, where N represents the total number of turns, l the length of the coil, and A the equivalent area. N and l could be accurately obtained, but A may easily have been ± 10 per cent. wrong. Substituting in the above formula we obtain $L \doteqdot 268$ microhenries.

This inductance was measured as described above, but with this modification, which is, we think, an improvement when the greatest possible accuracy is desired. A balance was obtained with steady currents on a sensitive bridge by varying a high resistance S used as a shunt to the coil C. A piece of copper wire W of the same cross-section as C and of slightly longer length was taken, and by means of a rock-over switch could be readily substituted for C. The length of W, about 30 feet, was then carefully adjusted until it produced precisely the same (almost zero) effect as C itself. This length was then (while still connected to the switch) wound non-inductively and then placed in contact with the coil. The shunt S was now removed and the rock-over switch together with C and W were transferred to the inductance-bridge and placed

in the arm R (see fig. 2). We are now in a position to
dispense altogether with the steady currents in the inductance

Fig. 2.
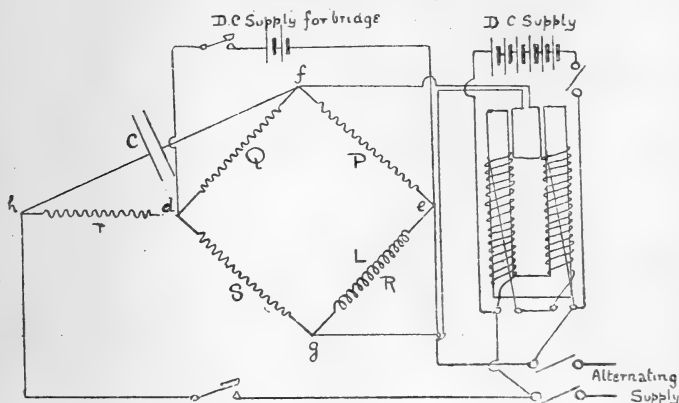
measurement. The bridge is balanced with W and 30 or
40 ohms non-inductive resistance in the arm, but without the
condenser connected to the bridge. The switch is then rocked
over, the condenser connected, and the bridge again balanced
for C by adjusting the resistance r. We found the experi-
ment hopeless with the air-leyden, $r = 0$ and $r = \infty$ giving
deflexions in the same direction. Obviously the capacity of
the air-leyden was too great. We therefore constructed a
rough air-condenser of two concentric tubes 34·4 cms. long,
the mean diameter being 6·3 cms. The tubes were separated
from one another by three small pieces of ebonite attached at
each end to the inner tube. The calculated capacity of this
condenser was 219 M.M.F.'s. On replacing the air-leyden by
this condenser we could easily measure the inductance, and
the following results were obtained using the moving coil C.

TABLE VII.

C.	S.	Q.	P.	r.	L.
219 M.M.F.'s	19·9	1869·3	4165·5	240·7	·000241 henries
	19·9	984·7	2112·7	496·8	·000244 ,,
	19·9	1181·7	2526·5	415·2	·000245 ,,

3 B 2

These results are in agreement with the admittedly rough calculation of the inductance from the dimensions of the coil. The sensitiveness was such that two microhenries could be detected with certainty.

By thus comparing a standard air-condenser with an inductance whose dimensions are sufficiently large and accurately measurable, we believe that " v " could be determined with considerable accuracy.

We can also use a slight modification of the method described for the measurement of coefficients of mutual induction. The results obtained by the method we hope to publish shortly.

We had hoped that the method would have been applicable to the accurate determination of electrolytic conductivities; but though we can get consistent results up to $\frac{1}{10}$ per cent., anything much beyond that seems at present hopeless owing to the apparently erratic motion of the coil. The fact is, we presume, that the capacity of the electrodes is not constant, and the apparatus being so good for measuring capacities becomes correspondingly inefficient for resistances involving variable capacities. This explanation is confirmed by reducing the area of the electrodes; but even with electrodes of only $\cdot 02$ cm.^{2} area or so, the results are only two or three times as consistent. Even with a balancing-cell * the results are not much better. Possibly, however, by making the ohmic resistance of the electrolyte used a negligible quantity the capacity of the electrodes might be measured instead.

Conclusion.—The apparatus and methods described will, we think, prove very useful for the calibration and adjustment of large and small capacities, and for the measurement of self-induction.

The Yorkshire College, Leeds.
  Sept. 28th, 1903.

XC. *The Kinetic Theory of Gases.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,

MR. BURBURY'S note on my "Kinetic Theory of Gases developed from a new standpoint" seems to call for a reply, although his criticisms are so numerous and far reaching that I cannot ask for space in which to attempt to answer them at all fully.

* Phil. Mag. Jan. 1897.

A large part of his criticism, however, centres round the assumption which he has called assumption A. He suggests that I make this assumption unawares, that I prove it to be both true and untrue, and so on. Now the whole point of my former paper as regards assumption A was that it is not a genuine assumption at all, but that it merely amounts to a licence to misapply the calculus of probabilities. It is, if I was right, as illogical to base a kinetic theory on this assumption, coupled with the laws of dynamics, as it would be to base a system of dynamics on the assumption that there is no causation in nature, coupling this assumption with the fundamental laws of dynamics. For the laws of dynamics imply causation with no greater certainty (I am still stating my own case) than they imply the negation of assumption A.

To establish my contention that assumption A is illogical, I attempted to show, in §§ 2 and 3 of my paper, that, when made, it leads to inconsistent results, and it is, I think, on the strength of this that Mr. Burbury charges me with the inconsistency of showing assumption A to be both true and untrue. If by making assumption A, I can prove this assumption to be both true and untrue, surely my contention that assumption A is illogical is sufficiently proved?

To establish my further contention that assumption A is unnecessary, I next attempt to develop the theory of gases without its help. Mr. Burbury now charges me with having made the assumption unawares. I wish that he had substantiated his charge by pointing to the definite stage in my argument at which he thinks the assumption is implied. Let me, however, recapitulate the steps of this argument freed from analytical details. At any instant $6N$ independent (in the usual algebraic sense) variables are required to define the state of the gas. The ranges of these $6N$ variables I propose to represent in a space of $6N$ dimensions. Each point of this space, then, represents a gas. I take a census of all the points in this space, and prove that the laws of equipartition &c. hold for all these points except an inappreciable minority lying on systems of singular points, lines, surfaces, &c. And now let us confine our attention to any particular gas in which the molecules are moving in accordance with the laws of dynamics. The changes taking place in the gas are represented by the motion of a point in my generalized space. Except when this point lies on one of the singular lines, surfaces, &c. just mentioned, equipartition of energy will hold in the gas. It follows from Liouville's Theorem, as I have shown, that there is no tendency of the representative points to crowd on to these singular lines, surfaces, &c. ; and

hence I argue that we are justified in ignoring the infinitesimal intervals of time which a representative point may spend on them, and in predicating equipartition of energy &c. to our unknown gas. This argument, I claim, assumes nothing but the laws of dynamics.

Mr. Burbury's objection to my proof is, in effect, that I assume all combinations of the 3 N velocities with the 3 N coordinates to be equally probable. Of course in constructing my generalized space, I regard the 6 N quantities as absolutely independent variables, but this is a different matter : I may make any possible construction I please, just as if I was proving a proposition of Euclid. If, now, all combinations of the 3 N velocities and 3 N coordinates are not equally probable—if, that is, the motion of the gas, in accordance with the laws of dynamics, tends to set up correlation between the 3 N velocities and the 3 N coordinates—then the fluid in my generalized space will tend to congregate about the regions in the generalized space at which this correlation obtains. This, I prove by Liouville's Theorem, does not occur. I therefore *prove* the absence of correlation between the velocities and coordinates : I cannot see that I assume it.

Yours faithfully,
J. H. JEANS.

XCI. *Notices respecting New Books.*

Manual of Advanced Optics. By C. RIBORG MANN, *Assistant Professor of Physics in the University of Chicago.* Chicago : Scott, Foresman and Co.   1902.   Pp. 196.

THIS excellent little manual of a laboratory course in advanced optics fills a distinct gap in scientific literature, and deserves a warm recognition.   It is just the sort of book which many advanced students must have been longing for.   While giving full practical instructions for the setting up and adjustments of the various forms of apparatus used, the author prefaces each experiment by an exposition of its theory.   The experiments are well chosen, and the necessary apparatus is not expensive.   The subjects dealt with are : limit of resolution, the double slit, the Fresnel mirrors and bi-prism, the Michelson Interferometer, the visibility curves, the prism spectrometer, total reflexion, the diffraction grating, the concave grating, polarized light, rotation of the plane of polarization, elliptically polarized light, reflexion of polarized light from homogeneous transparent substances, metallic reflexion, the spectrophotometer.   At the end of the book are two interesting chapters on " the development of optical theory " and " the trend of modern optics."

On p. 13, the term " wave-front " is applied to a plane the vibrations at different points of which are in different phases ; and on p. 69, in a reference, " meteorology " is, by a printers' error, used for " metrology."

Die Konstitution des Kamphers und seiner Wichtigsten Derivate. *Die Theoretischen Ergebnisse der Kampherforschung monographisch dargestellt von* Ossian Aschan. Braunschweig : Friedrich Vieweg und Sohn. 1903. Pp. xi + 117.

This valuable monograph will be found extremely useful by students of organic and physiological chemistry, and by others interested in the camphor group of compounds. Numerous references to original sources of information are given by the author.

Journal de Chimie Physique. Publié par M. Philippe-A. Guye, *Professeur de Chimie à l'Université de Genève.* Genève : Henry Kündig, 11, Corraterie. Paris : Gauthiers-Villars, 55, Quai des Grands Augustins. Tome I. No. 1. Juillet 1903. Pp. vii + 96.

Nothing could be more eloquent of the great activity now prevailing in the domain of Physical Chemistry than the recent appearance of several new journals devoted to this branch of science, and the establishment of a new Society of Electrochemists in our own country. Among the contributors to the new French journal we notice the names of most of the leaders in physical chemistry. Abstracts from other periodicals, lists of patents, and reviews of recent books form very useful features of the new periodical, and we wish it every success.

Papers on Mechanical and Physical Subjects. By Osborne Rey-nolds, *M.A., F.R S., LL.D., Mem. Inst. C.E., Professor of Engineering in the Owens College, and Honorary Fellow of Queen's College, Cambridge.* Volume III. *The Sub-Mechanics of the Universe.* Cambridge : At the University Press. 1903. Pp. xvii + 254.

Unlike its two predecessors, the present volume is likely to appeal to what we are afraid will be only an extremely limited circle of readers. It consists of a remarkable and very elaborate essay on a possible structure of the physical universe—the only possible one, according to the author. The intricate analysis by which the author seeks to establish his claim to the solution of the problem of the physical universe will prevent all but the most highly-trained mathematicians from following his arguments. While it would be rash to deny that anything from the pen of Prof. Osborne Reynolds is deserving of the closest and most painstaking study, one cannot help regretting that his remarkable gifts of mathematical analysis are not combined with those of clearness and elegance of style, and that his writings are characterized by an almost forbidding sternness of exposition. It also seems to us that in the present instance some of his conclusions are stated with undue dogmatism.

XCII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 636.]

November 4th, 1903.—Sir Archibald Geikie, D.Sc., F.R.S.,
Vice-President, in the Chair.

THE following communications were read :—
1. 'Metamorphism in the Loch-Lomond District.' By E.
Hubert Cunningham-Craig, Esq., M.A., F.G.S.

The area dealt with includes all the Highland rocks on either
side of the Loch, as well as the area lying to the eastward,
including the Trossachs. Each stage of the progressive meta-
morphism can be accurately determined, and each process can be
studied, as a rule, without confusing its effects with those of
another process. The rocks from the Leny-Grit Group and the
Aberfoil-Slate Group show dynamic metamorphism, which increases
on a higher stratigraphical horizon—the Beinn-Ledi Group; and at
Rudha Mor the beginning of the thermal type is seen. This is
quickly superseded by a constructive metamorphism, probably of
hydrothermal type, under which, combined with, or preceded by,
the increasing dynamic metamorphism, the rocks become more
highly crystalline, until all clastic structures are obliterated. The
segregation of like minerals into folia, the total recrystallization,
and the genesis of new mineral-groupings, result finally in the
production of coarsely-crystalline albite-gneisses from a series of
fine and coarse siliceous and felspathic grits. Contact with plutonic
igneous masses obliterates many of the results produced by hydro-
thermal, constructive, metamorphism.

2. 'On a New Cave on the Eastern Side of Gibraltar.' By
Henry Dyke Acland, Esq., F.G.S.

This cave, discovered in 1902, is situated a short distance south
of the eastern end of the tunnel, which pierces the Rock from the
Dockyard on the western side to 'Monkeys' Quarry' on the
eastern. It was opened by blasting operations ; and from
the opening thus made, 88 feet above sea-level, the floor falls to
the west. The main hall is about 70 feet high and 45 feet wide,
and has a smooth stalagmite-floor resting on breccia and a
stalactitic roof covering the limestone of the Rock. Its floor falls
to a point 19 feet above sea-level. The lower gallery descends at its
far end to little if anything short of sea-level. Its floor consists of
stalagmite resting on fine calcareous sand ; this on coarse sand,
followed by rubbly and calcareous grit, which in time rests on the
rock-floor of the cave at a depth of 15 feet. In the calcareous
grit are numerous well-rounded stones, some pierced by pholades.

At a depth of 13 feet were echinids and barnacles. Two other galleries were explored, and in these, as in the lower gallery, the walls are pitted to a height of 28 feet above sea-level. The author concludes that the cave existed at first as a fissure, to which the sea later obtained access through a large entrance for a long period; and during this period the Rock was elevated some 42 feet. The cave was closed to the sea at a period geologically recent; and the breccia and sand-slopes at this point on the eastern side of the Rock, which are 150 feet wide and reach to a height of 200–300 feet above sea-level, date from a still more recent period.

XCIII. *Intelligence and Miscellaneous Articles.*

ON THE INFLUENCE OF MAGNETIC FIELD ON THERMAL CONDUCTIVITY.

To the Editors of the Philosophical Magazine.

GENTLEMEN,— Bologna, Nov. 4, 1903.

I WOULD ask permission to say a few words in connexion with the interesting memoir published by Mr. Vincent J. Blyth in the number of May 1903 of your periodical (p. 529), which begins with the following words :—

" In a circuit composed of bismuth and another metal the thermo-electric E.M.F. is altered, if the bismuth be placed in a magnetic field. This alteration was observed by Leduc and by Righi, while using such a circuit as a means of measuring the difference of temperature between two points of a heat-conveying bar of bismuth."

Now I beg leave to state that in my experiments on the thermal conductivity of bismuth in a magnetic field (*Mem. della R. Accad. dei Lincei*, 4ª serie, t. iv., p. 433, *Il N. Cimento*, 3ª serie, t. xxiv. p. 5), I made use of thermoelectric elements in which bismuth did not enter at all, and which permitted me to measure independently the temperatures in three equidistant points of the bismuth bar.

It was therefore impossible that thermoelectric modifications produced in the bismuth by the magnetic field could have any influence on my results.

I shall be most thankful to you if you will publish these lines in your periodical.

Yours truly,
AUGUST RIGHI,
O. Prof. of Physics in the
University, Bologna.

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END OF THE SIXTH VOLUME.

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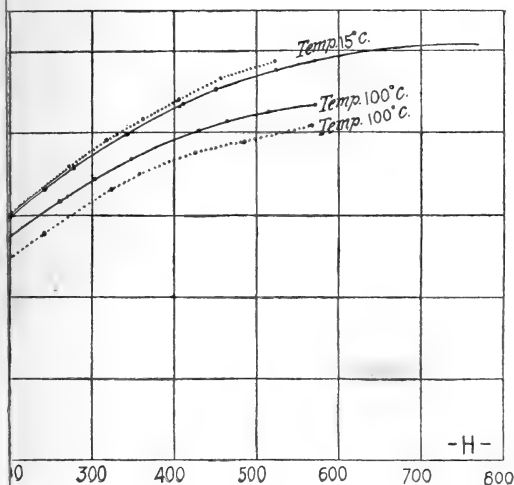

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Fig. 1.—Change of Resistance of Nickel wire under various Loads.
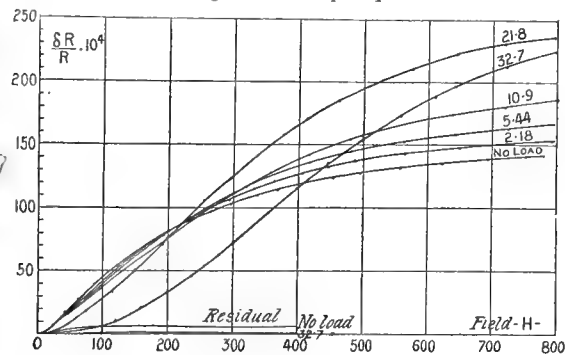Load given in kilos per sq. mm.

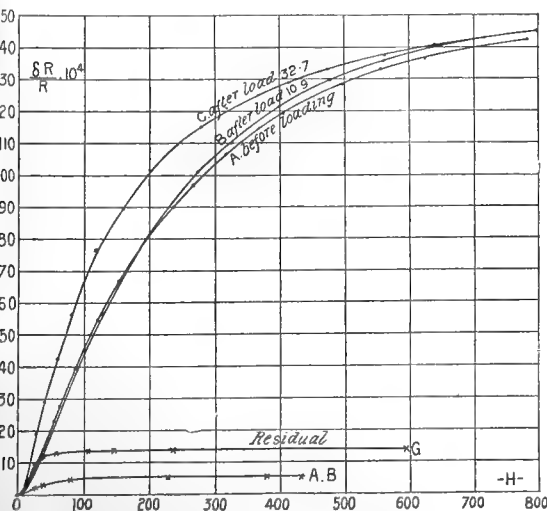Fig. 2.—After-effect of Stress on Change of Resistance in Nickel.

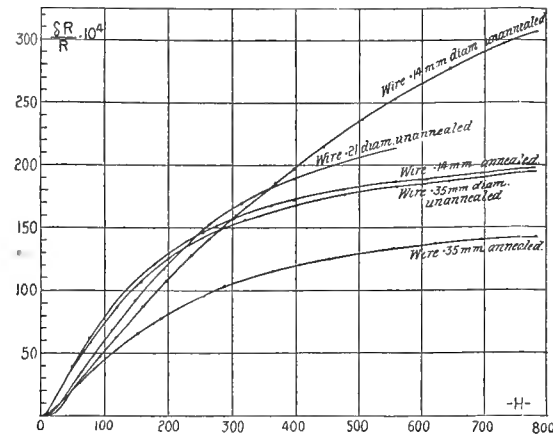Fig. 3.—Magnetic Change of Resistance of various specimens of Nickel wire.

Fig. 4.—Magnetic Change of Resistance in Iron at various Loads. Load given in kilos per sq. mm. Dotted curve shows after-effect of Load 24·5.
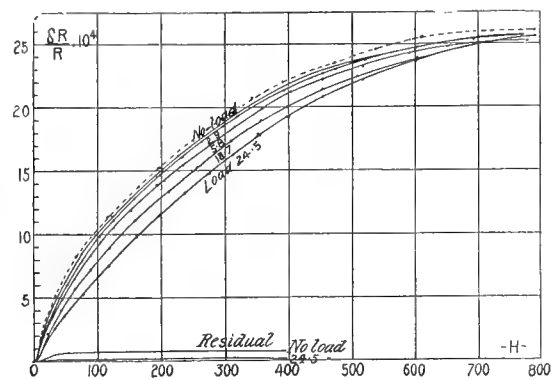
Fig. 5.—Magnetic Change of Resistance in Platinite. Load given in kilos per sq. mm.
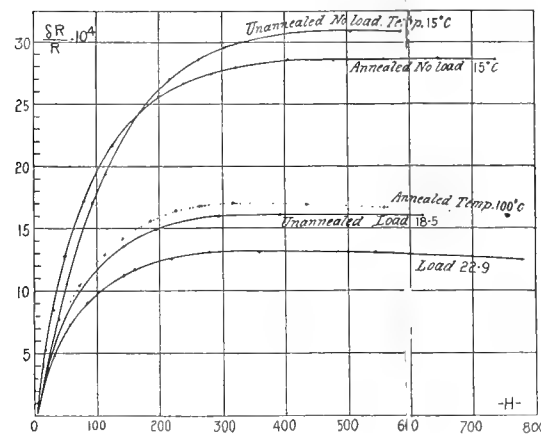
Fig. 6.—Magnetic Change of Resistance in Nickel at 15° C. and at 100° C. Full curves refer to wire ·35 mm. diameter, dotted curves to wire ·14 mm. diameter.
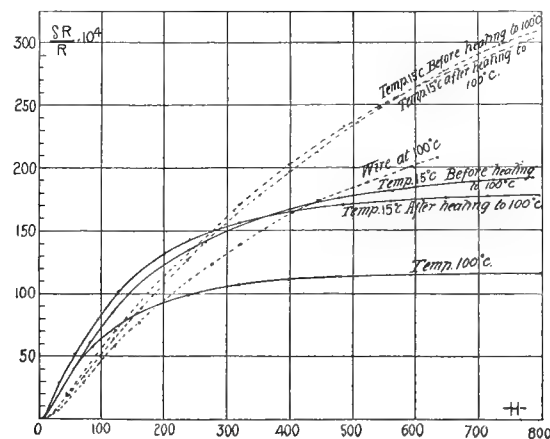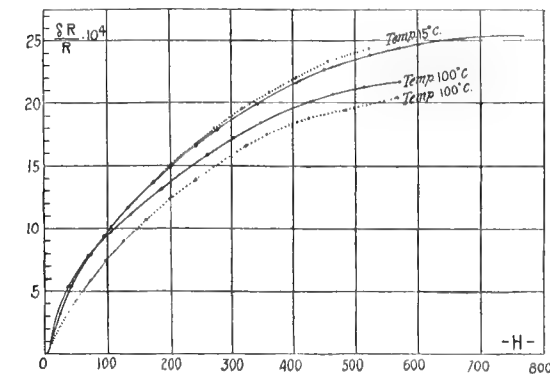

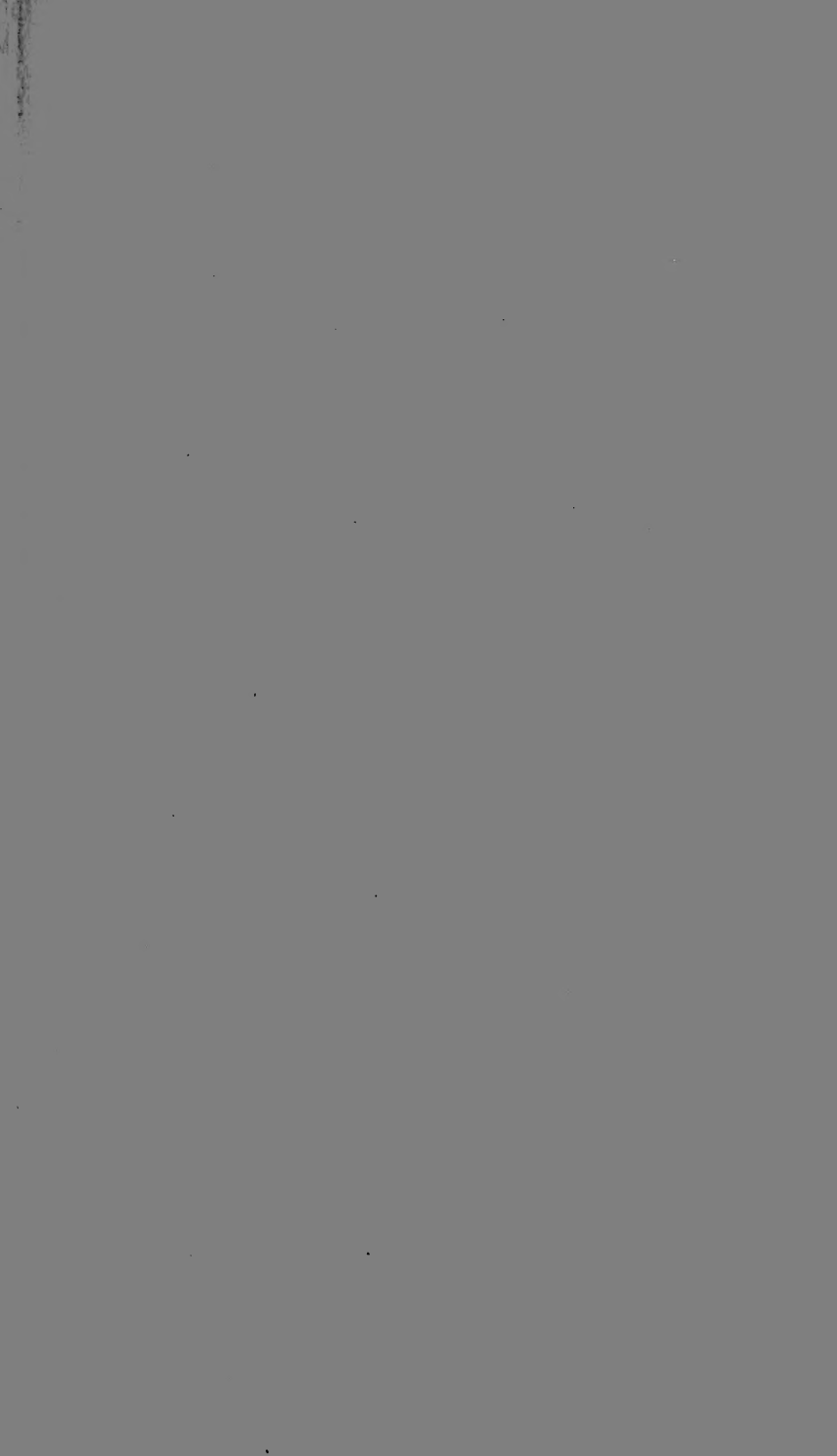Fig. 7.—Magnetic Change of Resistance in Iron at 15° C. and at 100° C. Full curves refer to soft wire, dotted curves to wire hardened by stretching.